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# Classical solutions by inverse scattering transformation in any number of dimensions. I. The gap equation and the effective action

J. Avan and H. J. de Vega
*Laboratoire de Physique Théorique et Hautes Energies-Paris, Université Pierre et Marie Curie, Tour 16-1er étage, 4 Place Jussieu, 75230 Paris Cedex 05-France**
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A method to find space-dependent extrema (soliton or instanton) of one-loop effective actions (local terms plus a logarithm of a functional determinant) is given. This method is based on a suitable inverse scattering transformation and can be used in any number of space dimensions, provided the field configurations depend on only one variable. The effective action of $(\vec{\phi}^{\,2})^2$ theory for the $1/N$ series in one, two, three, and four dimensions is worked out in detail. Explicit expressions for the effective action in terms of scattering data are derived. It is found that the gap equation for massless $(\vec{\phi}^{\,2})^2$ theory (in four dimensions) is analytically solvable for spherically symmetric fields.

## I. INTRODUCTION: INVERSE SCATTERING TRANSFORMATION

Saddle-point methods are widely used in quantum field theory and statistical mechanics. The knowledge of a solution of the classical field equations enables one to compute systematically perturbation theories in the coupling constant.

This is true for constant or extended (solitons, instantons) classical solutions. A step beyond classical solutions involves the study of stationary points of an effective action. This effective action is the generating functional of one-particle-irreducible (1PI) Green's functions.[1] It gives the energy of a static configuration in quantum field theory, and the Gibbs free energy in statistical mechanics. Moreover, such an effective action at the one-loop level (integral of local terms plus logarithm of a nonlocal functional determinant) appears in several physical problems: large-$N$ and mean-field approximations;[2-5] fermionic theories when the anticommuting variables have been integrated over;[6] and in connection with Gribov ambiguities in Yang-Mills theories.[7]

The search for extrema of such a one-loop effective action leads to a new type of nonlinear and nonlocal equation (sometimes referred to as a "gap equation") for which no general methods existed up to now. This equation reads, for instance (see below),

$$\left\langle x \left| \frac{1}{-\partial^2+m^2+v(\cdot)} - \frac{1}{-\partial^2+m^2} \right| x \right\rangle = \frac{v(x)}{8g} , \quad (1.1)$$

where $g$ is a coupling constant, and $v(x)$ the unknown solution.

The aim of this paper is to develop a method to extremize such effective actions [which amounts to solving (1.1)] and to find the corresponding solutions in $\nu$-dimensional Euclidean space for any integer $\nu$. This method is based on the fact that there exists a set of natural variables to express an effective action containing the determinant of a local differential operator $O$. These variables are the scattering data (SD) associated with this operator $O$. It often happens that it is easier to extremize $S_{\text{eff}}$ with respect to the SD rather than to the field variables.

Although functional determinants are always naturally related to scattering amplitudes,[8] we will restrict ourselves to configurations in $\nu$-dimensional space that depend only on one variable. In this paper, and the following one, we assume these configurations to be rotationally invariant. Other cases can be treated similarly, e.g., translationally invariant $v$, $v=v(x_1)$ (Ref. 9), etc.

To be specific, we start with the simplest possible case: an $N$-component scalar field with quartic coupling. However, it must be clear that our method can be extended to more general couplings[10] or to fields with nonzero spin, and local symmetries. In $\nu$-dimensional Euclidean space, the generating functional reads

$$Z(\vec{J}) = \int \int \mathscr{D}\vec{\Phi} \exp[-S(\vec{\Phi})] \exp\left[\int_{-\infty}^{+\infty} \vec{J}\cdot\vec{\Phi}\, dx\right] , \quad (1.2)$$

where $S$ is the action of the model:

$$S = \int_{-\infty}^{+\infty} \frac{1}{2}\left[\partial_\mu\vec{\Phi}\cdot\partial^\mu\vec{\Phi} + \mu^2\vec{\Phi}^2 + \frac{2g}{N}(\vec{\Phi}^2)^2\right] d^\nu x . \quad (1.3)$$

Using the Hubbard-Stratonovitch transformation,[11] and integrating over $\vec{\Phi}$, one obtains

$$Z(\vec{J}) = \int \int \mathscr{D}z \exp\left[-\frac{N}{2}S_{\text{eff}}(z)\right] \exp\left[-\int\int d^\nu x\, d^\nu y\, \vec{J}(x)\cdot\vec{J}(y)\left\langle x \left| \frac{1}{-\partial^2+\mu^2+z} \right| y \right\rangle\right] ,$$

 

where $S_{\rm eff}$ is the "one-loop effective action"

$$S_{\rm eff}(z)=\ln\det(-\partial^2+\mu^2+z)-\frac{1}{8g}\int_{-\infty}^{+\infty}z^2(x)d^\nu x \ . \tag{1.4}$$

A shift in $z$ allows us to rewrite this effective action [see Eqs. (2.4)—(2.8)]

$$S_{\rm eff}(v)=\ln\det\left[\frac{-\partial^2+m^2+v}{-\partial^2+m^2}\right]-\frac{m^{\nu-2}}{(4\pi)^{\nu/2}}\Gamma(1-\nu/2)\int_{-\infty}^{+\infty}d^\nu x\, v(x)-\frac{m^{\nu-4}}{8g_B}\int_{-\infty}^{+\infty}d^\nu x\, v^2(x)\ \ (m^2>0)\ . \tag{1.5}$$

This action is shown to be finite for $\mathrm{Re}\nu<6$ provided $g_B{}^{-1}$ contains the required counterterms.

We then look for extrema of this action. We shall restrict ourselves to spherically symmetric configurations. This assumption is not too restrictive because the lowest-extended solutions usually have a maximum of symmetry. The problem becomes one dimensional. We expand the ln det in partial waves:

$$\ln\det\left[\frac{-\partial^2+m^2+v}{-\partial^2+m^2}\right]=\sum_{l=0}^{\infty}\frac{(2l+\nu-2)\Gamma(l+\nu-2)}{l!\Gamma(\nu-1)}\ln\Delta(\lambda)\ \ (\lambda=l+\nu/2-1)\ , \tag{1.6}$$

where

$$\Delta(\lambda)=\ln\det\left[\frac{-\partial_r{}^2+m^2+v+(\lambda^2-\frac{1}{4})/r^2}{-\partial_r{}^2+m^2+(\lambda^2-\frac{1}{4})/r^2}\right]. \tag{1.7}$$

This expression indicates that the scattering data (SD) of the angular momentum for the potential $v$ are the natural variables in this case.[12] These SD are defined by the linear problem

$$\left[-\partial_r{}^2+m^2+v(r)-\frac{1}{4r^2}\right]\chi(r)=\frac{-\lambda^2}{r^2}\chi(r)\ . \tag{1.8}$$

Here $-m^2<0$ plays the role of the energy. For $m^2\neq0$ we can set $m^2=1$. In the case $m^2=0$, it is more convenient to work with the variable $x\equiv\ln r\in\mathbb{R}$. This leads to the linear problem

$$[-\partial_x{}^2+V(x)]\varphi=E\varphi\ , \tag{1.9}$$

where $V(x)\equiv e^{-2x}v(r)$ and $E=-\lambda^2\equiv k^2$. This linear problem is also useful in the one-dimensional case for $m^2\neq0$.[4,5]

