Classical solutions by inverse scattering transformation in any number of dimensions. II. Instantons and large orders of the $1/N$ series for the $(\vec{\phi}^{\,2})^2$ theory in ν dimensions $(1 \leq \nu \leq 4)$

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Instantons of the nonlocal effective action S_{eff} that generates a $1/N$ perturbative expansion for O(N)-symmetric $(\vec{\phi}^{\,2})^2$ theory are obtained for Euclidean spatial dimension $0 \leq \nu \leq 4$, through the inverse scattering transformation (IST). They are studied analytically to a large extent. In addition, variational methods are used when the IST does not provide a closed solution for all couplings. The values of the instanton action are given as a function of the coupling constant g for $\nu = 0$, 1, 2, 3, and 4, and $0 \leq g \leq +\infty$. The large orders of the $1/N$ perturbative expansion are thus estimated. It is found that the $1/N$ perturbation series can be resummed by a Borel transform in integer dimension $0 \leq \nu \leq 3$. In four dimensions, the $1/N$ perturbation series is not Borel-summable, owing to the existence of an instanton with real positive action, for physically relevant values of the renormalized coupling constant. It is concluded that $(\vec{\phi}^{\,2})^2$ theory in four dimensions is nonperturbatively unstable. The saddle-point equation of massless $(\vec{\phi}^{\,2})^2$ theory in the $1/N$ expansion is found to be completely integrable at least for spherically symmetric fields. Explicit instanton solutions are given for this case. A large-N estimate of the decay rate of the vacuum is given.

I. INTRODUCTION

The computation of saddle points for one-loop effective actions consisting of the sum of local terms, plus a functional determinant, is a very difficult task, since it amounts to solving a nonlinear and nonlocal equation called the "gap equation." However, the study of these saddle points is very useful in statistical mechanics and quantum field theory.[1] Therefore it would be very interesting to find a general method to solve these equations. It was possible, in some cases, to obtain solutions of a gap equation: the gap equation for the N-dimensional anharmonic oscillator in the $1/N$ expansion has been solved on infinite[2] and finite[3] time intervals. This study was recently extended to a general potential $V(\vec{\phi}^{\,2})$.[4] The resolution is based on the inverse scattering transformation[5] which enables one to express a nonlocal functional determinant as a local functional of the scattering data of an associated linear problem.[6] It is generally easier to search for saddle points of the effective action by extremizing it with respect to those variables. In the preceding paper[1] we used the inverse scattering transformation in the angular momentum.[7] The advantage of this particular inverse scattering transformation is that it can be used in any spatial dimension, provided that the fields are rotationally invariant. This transformation enabled us to express the effective action for $(\vec{\phi}^{\,2})^2$ theory (in particular the one that generates a $1/N$ expansion), in any spatial dimension, partially or completely, as a local functional of the scattering data. This procedure will allow us to investigate the specific problem of the large-order behavior of the $1/N$ perturbation expansion, in these models. It is known that this behavior is dominated by the instantons of the effective action.[8] The contribution of an instanton with effective action S_c to the Kth order of perturbation reads

$$W_K = C\frac{\Gamma(K+b/2)}{|S_c|^{K+b/2}}\cos[(K+b/2)\arg S_c][1+O(1/K)] , \tag{1.1}$$

where b is the number of zero modes.

To obtain an instanton, one has to solve the corresponding gap equation. In this case it reads

$$\left\langle x \left| \frac{1}{-\partial^2+m^2+v(\cdot)} - \frac{1}{-\partial^2+m^2} \right| x \right\rangle = \frac{v(x)}{8g} . \tag{1.2}$$

It will be much easier to find the instantons by using the effective action expressed in terms of the scattering data[1] (SD).

We recall these expressions.

One dimension: $r(k)$ is the reflection coefficient of the potential $v(x)$. $\{\kappa_j^2\}$ are the eigenvalues corresponding to this potential $(j=1,\ldots,N_B)$:

$$S=-2\sum_{j=1}^{N_B}F(x_j^2)-2\int_0^\infty k\frac{dk}{\pi}f(k)\ln[1-|r(k)|^2]\,, \tag{1.3}$$

where

$$F(x)=\arg\tanh(m/\sqrt{x}\,)+\frac{x^{3/2}}{3g}-(m^2-\mu^2)\frac{\sqrt{x}}{2g}\pm\frac{i\pi}{2}\,, \tag{1.4a}$$

$$f(k)=\frac{1}{4}\frac{dF}{dx}\bigg|_{x=i\sqrt{k}}\,, \tag{1.4b}$$

m is the positive solution of $m(m^2-\mu^2)=2g\mu^3$, g is the coupling constant.

Two dimensions: $D(\tau)$ is the norm of the Jost function of the spherically symmetric potential $v(r)$ (in angular momentum). $\{\lambda_K, K=1,\dots,N_B\}$ are the eigenvalues (zeros of the Jost function). $\{c_K, K=1,\dots,N_B\}$ are the corresponding normalization coefficients:

$$\begin{aligned}S_{\rm eff}=&\frac{4}{\pi^2}⨍_0^\infty ⨍_0^\infty \frac{\tau^2+\tau'^2}{(\tau^2-\tau'^2)^2}\ln D(\tau)\ln D(\tau')d\tau\, d\tau'+\ln D(0)\\&-\frac{8}{\pi}\int_0^\infty d\tau\sum_{K=1}^{N_B}\frac{\lambda_K\ln D(\tau)}{\tau^2+\lambda_K{}^2}+2\sum_{K=1}^{N_B}\ln\left[\frac{4\lambda_K{}^2}{c_K}\sin\pi\lambda_K\right]+\sum_{1\le K\ne L\le N_B}\ln\left[\frac{\lambda_K+\lambda_L}{\lambda_K-\lambda_L}\right]^2\\&\pm i\pi\left[1+2\sum_{K=1}^{N_B}E(\lambda_K)\right]-\frac{\pi}{4g}\int_0^\infty r^2v(r)dr\,,\end{aligned} \tag{1.5}$$

where $⨍$ denotes the principal-value integral.

Three dimensions:

$$\begin{aligned}S_{\rm eff}=&-2\int_0^\infty \tau\tanh\pi\tau\ln D(\tau)d\tau-2\pi\sum_{K=1}^{N_B}⨍_0^{\lambda_K}x\tanh\pi x\,dx\\&+\int_0^\infty r^2v(r)\left[1-\frac{\pi v(r)}{2g}\right]dr\pm i\pi\sum_{K=1}^{N_B}\left[\sum_{l=0}^{E(\lambda_K-1/2)}(2l+1)\right].\end{aligned} \tag{1.6}$$

Four Dimensions, $m^2\neq 0$:

$$\begin{aligned}S_{\rm eff}=&\frac{2}{\pi}\int_0^\infty \tau^2\ln D(\tau)[\ln 2-\tfrac{1}{2}+\mathrm{Re}\psi(i\tau)]d\tau\\&+\sum_{K=1}^{N_B}\left[\tfrac{2}{3}(\ln 2-\tfrac{1}{2})\lambda_K{}^3+⨍_0^{\lambda_K}x^2\left[2\psi(x+1)+\pi\cot\pi x-\frac{1}{x}\right]dx\right]\\&-\tfrac{1}{8}\int_0^\infty r^3\ln r[v(r)+2]v(r)dr+\left[\frac{96\pi^2}{g_R}-1\right]\tfrac{1}{16}\int_0^\infty r^3v(r)^2dr\pm i\pi\sum_{K=1}^{N_B}\sum_{l=0}^{E(\lambda_K-1)}(l^2+1)\,,\end{aligned} \tag{1.7}$$

where $\psi(x)=(d/dx)\ln\Gamma(x)$ and $E(\lambda)$ stands for the integer part of λ. g_R is the renormalized coupling constant (see Part I).

