PHYSICAL REVIEW D VOLUME 29, NUMBER 12 15 JUNE 1984

Relativistic rotator. III. Contraction limits and experimental justification

R. R. Aldinger, A. Bohm, P. Kielanowski,* M. Loewe, and P. Moylan
Center for Particle Theory, The University of Texas at Austin, Austin, Texas 78712
(Received 19 September 1983)

In this paper we give theoretical and experimental justification for the model of the quantum relativistic rotator (QRR) which was defined and analyzed in two preceding papers. We give purely theoretical arguments which involve the process of group contraction to show that the Hamilton operator for the QRR goes into the Hamilton operators for the structureless relativistic mass point and the nonrelativistic rotator in the elementary and nonrelativistic limits, respectively. We also give the experimental verification for the QRR by showing that the known meson resonances do form rotational bands.

I. CONTRACTION LIMITS OF THE QUANTUM RELATIVISTIC ROTATOR

A. Introduction

The ultimate justification of a new theory is based primarily, of course, on the theory's success in describing the experimental data. However, justification of a new theory is also based on whether it builds on old, well-established ideas rather than if it stands isolated.

In two previous papers (see Refs. 1 and 2) we defined and discussed the quantum relativistic rotator (QRR). In Sec. II of this paper we shall consider the experimental justification of the QRR model while in this section we give purely theoretical arguments for choosing Eq. (I 2.38) for the Hamiltonian of the QRR [or, equivalently, justification for choosing Eq. (I 2.37) for the constraint relation]. In particular, we shall show that correspondences exist between the QRR and two well-established quantum-mechanical models: The relativistic mass point (elementary particle) which is described by the irreducible representations of the Poincaré group[3] and the nonrelativistic rotator[4] whose space-time symmetry is described by the extended Galilei group and whose spectrum is described by a three-dimensional Euclidean group [for which one can take either $E(3)_{D_i,S_i}$ or $E(3)_{D_i,\Sigma_i}$ where D_i is the dipole operator, S_i is the intrinsic angular momentum which was defined in Sec. II of paper I, and Σ_i is the nonrelativistic spin angular momentum]. These two correspondences are given by group contractions.[5] The relativistic mass point is obtained in the elementary limit ($1/R=\lambda\to 0$, $\alpha\to\infty$) by contracting the de Sitter group (the group of motion in the micro—de Sitter space of radius R) into the Poincaré group:[6]

$$SO(4,1)_{B_\mu,J_{\mu\nu}} \xrightarrow[\lambda\alpha\to p]{\lambda\to 0,\ \alpha\to\infty} \mathscr{P}_{P_\mu,J_{\mu\nu}}$$

$$(p^2=\text{eignevalue of } P_\mu P^\mu)\ , \quad (1.1)$$

and the nonrelativistic rotator is obtained in the nonrelativistic limit ($1/c\to 0, p\to\infty$) by contracting the Poincaré group into the extended Galilei group:

$$\mathscr{P}_{P_\mu,J_{\mu\nu}} \xrightarrow[p/c\to m]{1/c\to 0,\ p\to\infty} \mathscr{G}_{P_i,H,M,G_i,J_i}\ . \quad (1.2)$$

B. Elementary limit

In the elementary limit ($\lambda\to 0$ and $\alpha\to\infty$) the QRR contracts into the relativistic mass point. The details of this particular contraction are summarized in Table I. In Table I, the generators and commutation relations of $SO(4,1)_{B_\mu,J_{\mu\nu}}$ (Ref. 7) are listed in the left-hand column, the generators and commutation relations of $\mathscr{P}_{P_\mu,J_{\mu\nu}}$ are listed in the right-hand column, and the $\to$ indicates the contraction limit, Eq. (1.1) between them. Also listed in Table I are the second- and fourth-order Casimir operators of $SO(4,1)_{B_\mu,J_{\mu\nu}}$ and $\mathscr{P}_{P_\mu,J_{\mu\nu}}$ along with their eigenvalues in the principal series representations. Here, the symbol $\overset{\text{irrep}}{=\!=}$ between a Casimir operator and a number means that the Casimir operator has that number as its eigenvalue in an irreducible representation.

The principal series representations of SO(4,1) (Ref. 7) are characterized by the pair (α,s), where s is a discrete parameter that can take on one of the values $s=0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots,$ and α is a continuous parameter that can take on values such that $\alpha^2>\frac{9}{4}-s(s+1)$. We have defined the QRR to be the physical system that is characterized by the eigenvalue $\lambda^2\alpha^2$ of the second-order Casimir operator of $SO(4,1)_{B_\mu,J_{\mu\nu}}$. For the QRR, mass and spin are related because of the $SO(4,1)_{B_\mu,J_{\mu\nu}}$ constraint, Eq. (I 2.37), or, equivalently, due to $C=\alpha^2$.