The Jost solution of Eq. (1.8) is defined as the regular solution at $r=0$:

$$\lim_{r\to0}r^{-\lambda-1/2}\chi(r,\lambda)=1\ \ (\mathrm{Re}\lambda>0)\ .$$

The Jost function reads

$$\lim_{r\to+\infty}e^{-r}\chi(r,\lambda)\equiv F(\lambda)\ . \tag{1.10a}$$

In the free case ($v\equiv0$) $F$ reads

$$F_0(\lambda)=\sqrt{2\pi}2^\lambda\Gamma(1+\lambda)\ . \tag{1.11}$$

The scattering data associated with $v(r)$ through (1.8) are

$$D(\tau)=\left|\frac{F(i\tau)}{F_0(i\tau)}\right|,\ \ 0\le\tau<+\infty\ ,$$

$$\{\lambda_K,K=1,\dots,N_B\}\ ,$$

where the discrete eigenvalues $\lambda_K$ are the positive real zeros of $F(\lambda)$,

$$\{c_K,K=1,\dots,N_B\}\ ,$$

which are the normalization coefficients of the respective eigenfunctions:

$$c_K=\frac{-\lambda_K{}^2}{F(-\lambda_K)F'(\lambda_K)}\ . \tag{1.12}$$

The Jost solution of Eq. (1.9) satisfies

$$\varphi(k,x)\underset{x\to-\infty}{\sim}e^{ikx}\ .$$

So the Jost function is given by

$$\lim_{x\to+\infty}e^{-ikx}\varphi(x)=[t(k)]^{-1}\ \ (\mathrm{Im}k>0)\equiv F(k)\ . \tag{1.13}$$

The scattering data associated with $V(x)$ through Eq. (1.9) are defined by

$$|t(k)|^2,\ \ 0\le k<\infty$$

$$\{\kappa_j,j=1,\dots,N_B\}\ ,$$

where the discrete eigenvalues $i\kappa_j$ are the zeros of $F(k)$ in $\mathrm{Im}k>0$,

$$\{c_j,\ j=1,\dots,N_B\}\ ,$$

which are the normalization coefficients for the respective eigenfunctions.

The introduction of these scattering data will enable us to express the nonlocal part of $S_{\rm eff}$ (functional determinant) as a local functional of those data. The sum in Eq. (1.6) can now be performed. Using a dispersion relation for the Jost function

$$\frac{F(\lambda)}{F_0(\lambda)}=\prod_{K=1}^{N_B}\frac{\lambda-\lambda_K}{\lambda+\lambda_K}\exp\left[\frac{2\lambda}{\pi}\int_0^\infty\ln D(\tau)\frac{d\tau}{\lambda^2+\tau^2}\right] \tag{1.14}$$

we obtain

$$\ln\det\left[\frac{-\partial^2+m^2+v}{\partial^2+m^2}\right]=\sum_{K=1}^{N_B}\varphi(\lambda_K,\nu)+\int_0^\infty\frac{d\tau}{\pi}\ln D(\tau)\rho(\tau,\nu)\ , \tag{1.15}$$

where

$$\varphi(\lambda,\nu)=-2\,\mathrm{arg\,tanh}\frac{2\lambda}{\nu-2}+\frac{4\lambda}{\nu-2}-2\int_0^\lambda\frac{s\,ds}{\Gamma(\nu-1)}\int_0^1\frac{t^{2-\nu}(1-t)^{\nu-2}}{\Gamma(3-\nu)}dt\int_0^1 x^{\nu/2-1}\frac{x^{-s}-x^s}{1-(1-t)x}dx\ , \tag{1.16a}$$

$$\rho(\lambda,\nu)\equiv i\frac{\partial\varphi(i\lambda)}{\partial\lambda}\ . \tag{1.16b}$$

Those expressions are obtained for $m^2\neq 0$. In the case when $\nu=1$, $m^2\neq 0$, and $\nu=4$, $m^2=0$ (this is the only massless case which we shall study), we use the scattering data associated to the problem (1.9), and similar trace identities.[13]

The local terms of $S_{\text{eff}}$ [Eq. (1.5)] in $v(x)$ will be reexpressed thanks to trace identities which link them to the scattering data, as explicitly as possible.[12]

We finally obtain, for the effective action in two, three, and four dimensions for spherically symmetric $v(r)$ and nonzero renormalized mass $m^2$, $m^2>0$.

*Two dimensions:*

$$S=\frac{4}{\pi^2}⨍_0^\infty⨍_0^\infty\frac{\tau^2+\tau'^2}{(\tau^2-\tau'^2)^2}\ln D(\tau)\ln D(\tau')d\tau\,d\tau'+\ln D(0)-\frac{8}{\pi}\int_0^\infty\left[\sum_{K=1}^{N_B}\frac{\lambda_K}{\tau^2+\lambda_K{}^2}\ln D(\tau)\right]d\tau$$
$$+2\sum_{K=1}^{N_B}\ln\left[\frac{4\lambda_K{}^2}{c_K}\sin\pi\lambda_K\right]+\sum_{1\le K\neq L\le N_B}\ln\left[\frac{\lambda_K+\lambda_L}{\lambda_K-\lambda_L}\right]^2\pm i\pi\left[\sum_{K=1}^{N_B}[1+2E(\lambda_K)]\right]-\frac{\pi}{4g_B}\int_0^\infty rv^2(r)dr\ ,$$

where $⨍$ denotes the principal-value integral.

*Three dimensions:*

$$S=-2\int_0^\infty\tau\tanh\pi\tau\ln D(\tau)d\tau-2\pi\sum_{K=1}^{N_B}⨍_0^{\lambda_K}x\tan\pi x\,dx+\int_0^{+\infty}r^2v(r)\left[1-\frac{\pi v(r)}{2g_B}\right]dr\pm i\pi\sum_{K=1}^{N_B}\left[\sum_{n=0}^{E(\lambda_K-1/2)}(2n+1)\right]\ .$$

*Four dimensions:*

$$S=\frac{2}{\pi}\int_0^\infty\tau^2\ln D(\tau)[\,\mathrm{Re}\psi(i\tau)+\ln 2-\tfrac{1}{2}]d\tau+\sum_{K=1}^{N_B}\left[\tfrac{2}{3}(\ln\tau-\tfrac{1}{2})\lambda_K{}^3+⨍_0^{\lambda_k}x^2\left[2\psi(1+x)+\pi\cot\pi x-\frac{1}{x}\right]dx\right]$$
$$-\tfrac{1}{8}\int_0^\infty r^3\ln r[v^2(r)+2v(r)]dr+\left[\frac{6\pi^2}{g_R}-\frac{1}{16}\right]\int_0^{+\infty}r^3v^2(r)dr\pm i\pi\left[\sum_{K=1}^{N_B}\sum_{n=1}^{E(\lambda_K)}n^2\right]\ .$$

*Here $E(x)$ means the integer part of $x$, $\psi(x)=(d/dx)\ln\Gamma(x)$, $g_B$ is the bare coupling constant in $\nu=2$ and 3 (no coupling-constant renormalization is needed here), and $g_R$ is the renormalized coupling constant in four dimensions* ($m^2\neq 0$) defined by $\Gamma_{\text{IV}}(0,0,0,0)=-g_R$ where $\Gamma_{\text{IV}}$ is here the four-point irreducible Green's function. In the massive one-dimensional case, and the massless four-dimensional case, with the aid of a dispersion relation analogous to (1.14), we get