Four dimensions, $m^2=0$: $r(k)$ and $\{\lambda_k\}$ have the same definition as $r(k)$ and $\{\kappa_j\}$ in dimension 1. The mass scale is set by an arbitrary quantity μ_0 (scale invariance):

$$\begin{aligned}S_{\rm eff}=&-\frac{1}{4\pi}\int_0^\infty dk\,k^2\ln[1-|r(k)|^2]\left[\frac{6\pi^2}{g_R'}+\mathrm{Re}\psi(ik)\right]\\&+\sum_{K=1}^{N_B}\left[\frac{32\pi^2}{g_R'}\lambda_k{}^3+⨍_0^{\lambda_k}dx\,x^2\left[2\psi(x+1)+\pi\cot\pi x-\frac{1}{x}\right]\right]\pm i\pi\sum_{K=1}^{N_B}\sum_{l=0}^{E(\lambda_k-1)}l^2\,,\end{aligned} \tag{1.8}$$

where g is the running coupling constant, taken at scale e^{x_0}; x_0 is the center of gravity of the field $v^2(x)$:

$$x_0=\int_{-\infty}^{+\infty}xv^2(x)dx\Big/\int_{-\infty}^{+\infty}v(x)^2dx\,.$$

We also recall the derivatives of $v(r)$ with respect to angular momentum scattering data:

$$\delta v(r)=\frac{1}{r}\frac{d}{dr}\left[\frac{1}{r}\left[-\frac{4}{\pi}\int_0^\infty\frac{\tau\sinh\pi\tau}{D(\tau)^3}\varphi(r,i\tau)\delta D(\tau)d\tau+\sum_{K=1}^{N_B}2\varphi_K{}^2(r)\delta c_K+4\sum_{K=1}^{N_B}c_K\varphi_K(r)\dot\varphi_K(r)\delta\lambda_K\right]\right]. \tag{1.9}$$

An interesting expression for the inverse mapping is

$$\delta\lambda_K = -\frac{c_K}{2\lambda_K}\int_0^\infty \varphi_K(r)^2 \delta v(r)dr \ . \tag{1.10}$$

In all those expressions, φ_K or $\varphi(r,i\tau)$ are the solutions $\varphi(r,\lambda)$ of the radial Schrödinger equation, with angular momentum λ equal to λ_K or $i\tau$, defined by

$$\lim_{r\to\infty} e^r\varphi(r,\lambda)=1 \ .$$

Finally we recall that

$$c_K = -\lambda_K{}^2/F(-\lambda_K)\dot{F}(\lambda_K) \ , \tag{1.11}$$

where $F(\lambda)$ is the Jost function.

First of all, we study the general features of those effective actions (Sec. II). Their main characteristic is the existence of infinite-action barriers, which separate the space of configurations into distinct homotopy classes. This allows a classification of instantons, according to the homotopy class to which they belong. Owing to the behavior (1.1), we shall only be interested in the "lowest-action instanton," which corresponds to a single bound-state potential with the lowest possible eigenvalue.

Since it does not seem possible to express the effective action in a completely closed form as a functional of the scattering data, variational methods through numerical computations of $S_{\rm eff}$ are also used. The introduction of the scattering data in the angular momentum considerably simplifies numerical computation of the effective action.

It is also possible to study analytically the gap equation (expressed through the scattering data) and its solutions for weak and strong couplings. The results obtained analytically are in very good agreement with the numerical simulations.

In Secs. III and IV, we study numerically and analytically the gap equation in zero (Sec. III), one, two, and three (Sec. IV) dimensions. In dimension $\nu=0$ and $\nu=1$, all saddle points are given in closed form.[7,9] In dimension $\nu=2$, we study the lowest-action instanton corresponding to a single-eigenvalued potential. For small g, the eigenvalue λ_1 behaves like $\alpha_1 g$; the normalization coefficient behaves as $\alpha_2 g$; the effective action reads $S=-2.95\cdots/g+\ln g\pm i\pi+O(1)$. For large g, C decreases like $g^{\epsilon-1}$ $(\epsilon\sim 0.3)$; S_c behaves like $2(1-\epsilon)\ln g-0.65$; λ_1 has a finite limit $\lambda_\infty=0.818\ldots$; this value λ_∞ can be exactly computed since $S_{\rm eff}$ has a closed form in terms of the scattering data, when $g=+\infty$.[11] In dimension $\nu=3$, again, $S_{\rm eff}$ cannot be expressed as a closed functional of the scattering data. The lowest instanton solution is studied analytically and numerically. When $g\to 0^+$, we get

$$\lambda_1=0.5+\beta_1 g \quad (\beta_1>0), \quad c_1\simeq 2.2 \ ,$$

$$S_c=-\frac{9.44\cdots}{g}+\ln g\pm i\pi+O(1) \ .$$

When g becomes large, we get

$$\lambda_1=1+O(g^{-1}), \quad c_1=O(g^{-1}) \ ,$$

$$S_c=0.94\cdots-115g^{-1}\pm i\pi+O(g^{-2}) \ .$$

In all these cases, the effective action of the lowest instanton has an imaginary part for any positive value of the coupling constant g. The $1/N$ perturbation series can therefore be resummed by a Borel transform.

We notice that in dimension $\nu=1$, the modulus of the effective action decreases as a function of g, which shows that the $1/N$ perturbation series becomes more divergent when g increases. On the contrary, in dimensions $\nu=2$ and $\nu=3$, $|S_c|$ increases for large g, when g increases. This effect is stronger in dimension $\nu=2$ where $|S_c\ (g=+\infty)|=+\infty$. This seems to be a typical feature of field theory.

Finally, we study the case of $(\vec{\phi}^{\,2})^2$ in four dimensions (Sec. V). When the renormalized mass is taken to be zero, the gap equation is analytically solvable through the scattering data. It is shown that the $1/N$ series is not Borel summable when the running coupling constant g_R' is such that $0\le g_R'\le 48\pi^2\gamma^{-1}$ where γ is the Euler constant. This comes from the existence of a real-action instanton. This instanton reads

$$v_c(r)=-\frac{8\lambda_0{}^2}{r^2}\left[\left[\frac{r}{r_0}\right]^{\lambda_0}+\left[\frac{r_0}{r}\right]^{\lambda_0}\right]^{-2} ,$$

where r_0 is an arbitrary length scale, and λ_0 is a solution of the transcendental equation (5.3); $0\le\lambda_0\le 1$.

Other instanton solutions can also be expressed analytically in a closed form [Eq. (5.5)].

Massive $(\vec{\phi}^{\,2})^2$ theory is studied in the limit of a small renormalized coupling constant g_R. For $g_R<0$, the infinite-N limit is not physically meaningful.[12] When $g_R>0$, the $N=\infty$ limit seems physically reasonable, but the theory has a real-action saddle point; hence its perturbation series in $1/N$ is not Borel-summable. This real-action instanton describes the instability of the vacuum $\vec{\phi}=0$. An estimate of the decay rate is found:

$$P=\mathrm{C}\,e^{-(N/2)S_c}[1+O(1/N)], \quad N\gg 1 \ .$$

The instability of $(\vec{\phi}^{\,2})^2$ theory in four dimensions has been argued before by other considerations.[12,10,13]

II. GENERAL FEATURES OF THE EFFECTIVE ACTION

We start to analyze the effective action in dimension ν that generates the $1/N$ expansion for $(\vec{\phi}^{\,2})^2$ theory (see Part I). For a spherically symmetric field, it can be seen from Eq. (I.2.15) that $S_{\rm eff}$ contains a sum of $d(\nu,l)\ln\Delta(\sigma_l)$ where $\sigma_l=l+\nu/2-1$, $l=0,1,2,\ldots$. So, each time a discrete eigenvalue $\lambda_K\ge\nu/2-1$ coincides with an integer (half integer) at $\nu=2,4,6$ $(\nu=3,5,\ldots)$, the effective action blows to $-\infty$. The lowest eigenvalues that make $S_{\rm eff}=-\infty$ are $\lambda_K^0=0,1,2,\ldots$ at $\nu=2$; $\lambda_K^0=\frac{1}{2},\frac{3}{2},\ldots$ at $\nu=3$ and $\lambda_K^0=1,2,3,\ldots$ at $\nu=4$.