In the elementary limit the second-order Casimir operator of $SO(4,1)_{B_\mu,J_{\mu\nu}}$ goes, according to Eq. (T1g) in Table I, into the second-order Casimir operator of $\mathscr{P}_{P_\mu,J_{\mu\nu}}$. This can be seen immediately by inserting Eq. (T1b) into Eq. (I 2.35) and taking the limit $\lambda\to 0$. Thus a principal series representation of $SO(4,1)_{B_\mu,J_{\mu\nu}}$ contracts into a physical irrep of the Poincaré group characterized by $(p>0,s)$, and the square of the momentum decouples from the spin. Also, the Hamiltonian of the QRR, Eq. (I 2.38), goes into the Hamiltonian of the relativistic mass point (elementary particle),

 ©1984 The American Physical Society

TABLE I. Elementary limit given by the group contraction from SO(4,1) to the Poincaré group.

	$SO(4,1)_{B_\mu, J_{\mu\nu}}$	$\mathscr{P}_{P_\mu, J_{\mu\nu}}$
(T1a)	$J_{\mu\nu}$	$\rightarrow J_{\mu\nu}$
(T1b)	$B_\mu = P_\mu + \frac{\lambda}{2}\{\hat{P}^\nu, J_{\mu\nu}\}$	$\rightarrow P_\mu$
(T1c)	$[J_{\mu\nu}, J_{\rho\sigma}] = -i(g_{\mu\rho}J_{\nu\sigma} + g_{\nu\sigma}J_{\mu\rho} - g_{\mu\sigma}J_{\nu\rho} - g_{\nu\rho}J_{\mu\sigma})$	
(T1d)	$[J_{\mu\nu}, B_\rho] = i(g_{\nu\rho}B_\mu - g_{\mu\rho}B_\nu)$	$\rightarrow [J_{\mu\nu}, P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu)$
(T1e)	$[B_\mu, B_\nu] = i\lambda^2 J_{\mu\nu}$	$\rightarrow [P_\mu, P_\nu] = 0$
(T1f)	$\Omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}B_\nu J_{\rho\sigma}$	$\rightarrow (P_\nu P^\nu)^{1/2}\hat{w}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu J_{\rho\sigma}$
	Casimir operators	
(T1g)	$\lambda^2 C = B_\mu B^\mu - \frac{\lambda^2}{2}J_{\mu\nu}J^{\mu\nu} \overset{\text{irrep}}{=\!=} \lambda^2\alpha^2$	$\rightarrow P_\mu P^\mu \overset{\text{irrep}}{=\!=} p^2$
	$(\lambda^2 C = P_\mu P^\mu + \frac{9}{4}\lambda^2 - \lambda^2 \hat{W})$	
(T1h)	$\lambda^2 C_{(4)} = \lambda^2 (J_{i0}J_i)^2 - \Omega_\mu \Omega^\mu$	$\rightarrow P_\mu P^\mu \hat{W} = -P_\mu P^\mu \hat{w}_\nu \hat{w}^\nu \overset{\text{irrep}}{=\!=} p^2 s(s+1)$
	$\overset{\text{irrep}}{=\!=} \lambda^2 (s-1)s(s+1)(s+2) + \lambda^2\alpha^2 s(s+1)$	

TABLE II. Contraction of the Poincaré group into the extended Galilei group.