*One dimension* [general $v(x)$]:

$$S=-2\sum_{j=1}^{N_B}P(\kappa_j{}^2)-2\int_0^\infty k\,dk\frac{p(k)}{\pi}\ln|F(k)|^2\ ,$$

where

$$P(x)=\mathrm{arg\,tanh}\frac{m}{\sqrt{x}}\pm\frac{i\pi}{2}+\frac{x^{3/2}}{3g}+\frac{[(m^2-1)x]^{1/2}}{2g}\ ,$$

$$p(k)=\frac{1}{4}\frac{dP(x)}{dx}2\sqrt{x}\,\bigg|_{x=-k}\ ,$$

$$F(k)\equiv[t(k)]^{-1}\ .$$

*Massless four dimensions* [spherically symmetric $v(r)$]:

$$S=\frac{32\pi^2}{g_R'(\mu_0/\mu)}\left[\sum_{K=1}^{N_B}\kappa_K{}^3\right]+\sum_{K=1}^{N_B}\int_0^{\kappa_K}x^2\left[2\psi(1+x)+\pi\cot\pi x-\frac{1}{x}\right]dx$$
$$-\frac{1}{4\pi}\int_0^{+\infty}k^2dk\ln[\,|F(k)|^2]\left[\frac{6\pi^2}{g_R'(\mu_0/\mu)}+\mathrm{Re}\psi(ik)\right]\pm\left[\sum_{K=1}^{N_B}\sum_{n=1}^{E(\lambda_K)}n\right]i\pi\ ,$$

where again

$E \equiv$ integer part,

$$\psi(x) \equiv \frac{d}{dx} \ln\Gamma(x) ,$$

$g_R'(\mu_0/\mu)$ is the running (renormalized) coupling constant defined at a scale $\mu$ such that

$$\ln(\mu_0/\mu) = \frac{\int_0^\infty dr\, r^3 v^2(r\mu_0) \ln(r\mu_0)}{\int_0^\infty dr\, r^3 v^2(r\mu_0)} .$$

$\mu_0$ is an arbitrary mass scale. We shall study in the following paper the properties of those actions, the solutions of the corresponding stationary-point equations (gap equation), and the application to the specific problem of $1/N$ perturbation series.

## II. LARGE-$N$ LIMIT. EFFECTIVE ACTION AND SCATTERING DATA

We want to study an $N$-field theory with O($N$)-invariant quartic coupling. The generating functional in $\nu$ Euclidean dimensions reads

$$Z(\vec{J}) = \int \int \mathscr{D}\vec{\Phi}(x) \exp\left[-S(\vec{\Phi}) - \int_{-\infty}^{+\infty} \vec{J}(x)\cdot\vec{\Phi}(x) d^\nu x\right] , \tag{2.1}$$

where

$$S(\vec{\Phi}) = \int_{-\infty}^{+\infty} d^\nu x \left[\tfrac{1}{2}\partial_\mu\vec{\Phi}\cdot\partial^\mu\vec{\Phi} + \frac{\mu^2}{2}\vec{\Phi}^2 + \frac{g}{N}(\vec{\Phi}^2)^2\right] .$$

Using the identity (Hubbard-Stratonovitch transformation)

$$\exp\left[-\frac{g}{N}\int_{-\infty}^{+\infty} d^\nu x (\vec{\Phi}^2)^2\right] = \int\int \mathscr{D}\alpha \exp\left[-\int_{-\infty}^{+\infty} d^\nu x \cdot \left[\alpha^2(x) - 2i\sqrt{g/N}\,\alpha(x)\vec{\Phi}^2(x)\right]\right]$$

and integrating over $(\vec{\Phi}^2)$ (Gaussian integration), we get

$$Z(z) = \frac{1}{Z_0}\int\int \mathscr{D}z(x) \exp\left[-\frac{N}{2}\left[\ln\det(-\partial^2+\mu^2+z) - \frac{1}{8g}\int_{-\infty}^{+\infty} z^2(x) d^\nu x\right] + (\text{source-dependent terms})\right] . \tag{2.2}$$

Saddle points of this functional integral are solutions of the so-called "gap equation"

$$\left\langle x \left| \frac{1}{-\partial^2+\mu^2+z(\cdot)} \right| x \right\rangle = \frac{z(x)}{4g} . \tag{2.3}$$

A constant solution $z_0$ obeys therefore

$$\frac{z_0}{4g} = \int_{-\infty}^{+\infty} \frac{d^\nu k}{(2\pi)^{\nu/2}} \frac{1}{k^2+\mu^2+z_0} . \tag{2.4}$$

Hence, setting $\mu^2+z_0=m^2>0$ we get

$$\frac{m^{\nu-2}\Gamma(1-\nu/2)}{(4\pi)^{\nu/2}} = \frac{m^2-\mu^2}{4g} . \tag{2.5}$$

Expansion around the saddle point $z_0$ generates the $1/N$ perturbation theory. The two-point function at leading order reads in momentum space

$$\langle\phi_a(k)\phi_b(k')\rangle = \frac{\delta(k+k')\delta_{ab}}{k^2+m^2} + O(1/N) . \tag{2.6}$$

This allows us to interpret $m^2$ as the renormalized mass for $N$ large. We keep $m^2>0$, so we shall always stay in the O($N$)-symmetric phase of the theory. The spectrum consists of an $N$-plet of massive scalars transforming under the fundamental representation of O($N$).

After a shift in the integration variable,

$$z \to (z+\mu_0{}^2-m^2) \equiv v , \tag{2.7}$$

we get from (2.2)—(2.4)

$$S_{\rm eff}(v)=\ln\det\left[\frac{-\partial^2+m^2+v(\cdot)}{-\partial^2+m^2}\right]-\frac{m^{\nu-2}}{(4\pi)^{\nu/2}}\Gamma(1-\nu/2)\int_{-\infty}^{+\infty}d^\nu x v(x)-\frac{m^{\nu-4}}{8g_B}\int_{-\infty}^{+\infty}v^2(x)d^\nu x\ , \tag{2.8}$$

where $g_B$ is the dimensionless bare coupling constant ($m$ is used as a mass unit). We will now show that $S_{\rm eff}$ can be made finite for $\mathrm{Re}\nu<6$ if $g_B$ contains an adequate counterterm. The ln det can be expanded as a sum of one-loop diagrams, as follows:

$$\ln\det\left[1+\frac{v(\cdot)}{-\partial^2+m^2}\right]=G(0)\int_{-\infty}^{+\infty}v(x)d^\nu x-\frac{1}{2}\int\int_{-\infty}^{+\infty}v(x_1)v(x_2)G^2(x_1-x_2)d^\nu x_1 d^\nu x_2+\cdots$$
$$+\frac{(-)^{n+1}}{n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}v(x_1)\cdots v(x_n)G(x_1-x_2)\cdots G(x_n-x_1)d^\nu x_1\cdots d^\nu x_n+\cdots\ , \tag{2.9}$$

where

$$G(x-y)=\left\langle x\left|\frac{1}{-\partial^2+m^2}\right|y\right\rangle\ . \tag{2.10}$$

The first term in (2.9) reads

$$G(0)\int_{-\infty}^{+\infty}v(x)d^\nu x=\frac{m^{\nu-2}\Gamma(1-\nu/2)}{(4\pi)^{\nu/2}}\int_{-\infty}^{+\infty}v(x)d^\nu x\ . \tag{2.11}$$

This term clearly cancels in (2.8), as it must. Since $v=0$ is a trivial saddle point of $S$ [due to the shift (2.7)], $\delta S/\delta v$ vanishes when $v=0$. When $\nu=2$, all terms with $n\geq 2$ are finite in (2.9), and so is $S_{\rm eff}$ in (2.8). Mass renormalization is enough here to get rid of UV divergences.

When $\nu\rightarrow 3$, no divergence at all appears in (2.8). This phenomenon is well known for any object expressed in terms of one-loop integrals such as $S_{\rm eff}$.

When $\nu\rightarrow 4$, besides mass renormalization, coupling-constant renormalization is also needed here since the second term in (2.9) has a pole:

$$\int_{-\infty}^{+\infty}V^2(x)\frac{d^\nu x}{(2\pi)^{\nu/2}}\int_{-\infty}^{+\infty}\frac{d^\nu k}{(k^2+m^2)^2}\frac{1}{(2\pi)^{\nu/2}}=\left[\frac{m^{\nu-4}}{16\pi^2(\nu-4)}+\text{finite terms}\right]\int_{-\infty}^{+\infty}v^2(x)d^\nu x\ . \tag{2.12}$$

We therefore choose

$$\frac{1}{8g_B}=\frac{1}{16\pi^2}\frac{1}{\nu-4}+\frac{1}{8g_F}\ , \tag{2.13}$$

where $g_F$ is a finite coupling constant.