These singularities show more explicitly in the expressions of $S_{\rm eff}$ in terms of the scattering data given in the Introduction. One sees in these expressions that the effective action has a logarithmic singularity each time an eigenvalue λ_K coincides with a λ_K^0. The singular part of $S_{\rm eff}$ reads

$$D(\nu,k)\ln(\lambda_K-\lambda_K^0) \ , \tag{2.1}$$

where

$$D(\nu,k)=\sum_{l=0}^{\lambda_K^0-\nu/2+1} d(\nu,l) \tag{2.2}$$

and $d(\nu,l)$ is the degeneracy of angular momentum in ν-dimensional space. These infinite-action barriers actually determine in a crucial way the structure of the action. As a consequence, we shall develop resolution methods which will be adapted to the presence of those barriers. They separate the space of field configurations in homotopy classes defined by a set of integers $\{l_1,\ldots,l_{N_B}\}$. Those integers define the intervals $(l_i+\nu/2-1, l_i+\nu/2)$ where the eigenvalues $\{\lambda_1,\ldots,\lambda_{N_B}\}$ lie. Moreover the imaginary part of the effective action is exactly determined by the class $l_i, 1\le i\le N_B$ to which the configuration belongs. We get

$$\mathrm{Im}S_{\mathrm{eff}}=\pm\pi\sum_{k=1}^{N_B}\sum_{l=0}^{l_k} d(\nu,l) . \tag{2.3}$$

In this paper, we shall study the stationary points (instantons) of the effective action. As is known,[8] these instantons control the large-order behavior of the perturbative series in $1/N$. The contribution of an instanton to the Kth order of any physical quantity reads ($K\gg 1$)

$$C\frac{\Gamma(K+b/2)}{|S_c/2|^{K+b/2}}\cos[(K+b/2)\arg S_c][1+O(K^{-1})] , \tag{2.4}$$

where S_c is the effective action (generally complex) of the instanton and b depends on the number of zero modes, and on the exact nature of the computed quantity. C contains the small-fluctuation contribution around the instanton, and the collective-coordinate-transformation Jacobian.

It is therefore clear the the dominant contribution will come either from an instanton with real action (which would imply that the $1/N$ series cannot be Borel-summable) or from the saddle point with the smallest possible $|S_c|$ (S_c complex). One can expect to have a stationary point in each homotopy sector. We also expect to find the dominant instanton in the lowest nontrivial homotopy sector. Namely, this instanton will have one eigenvalue ($N_B=1$) and this eigenvalue will lie in the interval $(\nu/2-1,\nu/2)$. For $\nu=4$, we find in addition a real-action instanton with $0\le\lambda_1\le 1$. In one dimension, and four dimensions with vanishing physical mass $m^2=0$, we can obtain explicitly the instanton solution in closed form, by extremizing the effective action with respect to each scattering data. The inverse scattering transformation gives us the corresponding field configurations. In the other cases (dimensions $\nu=2,3,4, m^2\neq 0$), it is only possible to get the behavior of the scattering data corresponding to the instanton in limiting cases (weak and strong couplings). For intermediate couplings we use a variational method. We consider trial functions $v_t(r)$ depending on some relevant parameters (e.g., range, depth, . . .). We compute the scattering data for a given configuration $v_t(r)$ by numerically solving the Schrödinger equation (1.11). The numerical computation of S_{eff} is then straightforward, and the research of extrema can be easily done.

III. ZERO DIMENSION[9]

In this case, the functional integral becomes an ordinary N-dimensional Riemann integral:

$$Z=\int\frac{d^N x}{(2\pi)^{N/2}}\exp\left[-\tfrac{1}{2}\vec{x}^{\,2}+\frac{g}{N}(\vec{x}^{\,2})^2\right] . \tag{3.1}$$

One gets after angular integration

$$Z=\frac{2(N/2)^{N/2}}{\Gamma(N/2)}\int_{-\infty}^{+\infty}e^{-NF(t)}dt , \tag{3.2}$$

where

$$F(t)=\tfrac{1}{2}(e^t+2g\,e^{2t}-t) . \tag{3.3}$$

$F(t)$ has one real saddle point, around which an expansion generates the $1/N$ perturbation series, and a pair of complex-conjugated saddle points that control the large-order behavior of the $1/N$ series. One gets for the Kth order

$$A_K=\frac{\sqrt{e}}{\pi}\sin(K\theta)\frac{\Gamma(K)}{\rho^K}\frac{(1+16g)^{1/2}-1}{(16g)^{1/2}} , \tag{3.4}$$

where

$$\rho=(2^2+\pi^2/4)^{1/2} , \tag{3.5a}$$

$$\cos\theta=z/\rho , \tag{3.5b}$$

$$z=-\frac{(1+16g)^{1/2}}{16g}-\tfrac{1}{2}\ln\frac{(1+16g)^{1/2}+1}{(1+16g)^{1/2}-1} . \tag{3.5c}$$

We explicitly find in this exactly solvable case an expression with the general structure (2.4).

IV. ANALYTIC AND NUMERICAL STUDY. ONE, TWO, AND THREE DIMENSIONS

A. One dimension[11]

In this solvable case, the scattering data of the stationary point come from extremizing the effective action given in the Introduction [(1.3)]. We obtain

$$|t(k)|=1,\;\; r(k)=0 , \tag{4.1a}$$

$$\kappa_1=\left[\frac{3m^2-1}{2}\right]^{1/2} , \tag{4.1b}$$

where m is the positive root of $m(1-m^2)=-2g$. The instanton follows from (4.1) by inverse scattering transformation:

$$v_c(x)=-2\kappa^2\mathrm{sech}^2(\kappa x) . \tag{4.1c}$$

Its effective action reads

$$S_c=-\ln(m^2-1)+2\ln\left[m+\left[\frac{3m^2-1}{2}\right]^{1/2}\right]-\frac{2(3m^2-1)}{3\sqrt{2g}}\pm i\pi , \tag{4.1d}$$

where

$$m^2=\tfrac{2}{3}+[g+(g^2-\tfrac{1}{27})^{1/2}]^{2/3}+[g-(g^2-\tfrac{1}{27})^{1/2}]^{2/3} . \quad (4.1e)$$

Limiting cases. When $g\to 0^+$,

$$\kappa_1=1+O(g) , \quad (4.2)$$

$$v_c(x)=\frac{-2}{(\cosh x)^2} , \quad (4.3)$$

which is precisely the square of the instanton for $(\vec{\phi}^{\,2})^2$ theory in g-perturbative expansion[14]

$$S_c=\frac{-2}{3g}+\ln\left[\frac{g}{4}\right]+O(g) . \quad (4.4)$$

It is known that the value $-\frac{2}{3}$ controls the high orders of the perturbation series in g.[14] This feature of instantons, exchange of limits $N\to\infty$ and $g\to 0$, appears in all dimensions. When $g\to+\infty$,

$$\kappa_1{}^2=3g^{2/3}+O(g^0) , \quad (4.5)$$

$v_c(x)$ becomes very deep and narrow (the depth is of order $\kappa_1{}^2=3g^{2/3}$, and the range $\kappa_1{}^{-1}\sim g^{-1/3}$). We recall that as long as g remains positive, $\kappa_1>m$; the effective action of the instanton has an imaginary part $\pm\pi$. This means that the $1/N$ perturbation theory can be Borel summable, due to the oscillating phase of $(S_c)^{-K}$.[14] The real part of S_c is plotted as a function of g in Fig. 1(a).

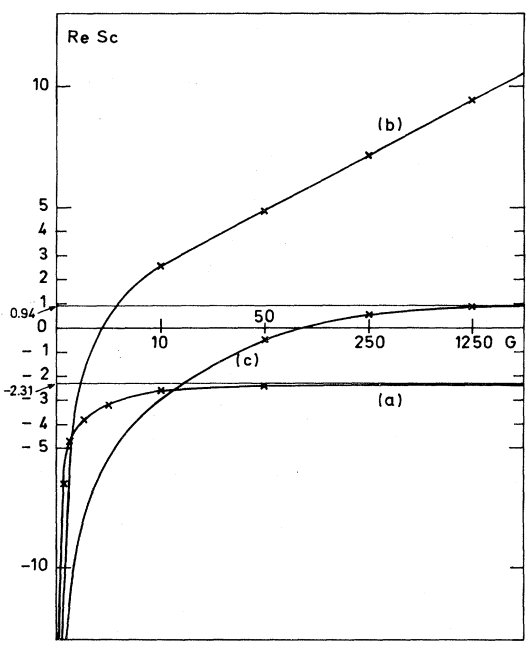

FIG. 1. Real part of the lowest-instanton effective action as a function of the coupling constant g. Curves (a), (b), and (c) correspond to one, two, and three space dimensions, respectively. The imaginary part of $S_{\rm eff}$ equals $\pm\pi$ for all g.