	$\mathscr{P}_{P_\mu, J_{\mu\nu}}$		$\mathscr{G}_{P_i, H, M, G_i, J_i}$
(T2a)	$J_i = \frac{1}{2}\epsilon_{ijk}J_{jk}$		$\rightarrow J_i$
(T2b)	$K_i = J_{i0}$	$G_i^{(c)} = \frac{1}{c}K_i$	$\rightarrow G_i$
(T2c)	P_i		$\rightarrow P_i$
(T2d)	P_0	$M^{(c)} = \frac{1}{c}(P_0{}^2 - \vec{P}^2)^{1/2}$	$\rightarrow M$
(T2e)		$H^{(c)} \equiv cP_0 - M^{(c)}c^2$	$\rightarrow H$
(T2f)	$[J_i, J_j] = i\epsilon_{ijk}J_k$		$\rightarrow [J_i, J_j] = i\epsilon_{ijk}J_k$
(T2g)	$[J_i, K_j] = i\epsilon_{ijk}K_k$	$[J_i, G_j^{(c)}] = i\epsilon_{ijk}G_k^{(c)}$	$\rightarrow [J_i, G_j] = i\epsilon_{ijk}G_k$
(T2h)	$[J_i, P_j] = i\epsilon_{ijk}P_k$		$\rightarrow [J_i, P_j] = i\epsilon_{ijk}P_k$
(T2i)	$[J_i, P_0] = 0$	$[J_i, H^{(c)}] = 0$	$\rightarrow [J_i, H] = 0$
(T2j)	$[K_i, K_j] = -i\epsilon_{ijk}J_k$	$[G_i^{(c)}, G_j^{(c)}] = -i\epsilon_{ijk}\frac{1}{c^2}J_k$	$\rightarrow [G_i, G_j] = 0$
(T2k)	$[K_i, P_j] = i\delta_{ij}P_0$	$[G_i^{(c)}, P_j] = i\delta_{ij}\left[\frac{1}{c^2}H^{(c)} + M^{(c)}\right]$	$\rightarrow [G_i, P_j] = i\delta_{ij}M$
(T2l)	$[K_i, P_0] = iP_i$	$[G_i^{(c)}, H^{(c)}] = iP_i$	$\rightarrow [G_i, H] = iP_i$
(T2m)	$[P_i, P_j] = 0$		$\rightarrow [P_i, P_j] = 0$
(T2n)	$[P_i, P_0] = 0$	$[P_i, H^{(c)}] = 0$	$\rightarrow [P_i, H] = 0$
(T2o)	$\hat{w}^0 = \frac{1}{M^{(c)}c}\vec{P}\cdot\vec{J}$		$\rightarrow 0$
(T2p)	$\vec{\hat{w}} = \left[\frac{1}{M^{(c)}c^2}H^{(c)} + 1\right]\vec{J} + \vec{P} \times \frac{\vec{G}^{(c)}}{M^{(c)}}$		$\rightarrow \vec{J} - \frac{\vec{G}}{M} \times \vec{P} \equiv \vec{\Sigma}^{(\infty)}$
	Casimir operators		
(T2q)	$\frac{1}{c^2}P_\mu P^\mu = M^{(c)2} \overset{\text{irrep}}{=\!=} \frac{p^2}{c^2}$		$\rightarrow M^2 \overset{\text{irrep}}{=\!=} m^2$
(T2r)	$\hat{W} = -\hat{w}_\mu \hat{w}^\mu \overset{\text{irrep}}{=\!=} s(s+1)$		$\rightarrow \left[\vec{J} - \frac{\vec{G}}{M} \times \vec{P}\right]^2 = \vec{\Sigma}^{(\infty)2} \overset{\text{irrep}}{=\!=} s(s+1)$

$$\mathcal{H}^{\text{ep}}=\phi(P_\mu P^\mu - p^2)\ , \tag{1.3}$$

since $\lambda^2 C \to P_\mu P^\mu$ and $\lambda^2\alpha^2 \to p^2$ where p^2 can have any value greater than zero.

Therewith, we have shown that in the elementary limit the QRR goes into the relativistic mass point (elementary particle) described by the irreducible representation (p,s) of the Poincaré group, with any values of momentum p and spin s.

C. Nonrelativistic limit

In the nonrelativistic limit the QRR contracts into the nonrelativistic rotator. As a preparation for showing this, we shall first consider the well-known contraction Eq. (1.2), from the irrep (p,s) of the Poincaré group describing the relativistic mass point (elementary particle) into the irrep of the extended Galilei group[8] describing the nonrelativistic mass point. In this case, the Casimir operators and their eigenvalues in an irreducible representation of the extended (i.e., quantum mechanical) Galilei group are

$$M=\text{central element}=m\ ,$$

$$\vec{\Sigma}^{(\infty)2}=s(s+1),\quad U=0\quad(+\text{const})\ . \tag{1.4}$$

The details of this contraction are summarized in Table II, where our notation is established by listing the commutation relations and Casimir operators of the Poincaré and of the extended Galilei groups. $\vec{\Sigma}^{(\infty)}$ is defined in Eq. (T2p) and $\vec{\Sigma}^{(\infty)2}$ is a Casimir operator of $\mathcal{G}$.

It is important to note that in the contraction Eq. (1.2), one does not just take the limit $1/c \to 0$. If this were the case, then $G_i^{(c)} \to 0$ and one would not obtain a faithful representation of the extended Galilei group $\mathcal{G}$. Therefore, one must increase K_i along with c in such a way that $G_i^{(c)}=(1/c)K_i$ remains finite. This is accomplished by going through a sequence of representations (p,s) of $\mathcal{P}_{P_\mu,J_{\mu\nu}}$ taking $p\to\infty$ as $1/c\to 0$ in such a way that $(1/c)p\to m$ remains finite. During this process, the operator P_0 "increases to infinity," but the operator

$$H^{(c)}=c(P_0-mc)=c[(\vec{P}^2+m^2c^2)^{1/2}-mc]$$
$$=\frac{\vec{P}^2}{2m}+O\left[\frac{1}{c^2}\right] \tag{1.5}$$

remains finite and becomes the energy operator of the nonrelativistic mass point,

$$H^{(c)}\to H=\frac{\vec{P}^2}{2m}\ . \tag{1.6}$$

We shall now consider the contraction of the QRR into the nonrelativistic rotator.

