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Does the cancellation of quadratic divergences imply supersymmetry?

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We consider a large class of renormalizable field theories containing scalar particles and look for relations among the various coupling constants such that the one-loop contributions to the quadratic divergence for each scalar cancel. With the severe restriction that any such relation be invariant under the renormalization group, we show in all cases considered that whenever such relations exist, they are those of a supersymmetric theory. Non-Abelian and Abelian gauge theories as well as Yukawa theories are treated.

I. INTRODUCTION

Grand unified theories with large-energy-scale hierarchies have a well-known theoretical problem in that the parameters in the Higgs potential that determine the low-energy behavior undergo large corrections due to the strong cutoff dependence of quadratic divergences. Thus, the fine tuning of parameters necessary for the correct low-energy phenomenology seems very "unnatural."[1] If one adopts the philosphy that the correct theory should be "natural" then one is led to models such as hypercolor, where Higgs particles are composites of fermions, or to supersymmetric models, where no-renormalization theorems guarantee the absence of quadratic divergences.[2-4] Supersymmetry has received a great deal of attention recently because of this nice property. However, since there is still no solid experimental evidence for supersymmetry and since the no-renormalization theorems actually do much more than just cancel quadratic divergences, one wonders whether or not there might be a lesser symmetry that would render a theory natural without introducing a vast multitude of as-yet-unseen superpartners that are unavoidable in supersymmetric models. In this paper we attempt to shed some light on this question by parametrizing a large number of renormalizable field theories and placing the restriction on them that they be free of quadratic divergences. It will be seen that this restriction implies a system of relations between the coupling constants that is actually overconstrained when one demands that they also be invariant under the renormalization group. We will show in every case that either no solution exists or that the derived relations between the couplings are those of a supersymmetric theory. On the other hand, a chiral U(1) model which is supersymmetric will be shown to have no solution for the elimination of the quadratic divergence associated with a radiatively induced Fayet-Illiopoulos D term,[5] as will be discussed in Sec. IV.

In Sec. II of this paper we discuss the general problem of canceling quadratic divergences to one loop and outline our procedure for finding relations among coupling constants that will do it. In Sec. III we apply the procedure to models of the Wess-Zumino[2] type involving only spinors and scalars. In Sec. IV we treat Abelian gauge models, and in Sec. V we discuss some general non-Abelian gauge models. A summary of our results and our conclusions, which have already appeared in a letter,[6] is presented in Sec. VI. Details of our calculational technique are contained in the Appendix.

II. CANCELLATION OF QUADRATIC DIVERGENCES TO ONE LOOP

In general, for a theory with n coupling constants and p independent scalars, the cancellation of one-loop quadratic divergences yields the system of equations

$$\sum_{j=1}^{n} a_{ij} g_j{}^2 = 0, \quad i=1,\ldots,p \tag{2.1}$$

where we are using the convention that any coupling for a three-point interaction be written in the Lagrangian as g_i but any coupling for four-point interactions be written as a square g_j^2, e.g.,

$$L_{\text{int}} = -g_1\phi\bar{\psi}\psi - g_2{}^2\phi^4 .$$

This will make our loop expansion consistent with a perturbation expansion in small coupling constants and is necessary if one hopes to cancel fermion loops against scalar loops order by order.

In addition, one can obtain from the theory expressions for the n β functions to one loop,

$$16\pi^2\mu\frac{\partial}{\partial\mu}g_i{}^2 \equiv \beta_i = \sum_{j\geq k}^{n} b^i_{jk} g_j{}^2 g_k{}^2 , \quad i=1,\ldots,n \tag{2.2}$$

where μ is the renormalization scale. We evaluate these β functions using dimensional regularization in the minimal subtraction scheme as described in the Appendix. Now for Eq. (2.1) to have any meaning physically, it must be invariant under a change of scale μ. That is taking $\mu\partial/\partial\mu$ of Eq. (2.1), we require

$$\sum_{j,k\geq l} a_{ij}b^j_{kl}g_k{}^2g_l{}^2=0,\quad i=1,\ldots,p\ . \tag{2.3}$$

Since a_{ij} and b^i_{kl} are fixed in the theory, it is very unlikely that Eq. (2.3) will be satisfied in general. However, in some simple cases where Eq. (2.3) holds, we would have found the relationships between couplings that would eliminate the quadratic divergences consistently to one loop. If these relationships are those of a supersymmetric theory, then the no-renormalization theorems guarantee that the cancellations will work order by order to all orders. In all cases considered in this paper, the only solutions found were supersymmetric; but if some other set of relations had been found to work to one loop, we would then have to check for consistency in higher orders.

Let us suppose now the more likely case where Eq. (2.3) does not hold. The procedure is to eliminate p of the couplings by substituting Eqs. (2.1) into Eqs. (2.3). Now we have a new set of constraints on the remaining n-p independent couplings (which we relabel $k=1,\ldots,n$-p for notational convenience),

$$\sum_{k\geq l}^{n-p} C^{(0)}_{ikl}g_k{}^2g_l{}^2=0,\quad i=1,\ldots,p\ . \tag{2.4}$$

If all the g_k scale independently under the renormalization group, then $g_k{}^4$ will scale differently from $g_l{}^4$ and $g_k{}^2g_l{}^2$ ($l\neq k$). Therefore, if $C^{(0)}_{iaa}\neq 0$ for some given a, Eq. (2.4) cannot be satisfied unless a $g_a{}^4$ scale dependence arises from some of the other couplings (i.e., $g_a{}^2$ must not be an independent coupling). The order-by-order cancellations we seek then demand that we try to find a solution with $g_a{}^2$ expressed as the linear combination

$$g_a{}^2=\sum_{\substack{j=1\\ j\neq a}}^{n-p} G_jg_j{}^2\ , \tag{2.5}$$

where the G_j are unknown coefficients. Our procedure now is basically to iterate what we have done by adding Eq. (2.5) to the system of Eqs. (2.1) and its β-function equation

$$\beta_a=\sum_{j\neq a}^{n-p} G_j\beta_j \tag{2.6}$$

to the system of Eqs. (2.2). We can again form a system of equations analogous to Eq. (2.4),

$$\sum_{k\geq l}^{n-p-1} C^{(1)}_{ikl}g_k{}^2g_l{}^2=0,\quad i=1,\ldots,p+1\ , \tag{2.7}$$