B. Two dimensions

This case is not analytically solvable in general; nevertheless we can study the instanton behavior for limit coupling. We start with the small-coupling regime of instanton $v_c(r)$.

(a) $g\to 0^+$. We assume that the lowest instanton v_c and its eigenfunction $\varphi_K(r)$ have a nonsingular limit when $g\to 0^+$. First of all, the derivative of $S_{\rm eff}$ [Eq. (1.5)] with respect to c_K reads

$$0=\frac{\partial S_{\rm eff}}{\partial c_K}=-\frac{2}{c_K}+\frac{\pi}{g}\int_0^\infty dr\, v_c'(r)\frac{\varphi_K(r)^2}{r} . \quad (4.6)$$

Hence we obtain for $g\to 0^+$

$$c_K=\alpha_1 g , \quad (4.7)$$

where α_1 is a finite numerical constant:

$$\alpha_1=\left[\frac{\pi}{2}\int_0^\infty \frac{\varphi_K(r)^2}{r}v_c'(r)dr\right]^{-1} . \quad (4.8)$$

We will now study the gap equation $0=\delta S_{\rm eff}/\delta v(r)$. When $g\to 0^+$, the term $\pi r v(r)/2g$ which comes from the nonexplicit part of the effective action (1.5) has a singular behavior. This singular behavior can only be canceled by a singular contribution coming from the "explicit part" of the effective action; such a contribution is provided by the infinite-action barriers, and since we study the lowest instanton, it is natural to set ourselves near the $(\lambda_K=0,N_B=1)$ infinite-action barrier. In this case, the singular behavior comes from the single term $\ln D(0)$. It is easy to see that, near this barrier

$$\ln D(0)=\ln\lambda_K+O(1) . \quad (4.9)$$

The effective action can be rewritten in this limit

$$S_{\rm eff}=\ln\lambda_K-\frac{\pi}{4g}\int_0^\infty r v^2(r)dr+O(1) . \quad (4.10)$$

If we now derive (4.10) with respect to $v(r)$, we get

$$0=\frac{\delta S}{\delta v(r)}=-\frac{c_K}{2\lambda_K{}^2}\varphi_K(r)^2-\frac{\pi}{4g}r v(r)+O(1) . \quad (4.11)$$

Equations (4.6) and (4.11) have a common solution for $g\to 0^+$ if the first two terms cancel in Eq. (4.11). Indeed this implies $\lambda_1\to 0$ since we know that $c_1\to 0$ as $g\to 0^+$ (4.8). When $\lambda_1\to 0$:

$$c_1=\lambda_1/2\,|F'(0)|+O(\lambda_1{}^2) . \quad (4.12)$$

We assume that $F'(0)$ is not singular; therefore c_1/λ_1 has a finite limit when $g\to 0^+$. Moreover, we must have

$$\lambda_1=\beta g+O(g^2) , \quad (4.13)$$

$$\frac{\alpha}{2\beta^2}\varphi_1(r)^2=\frac{\pi}{4}r v(r) . \quad (4.14)$$

$r^{-1/2}\varphi_1(r)$ is then a solution of the radial Schrödinger equation with vanishing eigenvalue:

$$\left[-\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+1+\tilde{\alpha}\frac{\varphi_1{}^2(r)}{r}\right]\frac{\varphi_1(r)}{r^{1/2}}=0 . \quad (4.15)$$

This shows that up to a finite normalization ($\tilde{\alpha}$), $r^{-1/2}\varphi_1(r)$ is an instanton solution of the equation of the motion for the $(\vec{\phi}^{\,2})^2$ theory. This instanton controls the large orders in g of the $(\vec{\phi}^{\,2})^2$ theory. Its classical action is known numerically:[15]

$$S=2.95\cdots\ . \tag{4.16}$$

Therefore, S_c finally reads, for $g\to 0^+$,

$$\begin{aligned}S_c&=-\frac{\pi}{2g}\int_0^\infty r v_c{}^2(r)dr+\ln g+O(1)\\&=-\frac{2.95\cdots}{g}+\ln g+O(1)\ ,\end{aligned} \tag{4.17}$$

where we have used Eqs. (4.12)—(4.14) and (4.16).

(b) $g\to+\infty$. In the strong-coupling regime, we will assume that $v(r)$ is short ranged. This assumption will be checked analytically (in an indirect way), and also numerically. It is convenient to introduce in Eq. (4.6) (which is exact for any g) the Jost solution $f_K(r)$ [Eq. (I.1.10), and to rewrite c_K as see Eq. (1.11)]

$$c_K=-\lambda_K{}^2/F(-\lambda_K)F'(\lambda_K)\ . \tag{4.18}$$

We recall that

$$f_K(r)\underset{r\to 0}{\sim} r^{\lambda_K+1/2}[1+O(r)]\ .$$

The saddle-point equation (4.6) now reads

$$-\frac{2F'(\lambda_K)}{\lambda_K{}^2}+\frac{\pi F(-\lambda_K)}{g}\int_0^\infty \frac{f_K(r)^2}{r}V'(r)dr=0\ . \tag{4.19}$$

We assume that, for g large,

$$\left|\int_0^\infty \frac{dr}{r}v'(r)f_K(r)^2\right|<A_1 g^{\epsilon}\ , \tag{4.20}$$

where $\epsilon<1$. This amounts to requiring that, since $f_K(r)$ is perfectly regular and equals $r^{\lambda_K+1/2}$ around $r=0$, $v'(r)$ is not too singular in this region. The interest of introducing f_K instead of φ_K is precisely the fact that f_K has an exactly known behavior for $r\to 0$, while φ_K has not. From Eqs. (4.18)—(4.20), we get

$$c_K<A_2 g^{\epsilon-1}\ ; \tag{4.21}$$

c_K goes to zero when g goes to $+\infty$.

Let us now analyze the second saddle-point equation:

$$\frac{\delta S}{\delta D(\tau)}=0\ . \tag{4.22}$$

From dispersion relations for the Jost function

$$\frac{F(i\tau)}{F_0(i\tau)}=\prod_{K=1}^{N_B}\left[\frac{i\tau-\lambda_K}{i\tau+\lambda_K}\right]\exp\left[\frac{2i\tau}{\pi}\int_0^\infty\frac{\ln D(\tau')d\tau'}{\tau'^2-(\tau-i0)^2}\right], \tag{4.23a}$$

$$\frac{F(i\tau)}{F_0(i\tau)}=\prod_{K=1}^{N_B}\left[1+\frac{\lambda_K{}^2}{\tau^2}\right]\exp\left[\frac{2}{\pi}\int_0^\infty\frac{\tau'd\tau'\delta(\tau')}{\tau'^2-(\tau-i0)^2}\right], \tag{4.23b}$$

where $\delta(\tau)$ is defined by

$$\frac{F(i\tau)}{F_0(i\tau)}=D(\tau)e^{i\delta(\tau)} \tag{4.23c}$$

we get

$$0=\frac{\delta S_{\rm eff}}{\delta D(\tau)}=\frac{4}{\pi D(\tau)}\frac{d}{d\tau}\delta(\tau)-\frac{2\tau\sinh\pi\tau}{gD(\tau)^3}\int_0^\infty dr\, v(r)\frac{d}{dr}\left[\frac{\varphi(r,i\tau)^2}{r}\right]. \tag{4.24}$$

We see that if $\varphi^2(r,i\tau)$ is not singular around $r=0$, $\delta(\tau)$ must be of order $1/g$, since $\delta(\infty)=0$. In the limit

$$\delta(\tau)=0,\quad N_B=1,\quad D(\tau)=1+\lambda_K{}^2/\tau^2\ . \tag{4.25}$$

This corresponds to a rather singular potential, which indirectly checks the hypothesis that $v(r)$ is short ranged for large g; indeed $v(r)$ seems to be zero ranged in the limit $g\to+\infty$.