The energy operator of the nonrelativistic rotator is given by

$$H=\frac{\vec{P}^2}{2M}+\frac{1}{2I_B}\vec{\Sigma}^{(\infty)2}\ . \tag{1.7}$$

This corresponds to the case in which the Casimir operators $\Sigma^{(\infty)2}$ and U of $\mathcal{G}$ are related by

$$U=\frac{1}{2I_B}\Sigma^{(\infty)2}\ . \tag{1.8}$$

The same type of relation exists between the Casimir operators of the Poincaré group for the relativistic case in the form of the second-order Casimir operator of $SO(4,1)_{B_\mu,J_{\mu\nu}}$ and, as we shall see later, it survives the nonrelativistic limit to give Eq. (1.8).

As mentioned in Sec. I of paper I, the B_μ are generators of translations along the de Sitter sphere and the $\hat{B}_\mu$ are dimensionless generators of the corresponding de Sitter rotations ("de Sitter boosts"). In the nonrelativistic limit $(1/c\to 0)$ the operators B_μ increase to infinity. Therefore, we shall consider what the dimensionless operators $\hat{B}_\mu=(1/\lambda c)B_\mu$ go into. From Eq. (I 2.33), with the constant c properly restored, we have

$$\hat{B}_\mu\equiv\frac{1}{\lambda c}B_\mu=\frac{1}{\lambda c}P_\mu+\frac{1}{2M^{(c)}c^2}\{J_{\nu\mu},P^\nu\}\ . \tag{1.9}$$

The splitting of the Lorentz generators $J_{\mu\nu}$ into orbital angular momentum $M_{\mu\nu}=Q_\mu P_\nu - Q_\nu P_\mu$ and intrinsic angular momentum $S_{\mu\nu}$ [see Eq. (I 2.3a)] allows $\hat{B}_\mu$ to be written in the following way:

$$\hat{B}_\mu=\frac{1}{\lambda c}P_\mu+\frac{1}{2M^{(c)}c^2}\{Q_\nu P_\mu - Q_\mu P_\nu, P^\nu\}-M^{(c)}d_\mu\ , \tag{1.10}$$

where

$$d_\mu=S_{\mu\nu}\hat{P}^\nu\frac{1}{M^{(c)}c}=-\frac{1}{2M^{(c)2}c^2}\{S_{\nu\mu},P^\nu\} \tag{1.11}$$

is the vector operator of Eq. (I 2.9) which relates the particle position (position of charge) operator Q_μ to the c.m. position operator Y_μ as in Eq. (I 2.20). For the space components, $\mu=i=1,2,3$, Eq. (1.10) can be written

$$\hat{B}_i=\frac{1}{\lambda c}P_i+\frac{1}{2M^{(c)}c^2}\{Q_\nu P_i,P^\nu\}-\frac{1}{i}\frac{1}{2M^{(c)}c^2}P_i$$
$$-\frac{1}{2M^{(c)}c^2}\{M^{(c)2}c^2,Q_i\}-M^{(c)}d_i\ . \tag{1.12}$$

In the nonrelativistic limit, using

$$\frac{1}{\lambda c}P_i\to 0,\quad \frac{1}{2M^{(c)}c^2}\{Q_\nu P_i,P^\nu\}\to 0\ ,$$
$$\frac{1}{2M^{(c)}c^2}P_i\to 0\ ,$$

and Eq. (T2d), we obtain

$$\hat{B}_i\to -MQ_i - Md_i^{(\infty)}\equiv -MY_i^{(\infty)}\ , \tag{1.13}$$

where we have defined

$$d_i^{(\infty)}\equiv\lim_{c\to\infty} d_i \tag{1.14}$$

and, in analogy to the relativistic case,

$$Y_i^{(\infty)}\equiv Q_i + d_i^{(\infty)} \tag{1.15}$$

so that $Y_i^{(\infty)}$ is the nonrelativistic c.m. position operator. For the time component of $\hat{B}_\mu$ one obtains

$$\hat{B}_0\to\frac{1}{\lambda}M=RM\ . \tag{1.16}$$

Thus, $-\hat{B}_i/M^{(c)}$ goes into the nonrelativistic c.m. position operator $Y_i^{(\infty)}$ and $\hat{B}_0/M^{(c)}$ goes into the radius R of the micro—de Sitter space. M is the central element of the extended Galilei group and it has an arbitrary eigenvalue m.

In order to obtain the properties of $d_m^{(\infty)}$ and $Y_m^{(\infty)}$, we shall study the limit of Eq. (1.14) in detail. To do this, we first define

$$g_i^{(c)} \equiv \frac{1}{c} S_{i0} , \tag{1.17}$$

which in the nonrelativistic limit is to remain finite in analogy to Eq. (T2b):

$$g_i \equiv \lim_{c\to\infty} g_i^{(c)} . \tag{1.18}$$

Then, in analogy to Eqs. (T2f), (T2g), and (T2j), S_i and g_i satisfy the commutation relations of $E(3)_{S_i,g_i}$:

$$[S_i,S_j]=i\epsilon_{ijk}S_k,\quad [S_i,g_j]=i\epsilon_{ijk}g_k,\quad [g_i,g_j]=0 . \tag{1.19}$$