For higher dimensions ($\nu\geq 6$), new divergences appear in the expansion (2.9). For example, at $\nu=6$, one finds as residue $\sim[\frac{1}{2}(\partial_r v)^2+(v+1)^3-1]$. This cannot be canceled by the previous counterterms in (2.8) since $\phi^4$ theory is nonrenormalizable beyond $\nu=4$. Anyhow, it is possible to derive trace identities by computing the residue at $\nu=2,4,\ldots$ of $\ln\det[(-\partial^2+m^2+v)/(-\partial^2+m^2)]$ in two independent ways: first from the Feynman-diagram expansion (2.9) and then from the expansion of this ln det in terms of the scattering data [for a spherically symmetric $v(r)$].[5,12]

Now we go back to our problem. The extrema of effective action $S$ in (2.8) are general solutions of the nonlocal gap equation:

$$\left\langle x\left|\frac{1}{-\partial^2+m^2+v(\cdot)}-\frac{1}{-\partial^2+m^2}\right|x\right\rangle=\frac{m^{\nu-4}}{4g_B}v(x)\ . \tag{2.14}$$

No general method is known to solve such equations, except in one dimension.[3,5,10,14] We develop here a method to find spherically symmetric solutions of these equations. This method consists in expressing the effective action itself, (2.8), as explicitly as possible, in terms of the scattering data of a given problem, and then extremize $S_{\rm eff}$ with respect to those data. Since $v(r)$ is rotationally invariant, we can expand in partial waves:

$$\ln\det\left[\frac{-\partial^2+m^2+v(\cdot)}{-\partial^2+m^2}\right]=\sum_{l=0}^{\infty}d(\nu,l)\ln\Delta(l+\nu/2-1)\ , \tag{2.15}$$

where $d(\nu,l)$ stands for the degeneracy of angular momentum $l$ in a $\nu$-dimensional Euclidean space,

$$d(\nu,l)=\frac{(2l+\nu-2)\Gamma(l+\nu-2)}{l!\Gamma(\nu-1)} \tag{2.16}$$

and

$$\Delta(\lambda)=\det\left[\frac{-\partial_r{}^2+m^2+(\lambda^2-\frac{1}{4})/r^2+v(\cdot)}{-\partial_r{}^2+m^2+(\lambda^2-\frac{1}{4})/r^2}\right]. \tag{2.17}$$

It is now clear that the spectral problem (1.8) in the angular momentum provides the adequate scattering data in this case ($\nu=2,3,4;m^2\neq 0$). These scattering data read in this case (see Introduction)

$D(\tau)$: continuum contribution ,

$\{\lambda_K>0\}$: discrete eigenvalues ,

$\{c_K\}$: normalization coefficients .

We know[12] that

$$\Delta(\lambda)=\frac{F(\lambda)}{F_0(\lambda)}, \tag{2.18}$$

where $F(\lambda)$ is the Jost function for the potential $v$ in angular momentum variables [see Eq. (1.10)] and $F_0$ is the Jost function in the free case $v\equiv 0$ [see (1.11)]. Therefore we can write a dispersion relation for $\Delta(\lambda)$:

$$\Delta(\lambda)=\prod_{K=1}^{N_B}\frac{\lambda-\lambda_K}{\lambda+\lambda_K}\exp\left[\frac{2\lambda}{\pi}\int_0^\infty\frac{\ln D(\tau)}{\lambda^2+\tau^2}d\tau\right]. \tag{2.19}$$

In this way the sum over $l$ in (2.15) can be performed:

$$\ln\det\left[\frac{-\partial^2+m^2+v}{-\partial^2+m^2}\right]=\sum_{K=1}^{N_B}\varphi(\lambda_K,\nu)-\int_0^\infty d\tau\ln D(\tau)\frac{d\varphi}{d\lambda}\bigg|_{\lambda=i\tau}, \tag{2.20}$$

where

$$\varphi(\lambda,\nu)=-2\sum_{l=0}^{\infty}\frac{2l+\nu-2}{l!}\frac{\Gamma(l+\nu-2)}{\Gamma(\nu-1)}\operatorname{arg\,tanh}\frac{\lambda}{(l+\nu/2-1)}. \tag{2.21}$$

This function $\varphi$ will be reexpressed in a simpler form. Inserting in (2.21) the identities

$$\left[l+\frac{\nu}{2}-1\right]\operatorname{arg\,tanh}\frac{\lambda}{(l+\nu/2-1)}=\lambda+\tfrac{1}{2}\int_0^\lambda ds\left[\frac{1}{l+\nu/2-1-s}-\frac{1}{l+\nu/2-1+s}\right] \tag{2.22}$$

and

$$\frac{\Gamma(\nu-2+l)}{l!}=\frac{1}{\Gamma(3-\nu)}\int_0^1 t^{(\nu-3+l)}(1-t)^{2-\nu}dt, \tag{2.23}$$

we can sum over $l$ with the help of the formula

$$\sum_{l=1}^{\infty}\frac{t^l}{l+z}=\int_0^1\frac{x^3dx}{1-xt}\quad\text{for }z=\nu/2-1\pm s. \tag{2.24}$$

We finally obtain

$$\varphi(\lambda,\nu)=-2\operatorname{arg\,tanh}\frac{2\lambda}{\nu-2}+\frac{4\lambda}{\nu-2}-2\int_0^\lambda\frac{s\,ds}{\Gamma(\nu-1)}\int_0^1\frac{t^{2-\nu}(1-t)^{\nu-2}}{\Gamma(3-\nu)}dt\int_0^1 x^{\nu/2-1}\frac{x^{-s}-x^s}{1-(1-t)x}dx. \tag{2.25}$$

From now on we shall separate the cases $\nu=2$, $\nu=3$, and $\nu=4$ ($m^2\neq 0$).

## III. EFFECTIVE ACTION IN TWO, THREE, AND FOUR DIMENSIONS

### A. Two dimensions

We start from expressions (2.8), (2.20), and (2.25). To express $S_{\text{eff}}(\nu\to 2)$, we shall first of all study the local terms in $v$ from Eq. (2.8). Since the mass scale is naturally given by $m^2\neq 0$, we can set $m^2=1$ without loss of generality; it amounts to setting $v\equiv m^2v_0$ and $r\equiv r_0/m$ with $v_0$ and $r_0$ dimensionless.

The first local part reads

$$\Gamma(1-\nu/2)(\tfrac{1}{4})^{\nu/2}\nu\frac{1}{\Gamma(1+\nu/2)}\int_0^\infty r^{\nu-1}v(r)dr. \tag{3.1}$$

This generates three terms: The first one is the pole at $\nu=2$ from $\Gamma(1-\nu/2)$. It reads

$$\frac{2}{2-\nu}\frac{1}{2}\int_0^\infty rv(r)dr \; . \tag{3.2}$$

It can be reexpressed through the trace identity[12]

$$\int_0^\infty rv(r)dr = -4\sum_{K=1}^{N_B}\lambda_K + \frac{4}{\pi}\int_0^\infty d\tau \ln D(\tau) \; . \tag{3.3}$$

The second one comes from the expansion around $\nu=2$ of the constant factor in front of (2.2). It is also proportional to $\int_0^\infty rv(r)dr$, which allows one to reexpress it by the same trace identity (3.3).