The last saddle-point equation reads

$$0=\frac{\partial S}{\partial\lambda_K}=\frac{4}{\lambda_K}+2\pi\cot\pi\lambda_K-\frac{8}{\pi}\int_0^\infty\frac{\tau^2-\lambda_K{}^2}{(\tau^2+\lambda_K{}^2)^2}\ln D(\tau)d\tau+\frac{4}{g}\int_0^\infty dr\, v(r)\frac{d}{dr}\left[\frac{\varphi_K(r)\dot\varphi_K(r)}{r}\right]. \tag{4.26}$$

Assuming that

$$g^{-1}\int_0^\infty dr\frac{v'(r)}{r}\varphi_K(r)\dot\varphi_K(r)$$

goes to 0 when $g\to\infty$, using Eq. (4.25) and Ref. 16,

$$\int_0^\infty\frac{t^2-1}{(t^2+1)^2}\ln(1+t^{-2})dt=-\pi/2\ , \tag{4.27}$$

we obtain

$$\frac{4}{\lambda_\infty}+\pi\cot\pi\lambda_\infty=0\ . \tag{4.28}$$

The first root of Eq. (4.29), corresponding to the lowest instanton, is

$$\lambda_\infty^1=0.818\cdots\ . \tag{4.29}$$

We get the following pictures of the scattering data of the leading instanton in large-g limit. The eigenvalue λ_1 tends to the finite value $\lambda_\infty^1=0.818\cdots$, the normalization coefficient c_1 vanishes faster than $g^{\epsilon-1}$, and $D(\tau)$ has the form $1+(\lambda_\infty^1/\tau)^2$. Hence the effective action will be dominated by the term ($\ln c_1$) and goes to $+\infty$ when $g\to+\infty$, which confirms the conjecture of Ref. 11; we have shown here how this value is reached when the nonlinear σ model is obtained from $(\vec{\phi}^{\,2})^2$ theory by sending g to $+\infty$.

(c) Numerical results. A numerical survey of the effective action for g small indeed shows an extremum of action close to the infinite-action barrier $\{\lambda_1=0, N_B=1\}$. The normalization constant $\frac{1}{2}\pi\int rv_c(r)^2dr$ is computed and is found equal to 3.10, which coincides up to 5% with the analytic prediction (4.17). In the general case $(0\leq g<+\infty)$, we study numerically S_{eff} and we find an extremum of S_{eff} in the region of one-eigenvalued potentials with $\lambda_1<1$. This is done for a coupling constant g going from 10 to 1250. It is found that S_c can be represented as a function of g:

$$S_c=1.4\ln g-0.65\pm i\pi\ . \tag{4.30}$$

This shows that the inequality (4.22) can actually be replaced by an approximate equality for large g:

$$c_1\sim Ag^{\epsilon-1}\ . \tag{4.31}$$

Another result is that the behavior of the eigenvalue λ_1 can be approximated by

$$\lambda_1\approx 0.82-3g^{-1/2}\quad\text{for } g\gtrsim 10 \tag{4.32}$$

which confirms the analytically computed value of λ_1 when $g\to+\infty$, $\lambda_1=0.818\cdots$. A more qualitative result is that, as was assumed in the analytic discussion, the extremum configuration tends to a very deep and narrow potential when $g\to+\infty$.

The conclusion of our analytic and numerical studies is the following. There is indeed an instanton in the first nontrivial homotopic region. Its scattering data go from $c_1=O(g), \lambda_1=O(g)$ for g small to $c_1=O(g^{\epsilon-1})$, $\lambda_1=0.818\cdots$ for g infinite. The corresponding action always has an imaginary part $\pm\pi$ which means that the perturbative expansion in $1/N$ can be resummed by using a Borel transform. Its real part goes to $+\infty$ when g becomes large [see Eq. (4.30)]; the perturbation series, which is dominated by $\Gamma(K+\frac{3}{2})/S_c{}^{K+3/2}$ for K large, becomes therefore less divergent when g increases, and it would not be surprising that the conjecture[11] of a convergent $1/N$ series for the nonlinear σ model, which is known to be true for the S matrix[17] and for the form factors[18] should be correct for the Green's functions, too.

The effective action is plotted as a function of g in Fig. 1(b).

C. Three dimensions

As was the case in dimension 2, we cannot solve analytically the gap equation for this model. It is however possible to investigate the limit couplings $g\to 0^+$ and $g\to+\infty$.

(a) Analytic study: $g\to 0^+$. In this limit we shall assume that the one-eigenvalued extremum configuration has a finite-range potential; this implies that the normalization constant c_1 remains finite.

Indeed, if the potential has a finite range a, the integration from a to $+\infty$ in

$$c_K{}^{-1}=\int_0^\infty dr/r^2\varphi_K(r)^2$$

will be of order e^{-2a}. Clearly, if we want c_1 to become infinite, we need (as a necessary condition) $a\to+\infty$.

We shall now derive S_{eff} (1.6) with respect to $v(r)$. We obtain

$$0=\frac{\delta S_{\text{eff}}}{\delta v(r)}=\frac{c_1}{\pi}\tan\pi\lambda_1\varphi_1(r)^2+r^2\left[1-\frac{\pi r}{g}\right]$$

$$+\text{continuum contributions}\ . \tag{4.33}$$

Assuming that the continuum contributions do not have a singular behavior, and since we have supposed $c_1<+\infty$, we obtain, as before, that the $1/g$ singularity must be canceled by an equivalent singular behavior from the contribution of the eigenvalue. We then obtain

$$r^2v_c(r)=\alpha_1\varphi_1(r)^2+O(g)\ , \tag{4.34a}$$

$$\lambda_1=0.5+\alpha_2 g+O(g^2)\ . \tag{4.34b}$$

(We call this the first solution.) Since the potential $v(r)$ is nonsingular, has an eigenvalue λ_1, and a fixed sign [due to (4.34a)], it is necessarily negative. From (4.34b) and (4.33), it is easy to see that $\lambda_1\geq 0.5$ (or $\alpha_2\geq 0$), which means that at least for g small the "first instanton" lies in the complex-action sector. The normalization constant c_1 is given by

$$c_1{}^{-1}=\int_0^\infty\frac{dr}{r^2}\varphi_1(r)^2=\frac{1}{\alpha_1}\int_0^\infty v_c(r)dr \tag{4.35}$$

which is a finite constant, since $v_c(r)$ is assumed to be nonsingular. From (4.35a), we obtain at once

$$\left[-\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+\alpha_1\frac{\varphi_1(r)^2}{r^2}\right]\frac{\varphi_1(r)}{r}=0 \tag{4.36}$$

which shows again, as in one and two dimensions, that $-v(r)$ is the square (up to a normalization) of the instanton solution for the $(\vec{\phi}^{\,2})^2$ theory. This solution controls the large orders of the perturbation expansion in g; its classical action is known numerically:[15]

$$S=-9.44\cdots\ . \tag{4.37}$$

Therefore, the effective action of the extremum configuration will finally read

$$S=-\frac{9.44\cdots}{g}+\ln(g)+O(1)\pm i\pi\ . \tag{4.38}$$

The $\ln g$ is provided by the contribution of $-2\pi\int_0^{\lambda_K}x\,dx\,\tan\pi x$ when $\lambda_K\to\frac{1}{2}$.