Inserting Eq. (T2e) into Eq. (1.11) gives

$$d_\mu = S_{\mu 0}\left[\frac{1}{c}H^{(c)}+M^{(c)}c\right]\frac{1}{M^{(c)2}c^2}+S_{\mu j}P^j\frac{1}{M^{(c)2}c^2} , \tag{1.20}$$

which in the limit $1/c \to 0$ gives

$$d_i = c g_i^{(c)}\left[\frac{1}{c}H^{(c)}+M^{(c)}c\right]\frac{1}{M^{(c)2}c^2}+S_{ij}P^j\frac{1}{M^{(c)2}c^2} \to g_i\frac{1}{M} , \tag{1.21}$$

and

$$d_0 = S_{0i}\frac{P^i}{M^{(c)2}c^2} \to 0 . \tag{1.22}$$

This shows that the contraction limit of d_i, Eq. (1.14), exists and is given by

$$d_i^{(\infty)} = \frac{1}{M} g_i . \tag{1.23}$$

With Eqs. (T2b), (T2d), (1.17), (1.23), and (1.15), it follows from Eqs. (I 2.3a) that

$$\vec{G} = M\vec{Y}^{(\infty)} . \tag{1.24}$$

The nonrelativistic limit of $\Sigma_{\mu\nu}$ is obtained from definition (I 2.13) by using Eqs. (T2e), (1.17), (1.22), and (1.23):

$$\Sigma_{ij} \to S_{ij} - d_i^{(\infty)}P_j + d_j^{(\infty)}P_i = \Sigma_{ij}^{(\infty)} , \tag{1.25}$$

$$\frac{1}{c}\Sigma_{i0} \to g_i - g_i = 0 . \tag{1.26}$$

The last equality in Eq. (1.25) follows from the definition in Eq. (T2p) and the fact that

$$\vec{J} - \frac{\vec{G}}{M}\times\vec{P} = \vec{J} - \vec{Y}^{(\infty)}\times\vec{P} = \vec{S}+\vec{Q}\times\vec{P}-\vec{Y}^{(\infty)}\times\vec{P} = \vec{S} - \vec{d}^{(\infty)}\times\vec{P} , \tag{1.27}$$

where we have used Eqs. (1.24), (I 2.3a), and (1.15).

The operators $d_i^{(\infty)}$ and $\Sigma_{ij}^{(\infty)}$ form the spectrum-generating group $E(3)_{d_i^{(\infty)},\Sigma_{ij}^{(\infty)}}$ of the nonrelativistic rotator. To see this we take the nonrelativistic limits of Eqs. (I 2.14), (I 2.15), and (I 2.16). With the constant c properly restored ($\hat{P}_\mu = P_\mu/M^{(c)}c$), these equations read

$$[d_\mu,d_\nu] = -i\frac{1}{M^{(c)2}c^2}\Sigma_{\mu\nu} , \tag{1.28}$$

$$[d_\mu,\Sigma_{\rho\sigma}] = i(g_{\mu\rho}d_\sigma - g_{\mu\sigma}d_\rho) - i(\Sigma_{\rho\sigma}-S_{\rho\sigma})\frac{P_\mu}{M^{(c)2}c^2} , \tag{1.29}$$

$$[\Sigma_{\mu\nu},\Sigma_{\rho\sigma}] = -i(g_{\mu\rho}\Sigma_{\nu\sigma}+g_{\nu\sigma}\Sigma_{\mu\rho}-g_{\mu\sigma}\Sigma_{\nu\rho}-g_{\nu\rho}\Sigma_{\mu\sigma}) + i\frac{1}{M^{(c)2}c^2}(P_\mu P_\rho\Sigma_{\nu\sigma}+P_\nu P_\sigma\Sigma_{\mu\rho}-P_\mu P_\sigma\Sigma_{\nu\rho}-P_\nu P_\rho\Sigma_{\mu\sigma}) , \tag{1.30}$$

and in the limit $1/c\to 0$ they contract into

$$[d_i^{(\infty)},d_j^{(\infty)}]=0 , \tag{1.31a}$$

$$[d_i^{(\infty)},\Sigma_{kl}^{(\infty)}]=i(g_{ik}d_l^{(\infty)}-g_{il}d_k^{(\infty)}) , \tag{1.31b}$$

$$[\Sigma_{ij}^{(\infty)},\Sigma_{kl}^{(\infty)}]=-i(g_{ik}\Sigma_{jl}+g_{jl}\Sigma_{ik}-g_{il}\Sigma_{jk}-g_{jk}\Sigma_{il}) , \tag{1.31c}$$

which are the commutation relations of $E(3)_{d_i^{(\infty)},\Sigma_{ij}^{(\infty)}}$. In the same way, we find

$$[d_i^{(\infty)},S_{kl}]=i(g_{ik}d_l^{(\infty)}-g_{il}^{(\infty)}d_k^{(\infty)}) , \tag{1.32}$$

which follows from the contraction of Eq. (I 2.11). This, together with Eqs. (1.31a) and (I 2.2), shows that $d_i^{(\infty)}$ and S_{ij} also satisfy the Lie algebraic relations of E(3). A connection is made with Sec. I of paper I by taking

$$D_i = d_i^{(\infty)} . \tag{1.33}$$

Therewith, in the nonrelativistic limit we have recovered the spectrum-generating group $E(3)_{d_i^{(\infty)},\Sigma_{ij}^{(\infty)}}$ of the dipole operator and spin angular momentum, and also $E(3)_{d_i^{(\infty)},S_{ij}}$ generated by the dipole operator and the intrinsic angular momentum.