but this time there are $p+1$ equations involving $n-p-1$ independent couplings, and there are $n-p-1$ free parameters G_j. If the assumption that the $n-p-1$ couplings are independent is true, then each $C^{(1)}_{ikl}$ must vanish, which yields as many as $(p+1)(n-p-1)(n-p)/2$ constraints on the $n-p-1$ free parameters. It should be clear that in general these systems are severely overconstrained. Furthermore, the couplings may have additional constraints such as positivity and reality from physical demands on the Lagrangian. If one can find a set of parameters G_j consistent with all of the constraints, the analysis is completed for one-loop cancellation, but if no physical solution exists to make $C^{(1)}_{ibb}=0$ for some given b, then one must continue the procedure. Again, expand the new dependent coupling g_b in terms of the $n-p-2$ remaining independent couplings to obtain a new system of equations that will have $2(n-p-2)$ free parameters and as many as $(p+2)(n-p-2)(n-p-1)/2$ constraints. These calculations may have to be repeated several times, possibly even to the point where there is only one remaining independent coupling, $n-p-1$ free parameters, and $n-1$ constraints, before a solution can be found. Because this procedure can be quite tedious by hand, we have limited our models considered in this paper to less than ten couplings. With the aid of a computer, however, this need not be such a serious restriction on our method.

III. WESS-ZUMINO-TYPE MODELS

Let us first consider a simple model of one complex scalar particle and one Majorana spinor with an interaction Lagrangian

$$L_{\rm int}=-g_1(\phi\psi^tC\psi_L+\phi^*\psi^tC\psi_R)-g_2{}^2(\phi^*\phi)^2\ , \tag{3.1}$$

where the subscripts L and R denote left and right helicity projections. We are imposing symmetry under the transformation

$$\begin{aligned}&\phi\rightarrow e^{2i\alpha}\phi\ ,\\ &\psi\rightarrow e^{i\alpha\gamma_5}\psi\ .\end{aligned} \tag{3.2}$$

The quadratic divergence of the boson mass can be eliminated to one loop if there is an equality

$$g_2{}^2=g_1{}^2 \tag{3.3}$$

which must be invariant under the renormalization group. The two β functions are computed, as discussed in the Appendix, and we find

$$\beta_1=12g_1{}^4\ , \tag{3.4}$$

$$\beta_2=20g_2{}^4-16g_1{}^4+8g_1{}^2g_2{}^2\ . \tag{3.5}$$

It is then obvious that Eq. (3.3) is a consistent solution. In fact, this is just the massless supersymmetric Wess-Zumino model.[2]

Now let us remove our symmetry restriction so that we may have a mass term for the spinor. The most general Lagrangian becomes a bit more complicated:

$$\begin{aligned}-\mathscr{L}_{\rm int}=&g_0\phi\psi^tC\psi_L+g_0^*\phi^*\psi^tC\psi_R+g_1\phi\psi^tC\psi_R\\ &+g_1^*\phi^*\psi^tC\psi_L+g_2{}^2(\phi^*\phi)^2+m_b{}^2\phi^*\phi\\ &+\tfrac{1}{2}m_f\psi^tC\psi+(g_3{}^2\phi^*\phi^3+g_4{}^2\phi^4+g_5\phi^*\phi^2+g_6\phi^3\\ &+\tilde{m}_b{}^2\phi^2+v\phi+\text{H.c.})\ .\end{aligned} \tag{3.6}$$

For this case we have three quadratic-divergence constraints:

$$g_2{}^2=|g_1|^2+|g_0|^2 \tag{3.7}$$

from corrections to the $m_b{}^2$ parameter,

$$3g_3{}^2=4g_1g_0 \tag{3.8}$$

for the $\tilde{m}_b{}^2$ parameter, and

$$m_f(g_1+g_0)=g_5 \tag{3.9}$$

for the v parameter. We see that adding the fermion mass term and scalar cubic term actually creates a new source of quadratic divergences in Eq. (3.9). To examine the renormalization-group invariance of Eq. (3.7) we will need the following β functions:

$$\beta_0=12\,|g_0|^2(|g_1|^2+|g_0|^2)\,, \tag{3.10}$$

$$\beta_1=12\,|g_1|^2(|g_1|^2+|g_0|^2)\,, \tag{3.11}$$

$$\beta_2=20g_2{}^4-16\,|g_0|^4-16\,|g_1|^4-32\,|g_1|^2|g_0|^2 +54\,|g_3|^2+144\,|g_4{}^2|^2+8g_2{}^2|g_1|^2+8g_2{}^2|g_0|^2\,. \tag{3.12}$$

It is then easy to see that $\mu\partial/\partial\mu$ of Eq. (3.7) implies

$$g_3{}^2=0\,, \tag{3.13}$$

$$g_4{}^2=0\,. \tag{3.14}$$

Since $g_3{}^2=0$, we can from Eq. (3.8) without loss of generality, set

$$g_0=0\,, \tag{3.15}$$

which gives us back the same interaction structure as before for the hard terms,

$$g_2{}^2=g_1{}^2\,. \tag{3.16}$$

Now we must consider the soft terms by taking $\mu\partial/\partial\mu$ of Eq. (3.9),

$$\mu\frac{\partial}{\partial\mu}g_5=\mu\frac{\partial}{\partial\mu}(m_f g_1)\,. \tag{3.17}$$

[Recall that $\beta_1=(4\pi)^2\mu(\partial/\partial\mu)\,|g_1|^2$ so that $(4\pi)^2\mu(\partial/\partial\mu)g_1=6\,|g_1|^2g_1$.] We compute $\mu(\partial/\partial\mu)m_f$ and $\mu(\partial/\partial\mu)g_5$ from the remaining theory and find

$$16\pi^2\mu\frac{\partial}{\partial\mu}m_f=4m_f\,|g_1|^2\,, \tag{3.18}$$

$$16\pi^2\mu\frac{\partial}{\partial\mu}g_5=20g_5g_2{}^2-16g_1\,|g_1|^2m_f+6g_5\,|g_1|^2\,. \tag{3.19}$$

Using Eqs. (3.11), (3.16), (3.18), and (3.19), it is clear that Eq. (3.17) is indeed satisfied, completing our solution which is the massive Wess-Zumino model. Inami *et al.*[7] have also found this result by eliminating the quadratic divergences arising in a two-loop calculation of the effective potential.