The third one comes from the expansion of $x^{\nu-1}$ around $\nu=2$. It will generate the integral $Q_1=\int_0^\infty r\ln rv(r)dr$. This is a typical renormalization effect. Despite $Q_1$ not appearing in the trace identities, it can nevertheless be expressed as a local functional of the scattering data. The derivation of this expression is given in Appendix A. We obtain

$$\begin{aligned}Q_1=\int_0^\infty r\ln rv(r)dr &= \frac{4}{\pi^2}\fint_0^\infty \fint_0^\infty \frac{\tau^2+\tau'^2}{(\tau^2-\tau'^2)^2}\ln D(\tau)\ln D(\tau')d\tau d\tau' \\ &\quad +\frac{4}{\pi}\int_0^\infty \ln D(\tau)[\,\mathrm{Re}\psi(i\tau)+\ln 2]d\tau - \frac{4}{\pi}\int_0^\infty d\tau \sum_{K=1}^{N_B}\frac{2\lambda_K}{\tau^2+\lambda_K{}^2}\ln D(\tau) \\ &\quad +2\sum_{K=1}^{N_B}\ln\left[\frac{\pi\lambda_K}{c_K\Gamma^2(\lambda_K)}2^{2(1-\lambda_K)}\right]+\sum_{1\le K\pm L\le N_B}{}' \ln\left[\frac{\lambda_K+\lambda_L}{\lambda_K-\lambda_L}\right]^2 .\end{aligned} \tag{3.4}$$

The second local part reads

$$Q_2=-\frac{\pi}{4g}\int_0^\infty rv^2(r)dr \; . \tag{3.5}$$

This term does not seem to have a local expression in terms of the scattering data. For instance, if we try to compute $\partial Q_2/\partial c_K$, we are left (after partial integration) with a term proportional to

$$\int_0^\infty \frac{dv(r)}{dr}\frac{\varphi_K{}^2(r)}{r}dr \; .$$

In contrast with $\partial Q_1/\partial c_K$ (see Appendix A), this term does not show as a total derivative. The ( ln det) term in (2.8) will be obtained from (2.20) and (2.25). We compute $\varphi(\lambda,\nu)$ in (2.25) when $\nu\to 2$. Several terms must be considered.

$-2\arg\tanh[2\lambda/\nu-2]$ is replaced by $\pm i\pi$, according to the prescription $\lambda=\lambda_0\pm i\epsilon$. The derivative of the term, which appears in the continuum contribution (2.20), generates a $\delta(\tau)$. This leads to the appearance of a term $\ln D(0)$ in the effective action.

$4\lambda/\nu-2$ generates a pole term which will read

$$\frac{4}{\nu-2}\left[\sum_{K=1}^{N_B}\lambda_K-\frac{1}{\pi}\int_0^\infty \ln D(\tau)d\tau\right] . \tag{3.6}$$

This cancels as expected, with the pole generated by the first local term in $v$ in (3.3) and (3.4).

Finally we have to compute the remaining finite part of $\varphi$:

$$\varphi_k(\lambda)=-2\int_0^\lambda s\,ds\int_0^1\int_0^1\frac{x^{-s}-x^s}{1-(1-t)x}dx\,dt$$

with $\varphi_k(0)=0$. Differentiating with respect to $\lambda$ we get[15]

$$\frac{\partial\varphi_R}{\partial\lambda}=4\gamma+2\left[2\psi(\lambda)+\pi\cot\pi\lambda+\frac{1}{\lambda}\right],$$

hence

$$\varphi_R(\lambda_K)=4\gamma\lambda_K+4\ln\Gamma(\lambda_K)+2\ln\left[\frac{\sin\pi\lambda_K}{\pi}\right]+2\ln\lambda_K\pm i\pi\mathcal{N} , \tag{3.7}$$

where $\mathcal{N}=2E(\lambda_K)$ (where $E$ stands for the function "integer part"). Combining Eqs. (3.1)–(3.7), we finally obtain

$$\begin{aligned}S_{\mathrm{eff}}&=\frac{4}{\pi^2}\fint_0^\infty\fint_0^\infty\frac{\tau^2+\tau'^2}{(\tau^2-\tau'^2)^2}\ln D(\tau)\ln D(\tau')d\tau d\tau'+\ln D(0)\\ &\quad -\frac{8}{\pi}\int_0^\infty\left[\sum_{K=1}^{N_B}\frac{\lambda_K\ln D(\tau)}{\lambda_K{}^2+\tau^2}\right]d\tau+2\sum_{K=1}^{N_B}\ln\left[4\frac{\lambda_K{}^2}{c_K}\sin\pi\lambda_K\right]+\sum_{1\le K\ne L\le N_B}\ln\left[\frac{\lambda_K+\lambda_L}{\lambda_K-\lambda_L}\right]^2\pm i\pi\mathcal{N}-\frac{\pi}{4g}\int_0^\infty rv^2(r)dr ,\end{aligned} \tag{3.8}$$

where $\mathcal{N}$ is equal to

$$\sum_{K=1}^{N_B} 1 \left[ \text{one } i\pi \text{ for each eigenvalue, coming from } -2\,\text{arg}\tanh\left[\frac{2\lambda}{\nu-2}\right]\right] + \sum_{K=1}^{N_B} 2E(\lambda_K) \tag{3.9}$$

[which comes from the poles in $\sum_K \varphi(\lambda_K 0)$].

### B. Three dimensions

In this case, we can set directly $\nu=3$ in the local terms in $v$ of Eq. (2.8). No trace identity is available to express an integral of $r^2 v(r)$ or $r^2 v^2(r)$ in terms of the scattering data. The local term remains of the form

$$Q_3 = \int_0^\infty r^2 \left[ v(r) - \frac{\pi}{2g} v^2(r) \right] dr \ . \tag{3.10}$$

The ( ln det) can be obtained from (2.20) and (2.25) when $\nu \to 3$. Using the fact that[16] $\lim_{\epsilon\to 0} \epsilon t^{-1-\epsilon} = \delta(t)$, we obtain

$$\varphi(\lambda,3) = -2\,\text{arg}\tanh 2\lambda + 4\lambda - 2\int_0^\lambda s\,ds \int_0^1 \frac{x^{-s+1/2} - x^{s+1/2}}{1-x} dx \ .$$

From the definition and properties of $\psi(x) = (d/dx)\ln\Gamma(x)$,[15] we finally get

$$\varphi(\lambda,3) = -2\pi \int_0^\lambda x\,dx\,\tan\pi x \ . \tag{3.11}$$

Hence the effective action reads in three dimensions:

$$S_{\text{eff}} = 2\int_0^\infty (-\tau\tanh\pi\tau)\ln D(\tau)\lambda\tau - 2\pi \sum_{K=1}^{N_B} \fint_0^{\lambda_K} x\tan\pi x\,dx + \int_0^\infty r^2 v(r)\left[1 - \frac{\pi v(r)}{2g}\right] dr \pm i\pi\mathcal{N} \ , \tag{3.12}$$

where

$$\mathcal{N} = \sum_{K=1}^{N_B} \left[ \sum_{n=0}^{E(\lambda_K - 1/2)} (2n+1) \right]$$

[due to the poles in the integrand of (3.11) when $x = n + \frac{1}{2}$, $n$ integer].

### C. Four dimensions ($m^2 \neq 0$)

We start again from (2.8), (2.20), and (2.25) to obtain the expression for the effective action in terms of the scattering data, and we set $m^2=1$.

The first local term in (2.8) will give, as in the case $\nu=2$, three contributions: a pole ( proportional $[1/(\nu-4)]\int_0^\infty r^3 v(r) dr$), a finite term proportional to $\int_0^\infty r^3 v(r) dr$, and another finite term proportional to $\int_0^\infty r^3 \ln r v(r) dr$.