(b) Analytic study: $g\to+\infty$. First of all, we derive S_{eff} with respect to c_K. It gives

$$0=\frac{\partial S_{\text{eff}}}{\partial\lambda_K}=\int_0^\infty dr\,r\left[1-\frac{\pi v_c(r)}{2g}\right]\frac{d}{dr}\left[\frac{\varphi_K(r)^2}{r}\right]\ . \tag{4.39a}$$

This equation can only be fulfilled if $v_c(r)/g$ does not vanish for $r\geq 0$ when $g\to+\infty$. If $v_c(r)/g\equiv O$ $(g=+\infty)$, Eq. (4.40a) would read, after partial integration, in this limit

$$\int_0^\infty\varphi_K(r)^2\frac{dr}{r}=0 \tag{4.39b}$$

which means that $\varphi_K(r)=0$ when $g=+\infty$; as we have already seen for

$$\int_0^\infty \frac{dr}{r^2}\varphi_K{}^2(r) ,$$

this implies that $v_c(r)$ has an infinite range. We shall here assume that the first instanton is a simple well-shaped potential with finite range. This will be confirmed by numerical computations. Hence, $v_c(r)$ is in fact a potential whose depth increases like g (or faster than g). Combining this behavior with the request of a single eigenvalue, it is clear that the range of v must vanish when $g \to \infty$ [here by range we mean the distance a such that $v(r) \ll 1$ if $r \gtrsim a$]. Therefore the difference between the free solution $\varphi_K^0(r)$ and the bound-state eigenfunction $\varphi_K(r)$ will be small for $r \gtrsim a$, since both functions have the same boundary condition

$$\lim_{r\to\infty} e^r\varphi_K(r) = \lim_{r\to\infty} e^r\varphi_K^0(r) = 1 , \tag{4.40}$$

and $v(r)$ can be neglected for $r \gtrsim a$. Hence the normalization coefficient of

$$\varphi_K(r), c_K = \left[\int_0^\infty \frac{\varphi_K(r)^2}{r^2} dr \right]^{-1}$$

will diverge when $a \to 0$ since $\int_0^\infty [\varphi_K^0(r)^2/r^2]dr$ diverges like $a^{-2\lambda_K}/2\lambda_K$ when a goes to zero. So, we conclude that $c_K \to 0$ when g goes to $+\infty$. Let us now compute $\partial S/\partial\lambda_K$ when $g \to +\infty$. We get

$$0 = \frac{\partial S}{\partial \lambda_K} = -2\pi\lambda_K \tan\pi\lambda_K - 4c_K \int_0^\infty r\,dr \left[1 - \frac{\pi v(r)}{g}\right] \frac{d}{dr}\left[\frac{\varphi_K\dot{\varphi}_K}{r}\right] . \tag{4.41a}$$

Assuming that c_K vanishes fast enough, we get when $g \to +\infty$

$$0 = -2\pi\lambda_K \tan\lambda_K\pi . \tag{4.41b}$$

The first solution of this equation is $\lambda_K = 0$. We interpret this value as corresponding to the trivial potential $v(r) \equiv 0$, which is a solution of the gap equation for any value of the coupling constant g. Should it correspond to a nontrivial solution, it would mean that a real-action instanton exists, and that the three-dimension $(\vec{\phi}^{\,2})^2$ theory is unstable for large values of g. As a matter of fact, we shall see that numerical computations do confirm the absence of such an instanton. Therefore, we obtain that the first nontrivial solution of Eq. (4.41b) is $\lambda_1 = 1$:

$$c_1 = \frac{-\lambda_1{}^2}{F(-\lambda_1)F'(\lambda_1)} \tag{4.42}$$

goes to zero as $g \to +\infty$. This can be interpreted as a singularity of F around $z = -1$ (we recall that F does not need to be analytic for $z < 0$). The instanton remains in the first complex-action sector, as is the case when $g \to 0^+$.

(c) Numerical results: $g \to 0^+$. We find an extremum of the effective action in the first nontrivial homotopic sector; it corresponds indeed to a nonsingular, finite-ranged potential $v(r)$, with a single eigenvalue

$$\lambda_1 = \tfrac{1}{2} + O(g) \tag{4.43}$$

with

$$O(g) > 0 . \tag{4.44}$$

The numerical value of the limit $\frac{1}{2}\pi\int_0^\infty dr\, r^2 v_c(r)$ is computed and found equal to 9.50, which coincides up to 1% with the analytic prediction (4.38). For the constant c_1, we get the value 2.2 in the limit $g \to 0^+$.

(d) General computation. We do not find any nontrivial extremum in the real-action sector $\lambda_K < 1/2$, confirming the interpretation of the solution $\lambda_K = 0$ previously given. For values of g ranging from 10 to 1250, we find an extremum in the first nontrivial homotopic sector. The results for the real part of the action are plotted in Fig. 1(c). We find that $\mathrm{Re}S_c$ can be accurately represented as

$$\mathrm{Re}S_c(g) = 0.94 - 115g^{-1} \tag{4.45}$$

for $g \gtrsim 150$. The eigenvalue λ_1 corresponding to this potential is also studied. We find it has a limit $\lambda_1 = 1$ when $g \to \infty$. Finally, we want to emphasize that, when g increases, we find that the potential $v_c(r)$ becomes deeper and narrower, which confirms our analytic conclusion.

Our general conclusion will be the following: first of all, the analytic results have been totally confirmed by numerical simulations. It then follows that the $1/N$ perturbation series is controlled by a complex-action instanton, and can therefore be resummed by using the methods of Borel transformation.

Another conclusion is that we now know, with a fairly good precision, the action of the instanton that controls the large orders in $1/N$ of $(\vec{\phi}^{\,2})^2$ when $g \to +\infty$. This limit is known to describe the phase transition of a Heisenberg spin system.[19] The knowledge of the first orders of the $1/N$ perturbation theory[20] and of the large-order behavior of the $1/N$ expansion will make it possible to compute numerical values of critical exponents directly in three dimensions, avoiding $2+\epsilon$ and $4-\epsilon$ expansions.

V. FOUR DIMENSIONS

A. Massless case

We recall that in this case the spherically symmetric instantons can be found analytically, owing to the separability of the action [see (1.8)]. It is more convenient here to introduce the reflection coefficient $r(k)$ rather than the transmission coefficient $t(k)$ or the Jost function $F(k)$ as continuous scattering data. The derivatives of S_{eff} will read [see Eq. (1.8)]

$$0 = \frac{\delta S}{\delta r(k)} = k^2 \left[\mathrm{Re}\psi(ik) + \frac{96\pi^2}{g_R'}\right] \frac{2r(k)^*}{1 - |r(k)|^2} , \tag{5.1}$$

$$0=\frac{\partial S}{\partial\lambda_j}=\lambda_j^2\left[\frac{96\pi^2}{g_R'}+2\psi(1+\lambda_j)+\pi\cot\pi\lambda_j-\frac{1}{\lambda_j}\right]. \tag{5.2}$$

Equation (5.1) immediately gives $r(k)=0$. Equation (5.2) has a trivial solution $\lambda_j=0$. This, together with $r(k)=0$, generates the trivial potential $v=0$.

We shall now study nontrivial solutions $\lambda_K\neq 0$. This leads to

$$\frac{96\pi^2}{g_R'}=-2\psi(1+\lambda_j)+\frac{1}{\lambda_j}-\pi\cot\pi\lambda_j\ . \tag{5.3}$$

The term on the right-hand side behaves like $2\gamma+O(\lambda_j^2)$ around $\lambda_j=0$.[16] It is always increasing, and has asymptotic vertical straight lines for each integer $\lambda_j>0$. The discussion of (5.3) is therefore straightforward.

$0<g_R'<48\pi^2/\gamma$. This is equivalent to the condition $96\pi^2/g_R'>2\gamma$.

Equation (5.3) has an infinite number of solutions. The first one is smaller than 1. Therefore it corresponds to a configuration of a real effective action. The corresponding instanton $V_c(x)$ reads

$$V_c(x)=-\frac{2\lambda_1^2}{\cosh^2\lambda_1(x-x_0)}\ . \tag{5.4a}$$

In terms of the variable r we obtain

$$v_c(r)=-\frac{8\lambda_1^2}{r^2}\left[\left[\frac{r}{r_0}\right]^{\lambda_1}+\left[\frac{r_0}{r}\right]^{\lambda_1}\right]^{-2}. \tag{5.4b}$$

In those formulas, x_0 is a term which is necessary to constrain the potential $V(x)$ and r_0 is an arbitrary scale, $r_0=e^{x_0}\mu_0$, where μ_0 is the arbitrary mass scale (see Part I).