From this result we see that soft breaking terms did not destroy the previous result for the massless theory because the β functions and original quadratic-divergence equation were not changed by them. Also, soft couplings that introduced no new quadratic divergences (g_6, $m_b{}^2$, v, $\tilde{m}_b{}^2$) remained unrelated to other coupling constants. Henceforth, we will not consider soft couplings in any of our models, but note that to all solutions we find we can add arbitrary soft terms as long as they do not generate any new quadratic divergences.

Next, let us return to our massless theory and add another complex scalar field so the Lagrangian becomes

$$-\mathscr{L}_{\text{int}}=g_1{}^2(\phi_1^*\phi_1)^2+g_2{}^2(\phi_2^*\phi_2)^2+(g_3{}^2\phi_1{}^2\phi_1^*\phi_2^*+\text{H.c.})+[g_4{}^2(\phi_2{}^2\phi_2^*\phi_1^*)+\text{H.c.}]+g_5{}^2(\phi_1\phi_1^*)(\phi_2\phi_2^*) +[g_6{}^2(\phi_1{}^2)(\phi_2^*)^2+\text{H.c.}]+g_7(\phi_1\psi^tC\psi_L+\phi_1^*\psi^tC\psi_R)+g_8(\phi_2\psi^tC\psi_L+\phi_2^*\psi^tC\psi_R)\,, \tag{3.20}$$

which is invariant under the generalized transformation of Eq. (3.2).

This model can be simplified substantially if we redefine new scalars as orthogonal linear combinations of ϕ_1 and ϕ_2 so that only one scalar interacts with the fermion. We note that these redefinitions are renormalization-group invariant to one loop. Thus, we can set $g_8=0$ without loss of generality.

Then to cancel the quadratically divergent contributions for $\phi_1^*\phi_1$, $\phi_2^*\phi_2$, and $\phi_1\phi_2^*$ counterterms, respectively, we need

$$4g_1{}^2+g_5{}^2-4g_7{}^2=0\,, \tag{3.21}$$

$$4g_2{}^2+g_5{}^2=0\,, \tag{3.22}$$

and

$$g_3{}^2+g_4{}^{2*}=0\,. \tag{3.23}$$

The β functions we will need for this model are

$$\beta_1=20g_1{}^4+g_5{}^4-16g_7{}^4+8g_1{}^2g_7{}^2 +4\,|g_6{}^2|^2+24\,|g_3{}^2|^2\,, \tag{3.24}$$

$$\beta_2=20g_2{}^4+g_5{}^4+4\,|g_6{}^2|^2+24\,|g_4{}^2|^2\,, \tag{3.25}$$

$$\beta_5=4g_5{}^4+8(g_1{}^2+g_2{}^2)g_5{}^2+4g_7{}^2g_5{}^2 +32\,|g_6{}^2|^2+12\,|g_3{}^2|^2 +12\,|g_4{}^2|^2+8(g_3{}^2g_4{}^2+g_3{}^{2*}g_4{}^{2*})\,, \tag{3.26}$$

$$\beta_7=12g_7{}^4\,. \tag{3.27}$$

From Eqs. (3.21) and (3.22) we find

$$g_1{}^2-g_2{}^2=g_7{}^2\,, \tag{3.28}$$

and taking $\mu\partial/\partial\mu$ of Eq. (3.28) using Eqs. (3.23), (3.24), (3.25), and (3.27) yields

$$g_2{}^2(g_1{}^2-g_2{}^2)=0\,. \tag{3.29}$$

If we choose $g_1{}^2=g_2{}^2$, then Eqs. (3.28) and (3.21) tell us we must have $g_7{}^2=0$ and $g_5{}^2=-4g_1{}^2$, but for these to be renormalization-group invariant, Eqs. (3.23), (3.24), and (3.26) imply

$$g_1{}^2=g_2{}^2=g_6{}^2=g_3{}^2=g_4{}^2=0$$

(i.e., a trivial theory). Therefore, we must satisfy Eq.

(3.29) by letting

$$g_2{}^2=0 \tag{3.30}$$

so that from Eq. (3.28) we get

$$g_1{}^2=g_7{}^2\ . \tag{3.31}$$

Using Eq. (3.25) we demand that Eq. (3.30) be renormalization-group invariant, which sets

$$g_5{}^2=|g_6{}^2|=|g_3{}^2|=|g_4{}^2|=0 \tag{3.32}$$

and gives us the supersymmetric Wess-Zumino model with just one interacting scalar.

Finally, we try adding an arbitrary number of fermions to the model, keeping just one complex scalar. We can parametrize the model as

$$-\mathscr{L}_{\rm int}=g_0{}^2(\phi^*\phi)^2-\sum_{i=1}^{n}g_i(\phi\psi_i^tC\psi_{iL}+\phi^*\psi_i^tC\psi_{iR})$$

$$-\sum_{j=1}^{m}\widetilde{g}_i(\phi\chi_j^tC\chi_{jR}+\phi^*\chi_j^tC\chi_{jL})\ , \tag{3.33}$$

where we have ϕ interacting with n left-handed fermions and m right-handed fermions. To eliminate the quadratic divergence we require

$$g_0{}^2=\sum_{i=1}^{n}g_i{}^2+\sum_{j=1}^{m}\widetilde{g}_j{}^2\ , \tag{3.34}$$

and this equation must be preserved under renormalization where the β functions are given by

$$\beta_0=20g_0{}^4-16\sum_{i=1}^{n}g_i{}^4-16\sum_{j=1}^{m}\widetilde{g}_j{}^4+8g_0{}^2\left[\sum_{i=1}^{n}g_i{}^2+\sum_{j=1}^{m}\widetilde{g}_j{}^2\right]\ , \tag{3.35}$$

$$\beta_i=8g_i{}^4+4g_i{}^2\left[\sum_{i=1}^{n}g_i{}^2+\sum_{j=1}^{m}\widetilde{g}_j{}^2\right]\ , \tag{3.36}$$

$$\widetilde{\beta}_j=8\widetilde{g}_j{}^4+4\widetilde{g}_j{}^2\left[\sum_{i=1}^{n}g_i{}^2+\sum_{j=1}^{m}\widetilde{g}_j{}^2\right]\ . \tag{3.37}$$

Setting

$$\beta_0=\sum_{i=1}^{n}\beta_i+\sum_{j=1}^{m}\widetilde{\beta}_j$$

implies the relation

$$\left[\sum_{i=1}^{n}g_i{}^2+\sum_{j=1}^{m}\widetilde{g}_j^2\right]^2=\sum_{i=1}^{n}g_i{}^4+\sum_{j=1}^{m}\widetilde{g}_j{}^4\ ,$$

which can be satisfied only if all the g_i's and $\widetilde{g}_j$'s are zero except one. Again, the solution is just the Wess-Zumino model.