For the Majorana representation[9] the second Casimir operator of $SO(3,1)_{S_{\mu\nu}}$ is zero:

$$S_i S_{i0} = 0 . \tag{1.34}$$

In the nonrelativistic limit this contracts, according to Eqs. (1.17) and (1.18), into the second Casimir operator

$S_i g_i$ of $E(3)_{S_i,g_i}$ (Ref. 10) such that Eq. (1.34) is retained:

$$S_i g_i = 0 \ . \tag{1.35}$$

Then, according to Eq. (1.23),

$$d_i^{(\infty)} S_i = \frac{1}{M} g_i S_i = 0 \ . \tag{1.36}$$

The same kind of condition follows from Eq. (1.25) for the spectrum-generating group $E(3)_{d_i^{(\infty)},\Sigma_{ij}^{(\infty)}}$:

$$\epsilon_{ijk} d_i^{(\infty)} \Sigma_{jk}^{(\infty)} = d_i^{(\infty)} \Sigma_i^{(\infty)} = 0 \ . \tag{1.37}$$

This shows that if we restrict ourselves to the Majorana representation, then the nonrelativistic limit of the QRR will be the dumbbell or the rigid-rod model with Eq. (I 1.6).

Just as the operators $d_i^{(\infty)}$ commute, it follows from Eqs. (1.13) and (I 2.6) that the nonrelativistic c.m. position operators $Y_i^{(\infty)}$ also commute:

$$[Y_i^{(\infty)}, Y_j^{(\infty)}] = 0 \ . \tag{1.38}$$

Thus, we have shown that in the nonrelativistic limit the "dipole" operator d_μ and the c.m. position operator Y_μ have the familiar properties. Their noncommutativity, Eqs. (I 2.14) and (I 2.26), is a relativistic effect.

In Table III some further details of the contraction of the QRR into the nonrelativistic rotator are summarized. According to Eqs. (1.13), (1.16), (1.24), and (T2b), we have

$$\frac{1}{\lambda c}\vec{B} = \hat{\vec{B}} \to -\vec{G}, \quad \frac{1}{c}\vec{K} \to \vec{G}, \quad \lambda \hat{B}_0 \to M \ . \tag{1.39}$$

Consequently, in the nonrelativistic limit the Casimir operator of $SO(4,1)_{B_\mu,J_{\mu\nu}}$ contracts into the square of the mass operator of the extended Galilei group:

$$\frac{\lambda^2}{c^2} C = \lambda^2 \hat{B}_0{}^2 - \lambda^2 \hat{\vec{B}}{}^2 + \frac{\lambda^2}{c^2}\vec{K}^2 - \frac{\lambda^2}{c^2}\tfrac{1}{2} J_{ij} J^{ij} \to M^2 \ . \tag{1.40}$$

In order that $(1/c)K_i$ and $\hat{B}_i$ do not go into the zero operator in the limit $1/c \to 0$, one must again go through a sequence of representations taking $\alpha^2 \to \infty$ $(\lambda^2\alpha^2 \to \infty)$ as $1/c \to 0$ such that $(\lambda^2\alpha^2/c^2) \to m^2$ remains finite.

We are now ready to show that the correct form of the energy operator of the nonrelativistic rotator is obtained in the limit $1/c \to 0$ from the Hamiltonian of Eq. (I 2.55), i.e., from the $SO(4,1)_{B_\mu,J_{\mu\nu}}$ Casimir operator in the form

$$\lambda^2 C = P_\mu P^\mu - \lambda^2 \hat{W} + \tfrac{9}{4}\lambda^2 \ . \tag{1.41}$$

In order to take the nonrelativistic limit of Eq. (1.41), we first write it out in more detail:

$$\lambda^2 C = \frac{1}{c^2} H^{(c)2} + 2M^{(c)} H^{(c)} + M^{(c)2} c^2 - \vec{P}^2 + \lambda^2 (\hat{w}^0)^2 - \hat{\vec{w}}{}^2 \lambda^2 + \tfrac{9}{4}\lambda^2 \ . \tag{1.42}$$

According to Eqs. (T2e), (T2o), and (T2p), we have

$$\frac{1}{c^2} H^{(c)2} \to 0, \quad (\hat{w}^0)^2 \to 0, \quad \hat{\vec{w}}{}^2 \to \vec{\Sigma}^{(\infty)2} \ . \tag{1.43}$$