The second local term in (2.8) gives three contributions due to the coupling-constant renormalization. One is a pole proportional to $[1/(\nu-4)]\int_0^\infty r^3 v^2(r) dr$, one is a finite part proportional to $\int_0^\infty r^3 v^2(r) dr$, and the last one is another finite part proportional to $\int_0^\infty r^3 v^2(r) \ln r\,dr$. It is useful to combine the poles, which allows us to reexpress the residue at $\nu=4$ through the trace identity:

$$\int_0^\infty r^3[v^2(r) + 2v(r)] dr = \tfrac{16}{3} \sum_{K=1}^{N_B} \lambda_K{}^3 + \frac{8}{\pi}\int_0^\infty d\tau\,\tau^2 \ln D(\tau) \ . \tag{3.13}$$

Finite local terms contain the expression

$$-\tfrac{1}{8}\int_0^\infty r^3[v^2(r) + 2v(r)]\ln r\,dr \tag{3.14}$$

which cannot be reexpressed as a local functional of the scattering data.

The ln det term follows from (2.25). Here it is easier to compute $\partial\varphi/\partial\lambda$ rather than $\varphi$ itself. In order to obtain the limit $\nu \to 4$, we use the properties of the distribution:[16]

$$\frac{t^{2-\nu}}{\Gamma(3-\nu)} = \delta'(t) + O(\nu-4) \ . \tag{3.15}$$

Partially integrating Eq. (2.25) and using the properties of $\psi(x)$,[15] we get

$$\varphi(\lambda,\nu\to 4) = \frac{1}{\nu-4}\lambda^3 + \int_0^\lambda s^2[\psi(1+s) + \psi(1-s)] ds + O(\nu-4) \ . \tag{3.16}$$

The contribution of the pole $\lambda^3/(\nu-4)$ to the ln det in (2.20) exactly cancels the pole in the local terms as expected.

The poles in the integrand of (3.16) will contribute to the imaginary part of the effective action as a factor

$$\pm i\pi\mathcal{N} \text{ where } \mathcal{N}=\sum_{n=1}^{E(\lambda)} n^2 \tag{3.17}$$

since $\varphi$ can be rewritten as

$$\varphi(\lambda)=\frac{\lambda^3}{\nu-4}+\int_0^\lambda s^2\left[2\psi(1+s)+\pi\cot\pi s+\frac{1}{s}\right]ds+O(\nu-4)\ . \tag{3.18}$$

Finally, to obtain an interesting expression for $S_{\text{eff}}$ we must define precisely the renormalized coupling constant in terms of a physically meaningful quantity. In the massive theory, we can define the renormalized coupling constant by $g_R/N=-\Gamma_{\text{IV}}(0,0,0,0)$ where $\Gamma_{\text{IV}}(0,0,0,0)$ stands for the one-particle-irreducible (1PI) four-point function at zero external momentum. To compute this four-point function at leading order in $1/N$, we come back to the definition of $S_{\text{eff}}$ [(2.8) and (2.13)]:

$$S_{\text{eff}}=\ln\det\left[\frac{-\partial^2+m^2+v}{-\partial^2+m^2}\right]-\frac{m^{\nu-2}}{(4\pi)^{\nu/2}}\Gamma\left[1-\frac{\nu}{2}\right]\int_0^\infty d^\nu x v(x)-m^{\nu-4}\left[\frac{1}{16\pi^2(\nu-4)}+\frac{1}{8g_F}\right]\int_0^\infty v^2(x)d^\nu x. \tag{3.19}$$

We set

$$\frac{1}{8g_F}=\frac{z_R}{32\pi^2}+\frac{1}{32\pi^2}(\ln 4\pi+1-\gamma) \tag{3.20}$$

for conveniency of the next computations. It then follows by a standard computation that the 1PI four-point function reads

$$\Gamma_{a_1a_2a_3a_4}(x_1,x_2,x_3,x_4)=\delta_{a_1a_2}\delta_{a_3a_4}\Gamma(x_1,x_2,x_3,x_4)+\text{ two other crossed terms.} \tag{3.21}$$

At leading order in $1/N$, $\Gamma$ reads

$$\Gamma(x_1,x_2,x_3,x_4)=\frac{2}{N}\delta(x_1-x_2)\delta(x_3-x_4)G(x_1,x_3)\ , \tag{3.22}$$

where $G$ is the propagator of the $v$ field:

$$G(x,y)=\left[\frac{\delta^2 S_{\text{eff}}}{\delta v(x)\delta v(y)}\right]^{-1}\ . \tag{3.23}$$

In Fourier space, $G$ reads [from (3.19)]

$$G(k)=\frac{(4\pi)^2}{z_R+(1+4m^2/k^2)^{1/2}\operatorname{arg\,tanh}\left[\frac{1}{(1+k^2/m^2)^{1/2}}\right]}\ . \tag{3.24}$$

When $k^2$ goes to 0 ($m^2\neq0$), $G(k)$ goes to a finite limit $16\pi^2/(z_K+1)$; and we finally obtain

$$g_R=-\frac{g6\pi^2}{1+z_R}\ . \tag{3.25}$$

This allows us to give the expression for the effective action of a massive theory in four dimensions, in terms of the scattering data, and the renormalized coupling constant, from Eqs. (2.8), (3.13), (3.14), (3.16), and (3.25):

$$S_{\text{eff}}=\frac{2}{\pi}\int_0^\infty \tau^2[\ln 2-\tfrac{1}{2}+\operatorname{Re}\psi(i\tau)]\ln D(\tau)d\tau+\sum_{K=1}^{N_B}\left[\tfrac{2}{3}(\ln 2-\tfrac{1}{2})\lambda_K{}^3+\fint_0^{\lambda_K}x^2\left[2\psi(1+x)+\pi\cot\pi x-\frac{1}{x}\right]dx\right]$$
$$-\tfrac{1}{8}\int_0^\infty r^3\ln r[2+v(r)]v(r)dr+\frac{1}{16}\left[\frac{96\pi^2}{g_k}-1\right]\int_0^\infty r^3v^2(r)dr\pm i\pi\sum_{K=1}^{N_B}\sum_{l=1}^{E(\lambda_K)}l^2\ . \tag{3.26}$$

## IV. EFFECTIVE ACTION IN ONE-DIMENSIONAL AND MASSLESS FOUR-DIMENSIONAL CASES

### A. One dimension

It is known that the one-dimensional effective action $S_{\text{eff}}$ can be reexpressed in a closed form as a local functional of the scattering data of the auxiliary problem (1.9) studied in the Introduction:

$$\left[-\frac{d^2}{dx^2}+V(x)\right]\varphi(x)=k^2\varphi(x)\ .$$

We recall that these data are the modulus of the Jost function $|F(k)|$, $k\in\mathbb{R}^+$; the set of eigenvalues (positive zeros of $F$) $\{\kappa_j, j=1,\ldots,N_B\}$; and the set of corresponding normalization coefficients $\{c_j, j=1,\ldots,N_B\}$ (see the Introduction). We obtain

$$S_{\rm eff}=-2\sum_{j=1}^{N_B}P(\kappa_j)+\int_0^\infty p(k)k\ln|F(k)|^2dk\ , \tag{4.1}$$

where

$$P(x)=\arg\tanh\frac{m}{x}\pm\frac{i\pi}{2}+\frac{x^3}{3g}-\frac{(m^2-\mu^2)^{1/2}}{2g}x\ ,p(k)=\frac{1}{4}\frac{dP}{dx}\bigg|_{x=i\sqrt{k}}\ , \tag{4.2}$$

$m$ is defined by the equation $(m^2-\mu^2)m=-2g$, where $\mu$ is the bare oscillator mass, and $g$ is the coupling constant.