IV. ABELIAN GAUGE THEORIES

In this section, we consider Abelian U(1) gauge theories. Our first example will exhibit a somewhat surprising result. We take for our interaction Lagrangian a supersymmetric chiral U(1) theory, but allow the couplings to be arbitrary. The particle content includes a U(1) gauge boson V_μ, a neutral Majorana spinor λ, a charged scalar A, and a negatively charged left-handed Dirac spinor ψ. Then the Lagrangian can be parametrized as

$$\mathscr{L}_{\rm int}=g_1V^\mu\bar{\psi}_-\gamma_u\left[\frac{1-\gamma_5}{2}\right]\psi_-+ig_1V^\mu A^*_-\overleftrightarrow{\partial}_\mu A_-+g_1{}^2V^\mu V_\mu|A_-|^2$$
$$-g_2{}^2|A_-|^4-ig_3\left[A^*_-\bar{\lambda}\left[\frac{1-\gamma_5}{2}\right]\psi_--A_-\bar{\psi}_-\left[\frac{1+\gamma_5}{2}\right]\lambda\right]\ , \tag{4.1}$$

and the β functions are given by

$$\begin{aligned}&\beta_1=2g_1{}^4\ ,\\&\beta_2=20g_2{}^4-12g_2{}^2g_1{}^2+4g_2{}^2g_3{}^2+6g_1{}^4-2g_3{}^4\ ,\\&\beta_3=4g_3{}^4-6g_3{}^2g_1{}^2\ .\end{aligned} \tag{4.2}$$

Now in order to eliminate the quadratic divergence, the couplings must satisfy

$$3g_1{}^2+4g_2{}^2-2g_3{}^2=0\ . \tag{4.3}$$

But this relation cannot be made renormalization-group invariant. Hence, we have no solution for eliminating the quadratic divergence. However, if we set

$$g_3{}^2=4g_2{}^2=2g_1{}^2\ , \tag{4.4}$$

we recover the supersymmmetric theory. The theory evidently still has a quadratic divergence. To see where this divergence comes from, let us construct the theory from a chiral superfield $\phi_-(A_-,\psi_-,F_-)$ and a vector superfield $V(V^\mu,\lambda,D)$,

$$\mathscr{L}=W^\alpha W_\alpha|_{\theta\theta}+\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}|_{\bar\theta\bar\theta}+\phi^\dagger_-e^{-2gV}\phi_-|_{\theta\theta\bar\theta\theta}+\kappa V|_{\theta\theta\bar\theta\bar\theta}\ . \tag{4.5}$$

Normally one does not include the $\kappa V|_{\theta\theta\bar\theta\bar\theta}$ supersymmetric term since in this case it breaks the U(1), gauge symmetry spontaneously. However, Witten[8] has pointed out that for some nonsemisimple gauge groups this term will be generated by loops even if it is not put in at the tree level. If we expand the Lagrangian in terms of components this will become clear:

$$\mathscr{L}=\tfrac{1}{2}D^2-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{i}{2}\bar{\lambda}\partial\lambda+F^*F+(\mathscr{D}_\mu A_-)^*(\mathscr{D}^\mu A_-)+i\bar{\psi}_-\mathscr{D}\left[\frac{1-\gamma_5}{2}\right]\psi_- -gDA_-^*A_-$$
$$-i\sqrt{2}g\left[A_-^*\bar{\lambda}\left[\frac{1-\gamma_5}{2}\right]\psi_- - A_-\bar{\psi}_-\left[\frac{1+\gamma_5}{2}\right]\lambda\right]+\kappa D\,. \quad (4.6)$$

Figure 1 shows the one-loop diagram that will generate the κD term from the $DA_-^*A_-$ interaction, and it is quadratically divergent. After eliminating the auxiliary field D using its equation of motion, one gets a mass term for the scalar field proportional to κ. Conventionally, mass terms arise from a $m\phi\phi\,|_{\theta\theta}$ F term and as such are guaranteed by the no-renormalization theorems to be free of vertex corrections. However, in this case where the mass term arises from a $\kappa V\,|_{\theta\theta\bar{\theta}\bar{\theta}}$ D term, the theorems do not apply and hence the quadratic divergence. It is easy to see that if we add more chiral fields ϕ_i with charges e_i to the theory, then more scalar loops can contribute in Fig. 1 and the total contribution will be

$$\sum_{i=1}^{n} e_i\int\frac{d^4k}{(2\pi)^4k^2}\,. \quad (4.7)$$

Therefore, this problem does not arise if we have a theory where the sum of the charges is zero.

Let us now introduce a less pathological but much more complicated U(1) model with particle content V_μ, the gauge boson, ψ, a Dirac spinor, A_- and A_+, charged scalars, A_0, a neutral complex scalar, and ψ_0 and λ, Majorana spinors. We also assume parity conservation and R-parity invariance analogous to Eq. (3.2) to make our parametrized Lagrangian a little more tractable:

$$\mathscr{L}_{\rm int}=g_1V^\mu\bar{\psi}\gamma_\mu\psi+ig_1V^\mu(A_-^*\overleftrightarrow{\partial}_\mu A_-)-ig_1V^\mu(A_+^*\overleftrightarrow{\partial}_\mu A_+)+g_1{}^2V_\mu V^\mu(A_-^*A_-+A_+^*A_+)$$
$$-g_2{}^2\left[(A_-^*-A_-)^2+(A_+^*A_+)^2\right]+g_3{}^2(A_-^*A_-)(A_+^*A_+)-ig_4\left[A_-^*\bar{\lambda}\left[\frac{1-\gamma_5}{2}\right]\psi-\text{H.c.}\right]$$
$$-ig_4\left[A_+\bar{\lambda}\left[\frac{1+\gamma_5}{2}\right]\psi-\text{H.c.}\right]-g_5\left[A_-\bar{\psi}\left[\frac{1-\gamma_5}{2}\right]\psi_0+\text{H.c.}\right]-g_5\left[A_+^*\bar{\psi}\left[\frac{1+\gamma_5}{2}\right]\psi_0+\text{H.c.}\right]$$
$$-g_6\left[A_0\bar{\psi}\left[\frac{1-\gamma_5}{2}\right]\psi+\text{H.c.}\right]-g_7\left[A_0\bar{\psi}_0\left[\frac{1-\gamma_5}{2}\right]\psi_0+\text{H.c.}\right]-g_8{}^2(A_0^*A_0)^2$$
$$-g_9{}^2\,|A_0|^2(\,|A_+|^2+|A_-|^2)-g_{10}{}^2(A_+^*A_-^*A_0A_0+\text{H.c.})\,. \quad (4.8)$$

There are only two independent relations among the couplings to remove the quadratic divergences for the $A_\pm$ scalars and the A_0 scalar, and they are, respectively,

$$3g_1{}^2+4g_2{}^2-g_3{}^2-2g_4{}^2-2g_5{}^2+g_9{}^2=0 \quad (4.9)$$

and

$$-g_6{}^2-2g_7{}^2+2g_8{}^2+g_9{}^2=0\,. \quad (4.10)$$

The β functions for the theory are given by

$$\begin{aligned}
\beta_1&=4g_1{}^4\,,\\
\beta_2&=6g_1{}^4+g_2{}^2(-12g_1{}^2+20g_2{}^2+4g_4{}^2+4g_5{}^2)+g_3{}^4-2g_4{}^4-2g_5{}^4+g_g{}^4\,,\\
\beta_3&=-12g_1{}^4+g_3{}^2(-12g_1{}^2+16g_2{}^2-4g_3{}^2+4g_4{}^2+4g_5{}^2)+4g_4{}^4+4g_5{}^4-2g_9{}^4-4g_{10}{}^4\,,\\
\beta_4&=g_4{}^2(-6g_1{}^2+5g_4{}^2-g_5{}^2+g_6{}^2)\,,\\
\beta_5&=g_5{}^2(-6g_1{}^2-g_4{}^2+5g_5{}^2+g_6{}^2+4g_6{}^2)\,,\\
\beta_6&=g_6{}^2(-12g_1{}^2+2g_4{}^2+2g_5{}^2+4g_6{}^2+4g_7{}^2)\,,\\
\beta_7&=g_7{}^2(6g_6{}^2+12g_7{}^2)\,,\\
\beta_8&=-2g_6{}^4+4g_6{}^2g_8{}^2-16g_7{}^4+8g_7{}^2g_8{}^2+20g_8{}^4+2g_9{}^4+2g_{10}{}^4\,,\\
\beta_9&=g_9{}^2(-6g_1{}^2+8g_2{}^2-2g_3{}^2+2g_4{}^2+2g_5{}^2+2g_6{}^2+4g_7{}^2+8g_8{}^2+4g_9{}^2)-4g_4{}^2g_6{}^2-4g_5{}^2g_6{}^2-16g_5{}^2g_7{}^2+8g_{10}{}^4\,,\\
\beta_{10}&=g_{10}{}^2(-6g_1{}^2-2g_3{}^2+2g_4{}^2+2g_5{}^2+2g_6{}^2+4g_7{}^2+4g_8{}^2+8g_9{}^2)-8g_5{}^2g_6g_7\,.
\end{aligned} \quad (4.11)$$

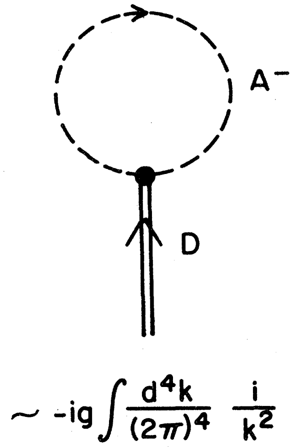

FIG. 1. One-loop graph generating the κD term which is quadratically divergent.

Before we can apply the procedure outlined in Sec. II, we must put β_{10} in the form of Eq. (2.2). To accomplish this, we define a fictitious parameter

$$g_x{}^2 = g_6 g_7 \tag{4.12}$$

with

$$\beta_x = g_x{}^2(-6g_1{}^2 + g_4{}^2 + g_5{}^2 + 5g_6{}^2 + 8g_7{}^2) . \tag{4.13}$$

This gives a system with eleven couplings which can be methodically reduced to find the unique solution which has only three independent couplings (g_1, g_5, g_7),

$$\begin{aligned} &2g_1{}^2 = 4g_2{}^2 = g_4{}^2 , \\ &g_3{}^2 = g_1{}^2 - g_5{}^2 , \\ &g_5{}^2 = g_6{}^2 = g_9{}^2 , \\ &g_7{}^2 = g_8{}^2 , \\ &g_{10}{}^2 = g_x{}^2 = g_5 g_7 . \end{aligned} \tag{4.14}$$

This corresponds to a supersymmetric U(1) theory with a vector superfield $V(V^\mu, \lambda)$ and three chiral superfields $\phi_+(\psi_+, A_+)$, $\phi_-(\psi_-, A_-)$, and $\phi_0(\psi_0, A_0)$, where g_1 is the gauge coupling and g_5 and g_7 are the Yukawa couplings for the $\phi_+\phi_-\phi_0|_{\theta\theta}$ and $\phi_0\phi_0\phi_0|_{\theta\theta}$ superfield terms, respectively.