In this $1/c \to 0$ contraction process we take $\lambda^2\alpha^2 \to \infty$ and $p^2 = m^2 c^2 \to \infty$, while keeping the difference $\lambda^2\alpha^2 - m^2 c^2$ finite. We can arrange to take $\lambda^2\alpha^2 - m^2c^2 \to \lambda a^2$, where a^2 is an arbitrarily chosen finite number. Then, from Eq. (1.42) we obtain

$$\lambda^2 a^2 = 2MH - \vec{P}^2 - \lambda^2 \vec{\Sigma}^{(\infty)2} + \tfrac{9}{4}\lambda^2 \ ,$$
$$H = \frac{\vec{P}^2}{2M} + \lambda^2 \frac{\vec{\Sigma}^{(\infty)2}}{2M} + \frac{\lambda^2}{2M}(a^2 - \tfrac{9}{4}) \ . \tag{1.44}$$

This is the energy operator of the nonrelativistic rotator, and it is identical to Eq. (I 1.7) up to the arbitrary constant $(\lambda^2/2M)(a^2 - \frac{9}{4})$, which we choose to be zero if we take

$$I_B = \frac{1}{\lambda^2} M = R^2 M \tag{1.45}$$

and if $\vec{S}$ in Eq. (I 1.7) represents the angular momentum in the center-of-mass frame.

TABLE III. Nonrelativstic limit of the rotator given by the group contraction from the Poincaré group to the Galilei group.

	Relativistic rotator	Nonrelativistic rotator
(T3a)	$SO(3,1)_{J_{\mu\nu}}$	$\to E(3)_{G_i J_i}$
(T3b)	$[J_i, \hat{B}_j] = i\epsilon_{ijk}\hat{B}_k$	$\to [M_i + S_i, Q_j + d_j^{(\infty)}] = i\epsilon_{ijk}(Q + d^{(\infty)})_k$
(T3c)	$\frac{1}{c}K_i = \frac{1}{c}M_{i0} + \frac{1}{c}S_{i0}$	$\to G_i = Q_i M + g_i$
(T3d)	$d_i = S_{i0}\left[\frac{1}{c}H^{(c)} + M^{(c)}c\right]\frac{1}{M^{(c)2}c^2} + S_{ij}P^j \frac{1}{M^{(c)2}c^2}$	$\to d_i^{(\infty)} = g_i \frac{1}{M}$
(T3e)	$[K_i, \hat{B}_j] = i\delta_{ij}\hat{B}_0$	$\to [G_i, Q_j + d_j^{(\infty)}] = 0$
(T3f)	$[\hat{B}_i, \hat{B}_j] = i\frac{1}{c^2}J_{ij}$	$\to [Q_i + d_i^{(\infty)}, Q_j + d_j^{(\infty)}] = 0$
(T3g)	$[\hat{B}_\mu, P_\nu] = iM^{(c)}\left[g_{\mu\nu} - \frac{P_\mu P_\nu}{M^{(c)2}c^2}\right]$	$\to [-MQ_i - Md_i^{(\infty)}, P_j] = iMg_{ij}$
		$[Q_i, P_j] = i\delta_{ij}$

If we return to the picture of the nonrelativistic rotator as a diquark dumbbell (mentioned in Sec. I of Ref. 1), then we can use Eqs. (1.44) and (1.45) to get an estimate of the "distance" between the two quarks in a meson. The moment of inertia for the two quarks, with masses m_1 and m_2 which are separated by the rigid distance x, is

$$I_B = x^2 \frac{m_1 m_2}{m_1 + m_2} . \tag{1.46}$$

From the first term in Eq. (1.44), M is the total mass which is equal to $m_1 + m_2$, so that Eq. (1.45) gives

$$x^2 = \frac{1}{\lambda^2} \frac{(m_1+m_2)^2}{m_1 m_2} . \tag{1.47}$$

Then for mesons consisting of quarks having nearly the same masses, we find

$$x \approx \frac{2}{\lambda} = 2R \approx 0.7 \times 10^{-13} \text{ cm} , \tag{1.48}$$

where we have used the phenomenological value of $\lambda = 1/R$ given in Eq. (I 3.31).

In conclusion, we have shown that the QRR model, specified by the value $\lambda^2\alpha^2$ of the second-order Casimir operator of $SO(4,1)_{B_\mu, J_{\mu\nu}}$, has the desired properties of a model for an extended relativistic rotating object: in the elementary limit ($1/R = \lambda \to 0$) it goes into the model of the relativistic mass point (elementary particle), and in the nonrelativistic limit ($1/c \to 0$) it goes into the model of the nonrelativistic rotator. The fact that the QRR model reduces to these two models, which have both been used successfully in their respective areas of physics, gives us hope that the QRR model will be useful for relativistic rotating objects. But, whether the QRR is realized in nature, i.e., whether a physical system really exists that is described to a sufficient degree of accuracy by this model, is a question which can only be answered by experiments. Comparison with the experimental data, as discussed in the following section, will indicate that this is indeed the case. Meson resonances appear as rotators in the same way that nuclei and molecules do, only that they are, perhaps, less rigid.