For large $N$, the energy levels of this quantum-mechanical problem with $N$ degrees of freedom read

$$E_{nl}=\frac{N}{2}m+lm+2n\left[\frac{3m^2-\mu^2}{2}\right]^{1/2}+O(1/N)\ , \tag{4.3}$$

where $n$ and $l$, respectively, stand for the principal and angular quantum numbers. This allows us to reinterpret $m$ as the angular quantum.[14]

### B. Four dimensions—$m^2=0$

We come back to the functional determinant in Eq. (2.8), which we expand in partial waves as in (2.15). Now we set $m^2=0$; $\Delta(\lambda)$ [in (2.15)] reads

$$\Delta(\lambda)=\det\left[\frac{-\partial_r{}^2+(\lambda^2-\frac{1}{4})/r^2+v(\cdot)}{-\partial_r{}^2+(\lambda^2-\frac{1}{4})/r^2}\right]\ . \tag{4.4}$$

The linear problem naturally associated with $\Delta(\lambda)$ is

$$\left[\partial_r{}^2-\frac{1}{4r^2}+v(\cdot)\right]\chi=-\frac{\lambda^2}{r^2}\chi\ . \tag{4.5}$$

Through the change of variables

$$x\equiv\ln\mu_0 r,\quad V(x)\equiv e^{2x}\frac{v(r)}{\mu_0{}^2},\quad \Psi(x)\equiv e^{-x/2}\chi(r)\ ,$$

where $\mu_0$ is an arbitrary mass scale, Eq. (4.5) is equivalent to

$$\left[-\frac{d^2}{dx^2}+V(x)\right]\Psi(x,\lambda)=-\lambda^2\Psi(x,\lambda)\ , \tag{4.6}$$

where

$$\lim_{x\to-\infty}V(x)=\lim_{r\to 0}r^2v(r)=0\ ,$$

$$\lim_{x\to+\infty}V(x)=\lim_{r\to+\infty}r^2v(r)=0\ .$$

We assume that $v(r)$ decreases faster than $r^{-2}$ for large $r$ so that $V(x)$ decreases exponentially (or at least very fast) for $x\to+\infty$. Hence we can identify $\ln\Delta(\lambda)$ with the logarithm of the Jost function defined in (1.11):

$$\ln\det\left[\frac{-\partial_r{}^2+(\lambda^2-\frac{1}{4})/r^2+v(\cdot)}{-\partial_r{}^2+(\lambda^2-\frac{1}{4})/r^2}\right]=\ln\det\left[\frac{-\partial_x{}^2+\lambda^2+V(\cdot)}{-\partial_x{}^2+\lambda^2}\right]\equiv\ln F(-\lambda^2,V)\ . \tag{4.7}$$

A dispersion relation for $F$ reads

$$F(-\lambda^2,V)=\prod_{j=1}^{N_B}\left[\frac{\lambda-\kappa_j}{\lambda+\kappa_j}\right]\exp\left[\frac{1}{\pi}\int_0^{+\infty}\frac{\lambda\,dk'}{\lambda^2+k'^2}\ln|F(k')|^2\right]\ , \tag{4.8}$$

where $\kappa_j$ is defined as in one dimension. Hence, from (2.15)

$$\ln\det\left[\frac{-\vec{\nabla}^2+v}{-\vec{\nabla}^2}\right]=\sum_{l=0}^{\infty}\sum_{j=1}^{N_B}d(\nu,l)\ln\left[\frac{l+\nu/2-1-\kappa_j}{l+\nu/2-1+\kappa_j}\right]+\text{continuum contribution.} \tag{4.9}$$

This leads to

$$\ln\det\left[\frac{-\vec{\nabla}^2+v(\cdot)}{-\vec{\nabla}^2}\right]=\sum_{j=1}^{N_B}\varphi(\kappa_j,\nu)-\frac{1}{\pi}\int_{-\infty}^{+\infty}dk\,\rho(k,\nu)\ln|F(k)|\ , \tag{4.10}$$

where $\varphi$ and $\rho$ are defined exactly as in (2.21). $\varphi$ is therefore given by expression (3.16).

The contribution from the local terms can be expressed in terms of the scattering data of problem (1.9) by means of trace identities. The linear term in $v$ from Eq. (2.8) vanishes, when $m^2=0$, due to the $(m^2)$ factor that appears in front of it.

We have given the bare coupling constant $g$ a dimension by introducing $1/g=\mu_0{}^{\nu-4}/g_B$ where $\mu_0$ is the mass scale introduced in the change of variables (4.6). $1/g_B$ must be renormalized as in (2.13) to cancel the UV divergences appearing in the ln det. Once this is done, the finite part of the quadratic term can be rewritten as

$$\int_0^{\infty}r^3v^2(r)dr=\int_{-\infty}^{+\infty}V^2(x)dx=\tfrac{16}{3}\sum_{j=1}^{N_B}\kappa_j{}^3+\frac{4}{\pi}\int_{-\infty}^{+\infty}\tau^2\ln|F(\tau)|^2d\tau\ , \tag{4.11}$$

$$\int_0^{\infty}r^3\ln\mu_0 r v^2(r)dr\equiv\int_{-\infty}^{+\infty}xV^2(x)dx\ . \tag{4.12}$$

There is no available trace identity for this last term. Let us introduce now the mean size $\mu^{-1}=R$ of the potential $v(r)$ through the identity

$$\ln\left[\frac{\mu}{\mu_0}\right]=\frac{-\int_0^{\infty}\ln(\mu_0 r)v^2(r)r^3dr}{\int_0^{\infty}r^3v^2(r)dr}\ . \tag{4.13}$$

In the $x$ variables, $\ln(\mu/\mu_0)$ corresponds to the center of gravity of the squared potential in the $x$ axis. From Eqs. (4.10), (2.8), (2.13), (3.16), and (4.11)–(4.13), we obtain for the effective action

$$S_{\text{eff}}=\sum_{j=1}^{N_B}\int_0^{\kappa_j}x^2\left[2\psi(1+x)+\pi\cot\pi x-\frac{1}{x}\right]+\frac{1}{4\pi}\int_0^{\infty}\tau^2\,\mathrm{Re}\psi(i\tau)\ln|F(\tau)|^2d\tau$$
$$+2\pi^2\left[\sum_{j=1}^{N_B}\frac{16\kappa_j{}^3}{3}+\text{continuum contribution}\right]\left[-\frac{1}{8g_f}-\ln\frac{\mu_0}{\mu}+\frac{1}{32\pi^2}(-\tfrac{1}{2}+\gamma-\ln\pi)\right]. \tag{4.14}$$

It is natural to introduce the running coupling constant. We define

$$\frac{6\pi^2}{g_R'(\mu/\mu_0)}\equiv(2\mu^2)\left[-\frac{1}{8g_F}-\ln\frac{\mu_0}{\mu}+\frac{1}{32\pi^2}(-2+\gamma-\ln\pi)\right] \tag{4.15}$$

so the effective action reads

$$S_{\text{eff}}=\sum_{j=1}^{N_B}\left[\frac{32\pi^2}{g_R'}\kappa_j{}^3+⨍_0^{\kappa_j}x^2\left[2\psi(1+x)+\pi\cot\pi x-\frac{1}{x}\right]dx\pm i\pi\sum_{n=1}^{E(\lambda_K)}n^2\right]$$
$$+\frac{1}{4\pi}\int_{-\infty}^{+\infty}\tau^2\ln|F(\tau)|^2\left[\mathrm{Re}\psi(i\tau)+\frac{96\pi^2}{g_R'}\right], \tag{4.16}$$

where $g_R'(\mu/\mu_0)$ is the effective coupling constant at the dimensionless scale $(\mu/\mu_0)$ associated with the field configuration $V(x)$ through (4.13). It must be remarked that this massless effective action has a closed form in terms of the scattering data associated with the linear problem (1.9). This will allow us to get exact analytic solutions (instantons) of the zero-mass version of the gap equation (2.14), provided that the scale $(\mu/\mu_0)$ remains fixed. So we shall get extrema or stationary points of the effective action *with a constraint* on the dilatations on $r$ or equivalently on the translations on $x$: $V(x)$ will be obtained up to a translation on $x$, through inverse scattering transformation; this translational degree of freedom will be suppressed by the constraint, and $V(x)$ will be uniquely determined.