V. NON-ABELIAN GAUGE THEORIES

In this section, we consider non-Abelian gauge theories with particle content V_μ^a, the gauge boson, ψ and λ, Weyl spinors, and A, a complex scalar. We treat first the general case of a compact non-Abelian group, choosing ψ and A to have the same representation while λ is chosen to be in the adjoint. We can write the Lagrangian as

$$\begin{aligned} \mathscr{L} = &-\tfrac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + i\bar\lambda^a\bar\sigma^\mu(\delta^{ac}\partial_\mu - g_1 C^{abc}V_\mu^b)\lambda^c \\ &+ |(\partial_\mu + ig_1 V_\mu^a T^a)A|^2 + i\bar\psi\bar\sigma^\mu(\partial_\mu + ig_1 V_\mu^a T^a)\psi \\ &- ig_2(\bar\psi\bar\lambda^a T^a A - A^\dagger T^a \lambda^a \psi) \\ &- g_3{}^2(A^\dagger T^a A)(A^\dagger T^a A) , \end{aligned} \tag{5.1}$$

where the matrices T^a are representations of the group generators and satisfy

$$[T^a, T^b] = iC^{abc}T^c . \tag{5.2}$$

With our choice of representations for the particles, the relationship between couplings that cancels the quadratic divergences turns out to be independent of group structure and is

$$3g_1{}^2 + 2g_3{}^2 - 2g_2{}^2 = 0 . \tag{5.3}$$

But for the β functions let us define some group constants

$$C^{abc}C^{bcd} \equiv S_1(G)\delta^{ab} , \tag{5.4}$$

$$T_{ik}^a T_{kj}^a \equiv S_2(A)\delta_{ij} , \tag{5.5}$$

$$\mathrm{Tr}(T^a T^b) \equiv S_3(A)\delta^{ab} . \tag{5.6}$$

Then we find

$$\beta_1 = g_1{}^4[-6S_1(G) + 2S_3(A)] , \tag{5.7}$$

$$\beta_2 = g_2{}^2[-6g_1{}^2 S_1(G) + g_2{}^2 S_3(A) + (3g_2{}^2 - 6g_1{}^2)S_2(A)] . \tag{5.8}$$

When calculating one-loop contributions to g_3, however, we encounter the complication that a new vertex structure might be generated. We have been assuming that our Lagrangian is renormalizable and therefore $(A^\dagger T^a A)(A^\dagger T^a A)$ should be the only allowable quartic structure generated. There are two possibilities; either our representations are such that no other vertex structures can arise (for example, if A and ψ are both in the fundamental representation) or the other vertex structures that are generated must be canceled by a new supplementary relation between couplings. Discussing the latter case first, the new relation needed is

$$4g_3{}^4 + 3g_1{}^4 - g_2{}^4 = 0 , \tag{5.9}$$

and β_3 is then given by

$$\begin{aligned} \beta_3 = &(-2g_3{}^4 + \tfrac{3}{2}g_1{}^4 - g_2{}^4)S_1(G) + 4g_3{}^4 S_3(A) \\ &+ 4g_3{}^2(2g_3{}^2 - 3g_1{}^2 + g_2{}^2)S_2(A) . \end{aligned} \tag{5.10}$$

Equations (5.9) and (5.3) immediately give the result

$$g_2{}^2 = 2g_1{}^2 = 4g_3{}^2 , \tag{5.11}$$

which is the supersymmetric solution.

Now in the case where we have representations that do not generate any different structures, β_3 has an additional contribution $(4g_3{}^4 + 3g_1{}^4 - g_2{}^4)[2S_3(A) + 2S_2(A) - S_1(G)]$. Taking $\mu\partial/\partial\mu$ of Eq. (5.3) then implies

$$(2g_1{}^2 - g_2{}^2)^2[-2S_1(G) + 5S_2(A) + 3S_3(A)] = 0 . \tag{5.12}$$

If $2g_1{}^2 - g_2{}^2 = 0$, we again immediately recover the supersymmetric relations Eq. (5.11). As for the other possibility,

$$[-2S_1(G) + 5S_2(A) + 3S_3(A)] = 0 , \tag{5.13}$$

we have checked this for fundamentals of SU(N) and SO(N) and found that no solutions exist for any N. Furthermore, even if such a representation could be found, higher-loop calculations would put still more constraints on the couplings that might not be satisifed by appealing to Eq. (5.13).

Let us now change the representations for the fields so they do not correspond to any supermultiplet structure. This time we choose the scalar A to be a singlet and let ψ and λ form a Dirac spinor Ψ in some arbitrary representation. Then our Lagrangian can be written as

$$\mathscr{L}=-\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a+i\overline{\Psi}\gamma^\mu(\partial_\mu+ig_1V^a_\mu T^a)\Psi$$
$$+\partial_\mu A^*\partial^\mu A-g_2{}^2|A|^4$$
$$-ig_3\left[\overline{\Psi}\left[\frac{1+\gamma_5}{2}\right]\Psi A-\overline{\Psi}\left[\frac{1-\gamma_5}{2}\right]\Psi A^*\right], \quad (5.14)$$

so that

$$\beta_1=[-\tfrac{22}{3}S_1(G)+\tfrac{8}{3}S_3(F)]g_1{}^4\,,$$
$$\beta_2=20g_2{}^4-D(F)g_3{}^4+4D(F)g_3{}^2g_2{}^2\,, \quad (5.15)$$
$$\beta_3=2[D(F)+1]g_3{}^4-12S_2(F)g_1{}^2g_3{}^2\,,$$

and cancellation of the quadratic divergence requires

$$4g_2{}^2=2D(F)g_3{}^2\,. \quad (5.16)$$

Applying $\mu\partial/\partial\mu$ to Eq. (5.16) and requiring invariance under Eq. (5.15) gives the relation

$$g_1{}^2=\left[\frac{1-2D(F)}{D(F)S_2(F)}\right]g_2{}^2\,, \quad (5.17)$$

where $D(F)$ is the dimension of the representation for Ψ. Since $g_2{}^2$ must be positive for a stable theory, g_1 becomes imaginary. Therefore, no satisfactory solutions exist for this model.