The most general spherically symmetric instanton is a reflectionless potential with any number N_B of bound states, all of them being solutions of (5.3). All those potentials admit closed-form expressions[21]

$$V(x)=-2\frac{\partial}{\partial x}\ln\det M\ ,$$

where M is a $N_B\times N_B$ matrix with elements

$$M_{ab}=\delta_{ab}+\frac{(2\lambda_a\lambda_b)^{1/2}}{\lambda_a+\lambda_b}\exp\left[\frac{\lambda_a+\lambda_b}{2}x+\gamma_a+\gamma_b\right]. \tag{5.5}$$

Here $\{\lambda_i;i=1,\ldots,N_B\}$ is the set of eigenvalues of V, and $\{\gamma_i\}$ are N_B arbitrary parameters. The action can be easily obtained in general; it reads

$$S_{\text{eff}}=\sum_{K=1}^{N_B}s(\lambda_K)\ ,$$

where

$$s(\lambda_K)=\int_0^{\lambda_K}dx\left[\frac{96\pi^2}{g_R}x^2+x^2\left[2\psi(1+x)+\pi\cot\pi x-\frac{1}{x}\right]\right]. \tag{5.6}$$

For the first instanton ($N_B=1,\lambda_1<1$) the effective action S_1 reduces to $s(\lambda_1)$. Equation (5.3) shows that the integrand in (5.6) vanishes when $x=\lambda_K$. Since $96\pi^2 g_R^{-1}>2\gamma$, the integrand is positive in the interval $[0,\lambda_1]$ and S_1 is therefore positive. The $1/N$ perturbation series will be controlled by a term which behaves like $K!S_1^{-K}$ and therefore keeps a constant sign when K is large. The methods of resummation by Borel transformation cannot be applied here: this indicates an instability in the theory itself. Notice that

$$V_1(r)\underset{r\to\infty}{\sim}-C_1r^{-2(1+\lambda_1)}\ , \tag{5.7a}$$

$$v_1(r)\underset{r\to 0}{\sim}-C_2r^{-2(1-\lambda_1)}\ , \tag{5.7b}$$

since $0\le\lambda_1\le 1$, $v_1(\infty)=0$, and $v_1(0)=\infty$. We can therefore interpret v_1 as an instanton describing the transition from the unstable vacuum $\vec{\phi}=0$ into the state of infinite field $\vec{\phi}$ and back to $\vec{\phi}=0$.

$g_R<0$ or $g_R>48\pi^2/\gamma$. In this case, $48\pi^2/g_R<\gamma$ and no solution of (5.3) exists in the interval [0,1]. However, in each interval $[n,n+1]$ with n a nonzero integer, there is a solution of (5.3), which means that there are in fact an infinite number of instantons. We shall study the first instanton: $N_B=1$, $\lambda_1<2$. It is possible to solve analytically (5.3) when g_R' is small. We obtain $\lambda_1=1-g_R/96\pi^2>1$, since g_R' small implies

$$g_R'<0\ . \tag{5.8}$$

The effective action takes the value

$$S_c(g_R')=\frac{32\pi^2}{g_R'}+\ln g_R'+O(1)\pm i\pi\ . \tag{5.9}$$

Those scattering data $[\lambda_1=1,r(k)\equiv 0]$ correspond to a potential $V(x)$ which reads

$$V(x)=-2\,\mathrm{sech}^2(x-x_0)\ , \tag{5.10}$$

where x_0 is determined by the constraint on $V(x)$ (see the Introduction).

Coming back to the variable r ($x-x_0=\ln r/r_0$) we get

$$v(r)=-2\sqrt{2}r_0^2\left[1+\frac{r^2}{r_0^2}\right]^{-2}. \tag{5.11}$$

This is exactly the square of the massless instanton of ϕ^4 theory: $\varphi_c=2\lambda\sqrt{2}/(1+\lambda^2r^2)$ up to a $\sqrt{g}$ factor. As in the first case ($g_R>0$ and $g_R<48\pi^2/\gamma$), the scale λ is $\lambda=\mu_0e^{x_0}$ with the same definition of μ_0 and x_0. Notice that this feature also appeared in the previous case, although we did not mention it. We also check that the dominant term in S_c in (5.9), for $g_R\to 0$, exactly gives the ϕ^4 massless instanton action (with our convention).[14] We see once more that, as in one, two, or three dimensions, a tight link exists between the limit $g\to 0^+$ of the $1/N$ instanton and the g instanton.

When g_R' goes to $+\infty$, we are left with the equation

$$-2\psi(1+\lambda_K)-\pi\cot\pi\lambda_K+1/\lambda_K=0\ .$$

The first solution to this equation is computed numerically. The effective action reads

$$S_c = \fint_0^{\lambda_1} x^2 dx \left[2\psi(x+1) - \frac{1}{x} + \pi \cot \pi x \right] \pm i\pi \ .$$

We obtain

$$\lambda_1 = 1.584 \cdots \ , \quad (5.12a)$$

$$S_c = 1.502 \cdots \ . \quad (5.12b)$$

The effective action of the real-action instanton is plotted in Fig. 2.

The conclusions are the following: (1) For $0 < g_R < 48\pi^2/\gamma$, there is a positive real-action instanton. The $1/N$ series is not Borel-summable. The theory is therefore unstable. (2) For $g_R < 0$ and $g_R > 48\pi^2/\gamma$, all instantons have a complex action. The $1/N$ perturbation series appear as Borel-summable in this case.

B. Massive $(\vec{\phi}^{\,2})^2_4$ theory

Before we begin any numerical or analytic discussion, we must discuss the consistency of the large-N approximation of the $(\vec{\phi}^{\,2})^2$ theory in four dimensions. It is known that, in one, two, or three dimensions, the theory remains well defined as long as both the squared mass m^2 and the unrenormalized coupling constant g are kept positive.[22] In four dimensions, we renormalize this coupling constant, and we want to know which is the region (m^2, g_R) where the large-N limit is well defined. The procedure is simple. One computes the effective potential $V(v)$ which is proportional to the effective action $S_{\text{eff}}[v]$ for v constant, with a given squared mass m^2 and a renormalized coupling constant g_R defined in part (I). Our effective potential is connected with the one in Ref. 12 through a simple renormalization-group transformation. We find that the large-N limit is meaningful if our renormalized coupling constant g_R is kept positive. Otherwise, we find for $g_R < 0$ a $1/N$ perturbation theory with tachyons. Therefore we shall always keep $g_R > 0$.

1. Analytic study; $g_R > 0$

Again, the gap equation is not solvable (at least explicitly), since the effective action S_{eff} does not have a closed form in terms of the scattering data (1.7). The only interesting limit here is the small-coupling one. The gap equation reads

$$0 = \frac{\delta S_{\text{eff}}}{\delta v(r)} = \frac{12\pi^2}{g_R} r^3 v(r) - \sum_{K=1}^{N_B} \frac{c_K \lambda_K}{2} \left[2\psi(\lambda_K+1) + \pi \cot \pi \lambda_K - \frac{1}{\lambda_K} \right] \varphi_K{}^2(r) + (\text{continuum contribution}) \ . \quad (5.13)$$

The same arguments as in dimensions 2 and 3 lead us to the conclusion that, for $g_R \to 0^+$, $N_B = 1$,

$$\lambda_1 = 1 + O(g) \ , \quad (5.14a)$$

$$\varphi_1(r)^2 \sim r^3 v(r) \ . \quad (5.14b)$$

From (5.14b) we obtain the following equation for φ_1:

$$\left[-\frac{d^2}{dr^2} - \frac{3}{r}\frac{d}{dr} + 1 - K\frac{\varphi_1(r)^2}{r^3} \right] \frac{\varphi_1(r)}{r^{3/2}} = 0 \ . \quad (5.15)$$

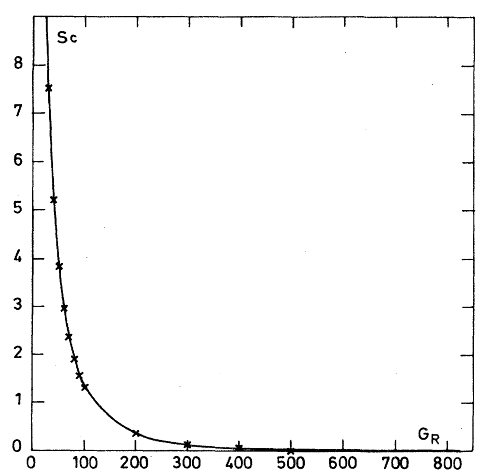

FIG. 2. Lowest-instanton effective action S_{eff} as a function of the running coupling constant g_R' in four space dimensions for $m^2 = 0$. This S_{eff} is always real.