## APPENDIX A: NEW TRACE IDENTITY

We want to obtain an explicit expression for $Q_1=\int_0^{\infty}r\ln rv(r)dr$ in terms of the scattering data of $v(r)$. An easy way to do that is to compute the derivatives of $Q_1$ with respect to those scattering data. We recall that the functional derivatives of $v(r)$ with respect to the scattering data can be explicitly obtained from the Gel'fand-Levitan-Marchenko equation:[12]

$$\delta v(r)=\frac{1}{r}\frac{d}{dr}\left\{\frac{1}{r}\left[-\frac{4}{\pi}\int_0^\infty \frac{\tau\sinh\pi\tau}{[D(\tau)]^3}[\varphi(r,i\tau)]^2\delta D(\tau)d\tau+4\sum_{K=1}^{N_B}\left[c_K\varphi_K(r)\frac{d}{d\lambda_K}\varphi_K(r)\delta\lambda_K+\tfrac{1}{2}\varphi_K{}^2(r)\delta c_K\right]\right]\right\}, \quad (A1)$$

where $\varphi$ is the regular solution of the radial Schrödinger equation (1.8) such that $\lim_{(r\to+\infty)}e^r\varphi(r,\lambda)=1$. This enables us to explicitly compute all derivatives of $Q_1$ with respect to scattering data. The first derivative reads

$$\frac{\delta Q_1}{\delta c_K}=\int_0^\infty \ln r\frac{d}{dr}\left[2\frac{\varphi_K{}^2(r)}{r}\right]dr . \quad (A2)$$

Partial integration leads to

$$\frac{\delta Q_1}{\delta c_K}=-2\int_0^\infty \frac{\varphi_K{}^2(r)}{r^2}dr=\frac{-2}{c_K} , \quad (A3)$$

where $\varphi_K(r)\equiv\varphi(r,\lambda_K)$ is the eigenfunction corresponding to the eigenvalue $\lambda_K$ of the linear problem (1.8). We recall

$$c_K{}^{-1}=\int_0^\infty \frac{\varphi_K{}^2(r)}{r^2}dr . \quad (A4)$$

Hence

$$Q_1=-2\ln c_K+Q_2(\lambda_K,D(\tau)) . \quad (A5)$$

We now compute

$$\frac{\delta Q_2}{\delta\lambda_K}=\int_0^\infty \ln r\frac{d}{dr}\left[\frac{4c_K}{r}\varphi_K(r)\frac{\partial}{\partial\lambda}\varphi(r,\lambda_K)\right]dr .$$

Again, through partial integration,we get

$$\frac{\delta Q_2}{\delta\lambda_K}=\int_0^\infty \frac{4c_K}{r^2}\varphi_K\frac{\partial}{\partial\lambda}\varphi(r,\lambda_K)dr . \quad (A6)$$

Now we write $\dot\varphi_K$ as

$$\lim_{\delta\lambda_K\to 0}\frac{\varphi(r,\lambda_K+\delta\lambda_K)-\varphi(r,\lambda_K)}{\delta\lambda_K}\equiv\dot\varphi_K(r) \quad (A7)$$

and we use the equality

$$\frac{d}{dr}W(\varphi(r,\lambda_1),\varphi(r,\lambda_2))=\frac{\varphi(r,\lambda_1)\varphi(r,\lambda_2)}{r^2}(\lambda_1{}^2-\lambda_2{}^2) , \quad (A8)$$

where $W$ is the Wronskian of the two solutions $\varphi(r,\lambda_1)$ and $\varphi(r,\lambda_2)$. We now set $\lambda_1\to\lambda_2$ in (A8), and using (A7) and (A8), we integrate exactly (A6). Using the behavior of $\varphi(r,\lambda)$ when $r\to 0$ (Ref. 12),

$$\varphi(r,\lambda)=\frac{F(\lambda)}{\lambda}r^{-\lambda+1/2}-\frac{F(-\lambda)}{\lambda}r^{\lambda+1/2} \quad (A9)$$

we finally get

$$\frac{\delta Q_2}{\delta\lambda_K}=\frac{4}{\lambda_K}-2\frac{F''(\lambda_K)}{F'(\lambda_K)} . \quad (A10)$$

We integrate (A10) with the help of

$$\frac{F(\lambda)}{F_0(\lambda)}=\prod_{K=1}^{N_B}\left[\frac{\lambda-\lambda_K}{\lambda+\lambda_K}\right]\exp\left[\frac{2\lambda}{\pi}\int_0^\infty \frac{\ln D(\tau)}{\tau^2+\lambda^2}d\tau\right] \quad (A11)$$

and

$$F_0(\lambda)=\frac{2^\lambda\Gamma(\lambda+1)}{\sqrt{2\pi}} , \quad (A12)$$

which leads to

$$Q_1=\sum_{K=1}^{N_B}\left[-2\ln c_K+2\ln\left[\lambda_K\frac{2^{-2\lambda_k}}{\Gamma^2(\lambda_K)}\right]+\frac{4}{\pi}\int_0^\infty d\tau\ln D(\tau)\left[\frac{-2\lambda_K}{\tau^2+\lambda_K{}^2}\right]\right]$$

$$+\sum_{1\le K\ne L\le N_B}\ln\left[\frac{\lambda_K+\lambda_L}{\lambda_K-\lambda_L}\right]^2+Q_3(D(\tau))\ . \tag{A13}$$

$Q_3$ follows from differentiating $Q_1$ with respect to $\ln D(\tau)$; and following the same scheme as above

$$\frac{\delta Q_1}{\delta D(\tau)}=\int_0^\infty \ln r\frac{d}{dr}\left[\frac{1}{r}\frac{\tau\sinh\pi\tau}{[D(\tau)]^3}[\varphi(r,i\tau)]^2\right]dr\ . \tag{A14}$$

By partial integration, and using Eq. (A8) for $\lambda_1=i\tau=\lambda_2$, we finally get

$$Q_1=\frac{4}{\pi^2}⨏_0^\infty ⨏_0^\infty\frac{\tau^2+\tau'^2}{(\tau^2-\tau'^2)^2}\ln D(\tau)\ln D(\tau')d\tau\, d\tau'$$

$$+\frac{4}{\pi}\int_0^\infty \ln D(\tau)[\,\mathrm{Re}\psi(i\tau)+\ln 2]d\tau-\frac{4}{\pi}\int_0^\infty\sum_{K=1}^{N_B}\frac{2\lambda_K}{\tau^2+\lambda_K{}^2}\ln D(\tau)d\tau$$

$$+2\sum_{K=1}^{N_B}\ln\left[\frac{\pi\lambda_K}{c_K[\Gamma(\lambda_K)]^2}2^{2(1-\lambda_K)}\right]+\sum_{1\le K\ne L\le N_B}\ln\left[\frac{\lambda_K+\lambda_L}{\lambda_K-\lambda_L}\right]^2 . \tag{A15}$$

The numerical constant $(\ln 4\pi)$ for each eigenvalue has been adjusted taking for $v$ and exactly solvable potential.

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*Laboratoire associé au CNRS no LA 280.

[1]See, for example, S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973), and references contained therein.

[2]See, for example, S. Coleman, in *Pointlike Structures Inside and Outside Hadrons,* proceedings of the International School of Subnuclear Physics, Ericé, 1979, edited by A. Zichichi (Plenum, New York, 1982), p. 11.

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