We also treated an SU(N) model with the complex scalar A and Weyl spinor ψ being in the fundamental representation, but in this example we put the Weyl spinor λ into the theory as a singlet. For this model,

$$\mathscr{L}=-\tfrac{1}{4}F^{\mu\nu}_aF^a_{\mu\nu}+i\overline{\psi}\overline{\sigma}^\mu(\partial_\mu+ig_1V^a_\mu T^a)\psi+i\overline{\lambda}\overline{\sigma}^\mu\partial_\mu\lambda+|(\partial_\mu+ig_1V^a_\mu T^a)A|^2-g_2{}^2(A^\dagger T^aA)(A^\dagger T^aA)-ig_3(\overline{\psi}\overline{\lambda}A-A^\dagger\psi\lambda)\,, \quad (5.18)$$

$$\beta_1=(-\tfrac{22}{3}N+1)g_1{}^4\,,$$
$$\beta_2=2\left[\frac{N^2+3N-4}{N}\right]g_2{}^4+3g_1{}^4\left[\frac{N^2+2N-2}{2N}\right]-g_3{}^4\left[\frac{4N}{N-1}\right]-6\left[\frac{N^2-1}{N}\right]g_2{}^2g_1{}^2+4g_2{}^2g_3{}^2\,, \quad (5.19)$$
$$\beta_3=(3+N)g_3{}^4-3g_3{}^2g_1{}^2(N^2-1)/N\,;$$

while the cancellation of the quadratic divergence implies

$$\tfrac{3}{2}g_1{}^2-\frac{2N}{N^2-1}g_3{}^2+g_2{}^2=0\,. \quad (5.20)$$

Once again, following the procedure of Sec. II, we found no consistent solution for any integer N.

Finally, we constructed another model with SU(2) gauge symmetry where we chose the complex scalar A to be in the adjoint representation and added a Dirac spinor ψ in the fundamental representation. Then the Lagrangian is

$$\mathscr{L}=-\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a+i\overline{\psi}\gamma^\mu\left[\partial_\mu+ig_1V^a_\mu\frac{\tau^a}{2}\right]\psi+|\partial_\mu A^a+g_1V^b_\mu\epsilon^{abc}A^c|^2-g_2{}^2(A^aA^a)(A^{*b}A^{*b})$$
$$-g_3{}^2(A^aA^{*a})(A^bA^{*b})-ig_4\left[\overline{\psi}\left[\frac{1+\gamma_5}{2}\right]\frac{\tau^a}{2}\psi A^a-\overline{\psi}\left[\frac{1-\gamma_5}{2}\right]\frac{\tau^a}{2}\psi A^{*a}\right] \quad (5.21)$$

with the β functions

$$\beta_1=-12g_1{}^4\,,$$
$$\beta_2=8g_2{}^4+3g_1{}^4-24g_2{}^2g_1{}^2+32g_2{}^2g_3{}^2+2g_2{}^2g_4{}^2+\tfrac{1}{4}g_4{}^4\,, \quad (5.22)$$
$$\beta_3=24g_3{}^4+16g_2{}^4+9g_1{}^4-24g_2{}^2g_1{}^2+16g_2{}^2g_3{}^2+2g_2{}^2g_4{}^2-\tfrac{1}{2}g_4{}^4\,,$$
$$\beta_4=\tfrac{5}{2}g_4{}^4-9g_4{}^2g_1{}^2\,,$$

and to eliminate the quadratic divergence we need

$$6g_1{}^2+4g_2{}^2+8g_3{}^2-g_4{}^2=0\ . \tag{5.23}$$

In this model also, we found no consistent solution for Eqs. (5.22) and (5.23). Thus, of all of the non-Abelian models we studied, the only solution we found was the supersymmetric one of Eq. (5.11).

VI. CONCLUSION

We have examined the question of whether supersymmetry is the only symmetry that has the nice feature of eliminating the quadratic divergences of a theory and found that in all of the models we studied a simple one-loop analysis was enough to show that supersymmetry was indeed unique in this respect. We considered a wide variety of models with quartic and Yukawa interactions as well as Abelian and non-Abelian gauge interactions. Our analysis consisted of finding the relations among coupling constants that would cancel the quadratic divergences of a theory to one loop and then demanding that these relations be renormalization-group invariant as determined by the one-loop β functions of the theory. This system of relations is in general overconstrained and no solution could be found in the models considered other than that of a supersymmetric theory. Soft terms can be added to any of the solutions found as long as they do not themselves create new quadratic divergences.

ACKNOWLEDGMENT

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APPENDIX

We present a concise and efficient method of calculating β functions to one loop. We choose the Landau gauge and use dimensional regularization with the minimal subtraction scheme. This will give us the fewest number of nonzero contributions and will make our β functions independent of masses. We first numerate all of the one-loop diagrams contributing to the renormalization of a particular vertex, say the one involving g_i. For each of these diagrams, we associate a factor M_j with the momentum-space integral of the jth diagram and a structure factor V_{ij} which includes all the coupling-constant factors as well as factors from group structure. Factors due to the combinatorics of a diagram are also included in V_{ij}. Thus, the factors M_j are independent of the model under consideration, although they can depend on the helicities of the fermions in the diagram. We have listed the values of M_j for all the diagrams that are of interest in this paper in Fig. 2. Note that only half the contribution of self-energy diagrams on external lines contribute to β functions, and this is taken into account in the values given for M_j.

If we denote the Feynman rule for the vertex associated with the coupling g_i as $g_i V_i$, where V_i can include helicity factors as well as group theory factors, we shall obtain for the β function the equation

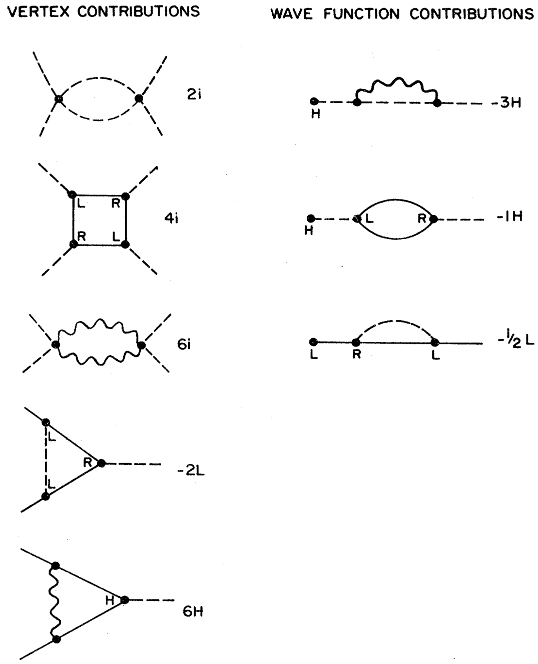

FIG. 2. Momentum-space integral factors used in calculating our β functions. We denote fermions with solid lines, scalars by dashed lines, and gauge particles by wavy lines. The letters L and R by a vertex stand for left and right helicity projectors, respectively. Diagrams that are zero, or can be obtained from interchanging L and R, or are used only in calculating β_{gauge}, are not given here. The letter H stands for either helicity.

$$16\pi^2\mu\frac{\partial}{\partial\mu}V_i g_i=\sum_j V_{ij}M_j\ . \tag{A1}$$

With a complicated group structure, a particular one-loop diagram can contribute to the renormalization of more than one bare vertex involving the same external lines. Using the appropriate projection operators on V_{ij}, this problem can be readily handled. We also add a note of caution when handling helicity projection matrices. The diagrams in Fig. 2 depend on the helicities, hence, in any particular model it may be necessary to add or subtract diagrams with $L\rightleftarrows R$ to get the same structure as the V_i under consideration.