This means that $r^{-3/2}\varphi_1$ must be an instanton solution of massive ϕ^4 theory in four dimensions. It is known that no such instanton exists with a finite action $S(\phi^4)$. In fact, the configuration that extremizes the classical action can be considered as the limit, when the scale goes to infinity, of the instanton of massless (ϕ^4) theory, with a mass "cutoff."[14,23] Although this limit is singular for the instanton solution itself ("virtual instanton"), the ϕ^4 action tends to the massless action value $32\pi^2$.

Here we deal with effective action; anyway we expect, for g_R small, a similar situation as in ϕ^4 theory (g_R perturbation theory). So the $1/N$ instanton is expected to have a singular limit when $g_R \to 0^+$.

An interesting feature of (5.13) is that, if we assume $V_c(r)$ to be negative and with the shape of a well, as it is for $\nu = 1$, 2, 3, and 4 (with $m^2 = 0$), the solution λ_1 attains $1-0$ from below when $g_R \to 0^+$. This means that the instanton—at least for g_R small—has a positive real action. This instanton describes the decay of the unstable vacuum ($\vec{\phi} = 0$) into infinite field configurations. Indeed we know that φ_1 behaves as $r^{\lambda_1 + 1/2}$ around $r = 0$ with $\lambda_1 < 1$. Hence, from Eq. (5.14), $v(r) \sim r^{2\lambda_1 - 2}$ and $V_1(r)$ goes to $-\infty$ when r goes to zero. The situation is exactly the same as in the massless case.

Two conclusions can be drawn from this study. (1) The massive and massless theories have the same qualitative behavior, at least for small g_R and g_R'. (2) The massive theory is unstable for $g_R > 0$ and g_R small.

2. Numerical results

We have studied the interesting case $g_R > 0$ and small, where an instanton of positive real action is present. For weak coupling, the computation of the effective action is simpler than for large coupling ($g_R \gtrsim 96\pi^2$), since the integrals over the continuous scattering data $D(\tau)$ [see Eq. (1.7)] are not dominant. This has been checked numerically for the instanton solution.

As trial functions, we have used two-parameter potentials

$$v(r) = -Pf(r/a) , \tag{5.16}$$

where $f(x)$ is a rapidly decreasing function. For instance, we used $f(x)=e^{-x}, e^{-x^2}, e^{-x^5}, \ldots$.

We want to emphasize that the parameters P and a are the essential variables in the extremization problem. Numerical checks have shown that other variables are essentially redundant. It is therefore possible to draw qualitative conclusions from a study of the effective action as a function of these two parameters only.

When a is small and P is large, we find that the effective action can be approximated by

$$S(P,a) = \frac{\alpha_1}{g_R} P^2 a^4 - \alpha_2 P^2 a^4 \ln a + \alpha_3 \ln[-P^2 a^4 (1-\Lambda a^2) + y_0] . \tag{5.17}$$

The first term comes from $6\pi^2 g_R^{-1} \int_0^\infty r^3 dr\, v(r)^2$; the second one comes from $\int_0^\infty r^3 dr \ln r (v^2+2v)$ when the depth P is much larger than 1. The third term takes into account the infinite-action barrier at $\lambda_1 = 1$. It has been checked numerically that the equation

$$P^2 a^4 (1-\Lambda a^2) = \text{const} \tag{5.18}$$

accurately represented the infinite-action barrier when $P \gg 1$ and $a \ll 1$.

Here the parameters α_1, α_2, α_3, Λ, and y_0 are positive constants depending on the exact shape of $v(r)$.

The study of $S(P,a)$ is easier in the variables a and $y = P^2 a^4 (1-\Lambda a^2)$. We get

$$S(a,y) = \frac{\alpha_1}{g_R} y (1+\Lambda a^2) - \alpha_2 y \ln a (1+\Lambda a^2) + \alpha_3 \ln |y - y_0| \tag{5.19}$$

when we have set $1/(1-\Lambda a^2) \simeq 1 + \Lambda a^2$ since $a \ll 1$.

The stationary-point equations read

$$0 = \frac{\partial S}{\partial a} = 2\frac{\alpha_1}{g_R} \Lambda y a - \frac{\alpha_2 y}{a} (1+\Lambda a^2 + 2\Lambda a^2 \ln a) , \tag{5.20}$$

$$0 = \frac{\partial S}{\partial y} = \frac{\alpha_1}{g_R} (1+\Lambda a^2) - \alpha_2 \ln a (1+\Lambda a^2) + \frac{\alpha_3}{y-y_0} . \tag{5.21}$$

We get from (5.20)

$$a = \left[\frac{\alpha_2}{2\alpha_1 \Lambda} \right]^{1/2} g_R + O(g_R) , \tag{5.22}$$

hence $a \to 0$ when $g \to 0^+$ which shows that our computation is consistent for g small, since we find $a \ll 1$. From (5.21) and (5.22), we get

$$y - y_0 = -\frac{\alpha_3}{\alpha_1} g_R + O(g_R{}^2 \ln g_R), \quad y < y_0 \tag{5.23}$$

and

$$P = \frac{2\sqrt{y_0}}{g_R \alpha_2} \Lambda . \tag{5.24}$$

From (5.22)–(5.24), we obtain the following picture of the instanton when $g_R \to 0^+$:

Its range a behaves as $\sqrt{g_R}$.

Its depth P behaves as $g_R{}^{-1}$.

Its eigenvalue λ_1 stays below 1; the action S has no imaginary part.

Its effective action behaves like

$$S_c \underset{g_R \to 0^+}{\sim} \frac{\alpha_1 y_0}{g_R} - (\alpha_2 y_0 - \alpha_3) \ln g_R + O(1) . \tag{5.25}$$

Numerical simulations show the existence of such an instanton for g_R small, with those qualitative features. The constant $\alpha_1 y_0$ is found numerically equal to $\simeq 330$. This must be compared with the analytical result of the g instanton of massless ϕ^4 theory:

$$32\pi^2 \simeq 320 . \tag{5.26}$$

Finally, it is interesting to note that (5.25) and (5.26) lead to

$$S_N \equiv g_R S_c = 330 \cdots + (\alpha_3 - \alpha_2 y_0) g_R \ln g_R . \tag{5.27}$$

If we replace g_R by the scale parameter $\lambda = 1/a$, we obtain, from (5.27) and (5.22),

$$S_N = 330 \cdots + C\lambda^{-2} \ln \lambda \tag{5.28}$$

which is exactly the action of a massless ϕ^4 instanton with size λ in massive ϕ^4 theory.[14,23]

The conclusion of this numerical study is that an instanton exists for $1/N$ perturbation theory, in the sector of real action. When $g \to 0^+$, this instanton tends to the massive virtual instanton, namely, the massless instanton when the scale is sent to $+\infty$. This indicates the existence of an instability for small coupling. This instability will probably also be present for larger couplings.

The decay rate of the vacuum will be given for large N (Ref. 24) by

$$P = e^{-(N/2) S_c} C [1 + O(1/N)] , \tag{5.29}$$

where C is an N-independent factor; S_c is given by (5.25) when g_R is small and positive, and $m^2 \neq 0$, and by Eq. (5.6) when $m^2 = 0$, and $0 \le g_R' \le 48\pi^2/\gamma$.

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