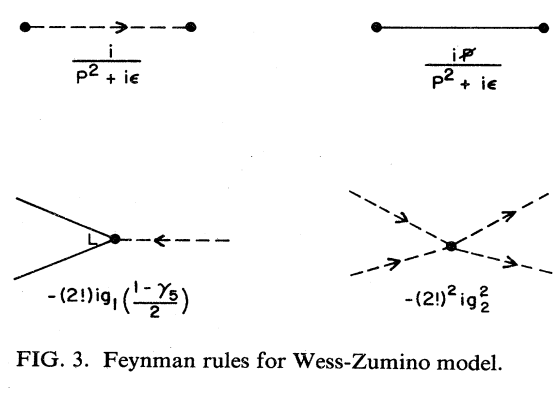

FIG. 3. Feynman rules for Wess-Zumino model.

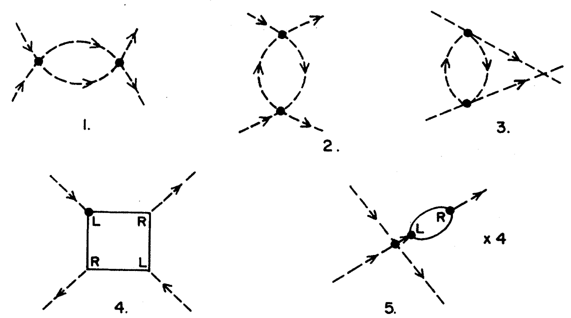

FIG. 4. One-loop contributions to the β functions for g_2^2. Diagram 5 is multiplied by 4 because there are four external legs to which it can be attached.

For the gauge coupling, the contributions to its β functions from gauge interactions and ghosts can be separated from the contributions of matter field loops and we can write[9]

$$16\pi^2\mu\frac{\partial g_{\text{gauge}}}{\partial\mu} = g^3_{\text{gauge}}\left[-\frac{11}{3}S_1(G) + \frac{1}{3}\sum_F S_3(F)n_F + \frac{1}{6}\sum_s S_3(s)n_S\right], \tag{A2}$$

where $S_1(G)$ and $S_3(A)$ are defined in Eqs. (5.4) and (5.6) for non-Abelian groups while for Abelian groups we define $S_1(G)=0$ and $S_3(A)=q_A{}^2$, where q_A is the charge in units of g_{gauge}. The sums are taken over all fermion representations F and all scalar representations S. The number of fermionic degrees of freedom n_F is 2 for a left-handed spinor or a Majorana spinor and 4 for a Dirac spinor. The number of bosonic degrees of freedom n_S is 1 for a real scalar and 2 for a complex scalar.

To fully illustrate this we calculate the β functions for the simple Wess-Zumino model,

$$\begin{aligned}\mathscr{L} = {} & \partial_\mu A^*\partial^\mu + \frac{i}{2}\psi^t C\partial\psi - g_1 A\psi^\dagger C\left[\frac{1-\gamma_5}{2}\right]\psi \\ & - g_1 A^*\psi^t C\left[\frac{1+\gamma_5}{2}\right]\psi - g_2{}^2(A^*A)^2\,,\end{aligned} \tag{A3}$$

with the Feynman rules given in Fig. 3. The one-loop contributions for $g_2{}^2$ are enumerated in Fig. 4. Then from Fig. 2, we find

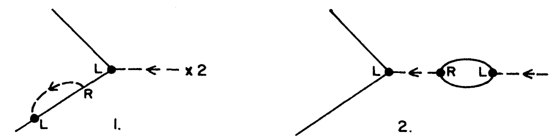

FIG. 5. One-loop contributions to the β functions for g_1. Diagram 1 is multiplied by 2 because there are two external legs to which it can be attached.

$$\begin{aligned} & M_1 = 2i\,, \\ & M_2 = 2i\,, \\ & M_3 = 2i\,, \\ & M_4 = 4i\,, \\ & 4\times M_5 = -4\,, \end{aligned} \tag{A4}$$

and from Fig. 3,

$$\begin{aligned} & V_2 = -4i, \quad V_{23} = (-4ig_2{}^2)^2 \\ & V_{21} = (-4ig_2{}^2)/2, \quad V_{24} = (-2ig_1)^4\,, \\ & V_{22} = (-4ig_2{}^2)^2, \quad V_{25} = (-4ig_2{}^2)(-2ig_1)^2/2\,, \end{aligned} \tag{A5}$$

where the factor $\frac{1}{2}$ in V_{21} and V_{25} is the combinatorial factor for an internal loop. Inserting these values into Eq. (A1), we recover Eq. (3.4)

$$16\pi^2\mu\frac{\partial}{\partial\mu}g_2{}^2 = 20g_2{}^4 - 16g_1{}^4 + 8g_1{}^2g_2{}^2\,. \tag{A6}$$

To compute β_1, we list the contributing one-loop diagrams in Fig. 5 for which we find the following values:

$$\begin{aligned} & V_1 = -2i(1-\gamma_5)/2, \quad 2M_1 = -\frac{(1-\gamma_5)}{4}\times 2\,, \\ & V_{11} = (-2ig_1)^3, \quad M_2 = -(1-\gamma_5)/2\,, \\ & V_{12} = (-2ig_1)^3/2\,, \end{aligned} \tag{A7}$$

where V_{12} again contains the combinatorial factor of $\frac{1}{2}$.

From Eq. (A1) we find

$$16\pi^2\mu\frac{\partial}{\partial\mu}g_1 = 6g_1{}^3\,, \tag{A8}$$

so that we recover Eq. (3.5)

$$\beta_1 = 16\pi^2\mu\frac{\partial}{\partial\mu}g_1{}^2 = 12g_1{}^4\,. \tag{A9}$$

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