II. EXPERIMENTAL EVIDENCE FOR THE QUANTUM RELATIVISTIC ROTATOR

Experimental evidence for the rotational bands in molecules[11] and nuclei[12] is given by their energy levels and the matrix elements (intensities) for the radiative transitions between them (mainly electric dipole moments for molecules and electric quadrupole and magnetic dipole moments for nuclei). For the higher-spin hadron resonances, radiative and weak decay rates and magnetic moments are unknown so that presently any evidence for hadronic rotational bands can only come from the mass levels where the data are rather sparse. But for the best-known class of meson resonances, those with normal s^P and positive $C_n P$, rotational bands are clearly visible in perfect analogy to the nonrelativistic rotator levels in nuclei and molecules.

According to our result, Eq. (II 4.13) with Eqs. (II 4.8) and (II 4.15), the irreducible representation space $\mathcal{H}^{\alpha}$ (Maj) describes a physical system which consists of different mass levels where each mass value is determined by the spin. The spin spectrum is determined by the SO(3,2) representation and, for the Majorana case, each spin occurs exactly once with alternating parity:

$$s^P = 0^+, 1^-, 2^+, 3^-, 4^+, \ldots , \quad \text{for the integer-spin case} (k_0 = 0, c = \tfrac{1}{2})$$

and (2.1)

$$s^P = \tfrac{1}{2}^+, \tfrac{3}{2}^-, \tfrac{5}{2}^+, \ldots , \quad \text{for the} \otimes\text{-integer-spin case} (k_0 = \tfrac{1}{2}, c = 0)$$

(or with opposite parity 0^-, 1^+, 2^-, etc.). This is the same as for the energy levels of the simple nonrelativistic rotator, where to each energy level there corresponds an irreducible representation of the rotation group. For our relativistic case, to each mass level there corresponds an irreducible representation $(m(s), s)$ of the physical Poin-

TABLE IV. Experimental data of all known meson resonances with normal s^P and positive $C_n P$.

		$Y=0$, $I=1$			$Y=0$, $I=0$			$Y=1$, $I=\frac{1}{2}$	
s^P	Particle name	m	Δm	Particle name	m	Δm	Particle name	m	Δm
1^-	ρ	0.769	0.154	ω	0.7826	0.0099	K^*	0.892	0.051
2^+	A_2	1.318	0.110	f	1.273	0.179		1.434	0.100
3^-	g	1.691	0.200	ω	1.67	0.166		1.775	0.14
4^+	δ	2.034	0.20	h	2.04	0.15		2.037	0.249
				ϵ	2.30	0.17			
5^-	ρ	2.35	0.20						
6^+	δ	2.45	0.32						
1^-	$\rho'(1600)$								
1^-	$\rho'(2150)$								

caré group $\mathscr{P}_{P_\mu, J_{\mu\nu}}$. Thus each hadron, which is described by this irreducible representation, is considered as a different level of the rotator. To test whether hadrons fall into rotational bands we will have to assign to each $\mathscr{H}^\alpha$ (Maj) a set of hadrons with increasing value of spin and mass but otherwise with identical or similar properties. Such a set we will call a tower. The best-known integer-spin towers are those with normal s^P and positive $C_n P$ consisting of [13]

$$(s^P)_{\square C_n P}=(1^-)_{N^+},(2^+)_{N^+},(3^-)_{N^+},(4^+)_{N^+},\ldots \quad (2.2)$$

In Table IV we list the ρ tower ($I=1$), the ω tower ($I=0$), and the K tower ($Y=1$). Although there are indications for other towers with s^P abnormal and $C_n P$ negative or positive, there are not enough recurrences in these towers (or in the ϕ tower) to perform a meaningful test on them. Therefore, we will restrict our tests to these three meson towers. It is important to note that we have listed all the known resonances from Ref. 14 with the properties of these towers and not just a subset of them conveniently selected for our purpose. Not all resonances listed in Table IV are entries in the meson table of Ref. 14, but some are entries in the meson data card listings. Notice that for the $I=0$ tower there are two 4^+ resonances in the meson data card listings. If this degeneracy is confirmed, then the Majorana representation is too simple to describe this tower and a larger representation of SO(3,2) must be chosen which allows for a degeneracy of the states with the same s^P. We have already mentioned, in Sec. III of Ref. 1, that this is likely to be the case and we should expect fine-structure effects to appear. Here our interest lies only in checking whether rotational bands do exist for the mesons and, therefore, we ignore these probable fine-structure effects[15] and omit the $\epsilon(2300)$ from our test.

When we performed fits of the ρ, ω, and K towers in order to determine the parameters λ^2 and α^2 in the mass formula

$$m^2=\lambda^2(\alpha^2-\tfrac{9}{4})+\lambda^2 s(s+1)$$
$$=\lambda^2\hat{\alpha}^2+\lambda^2 s(s+1)\,, \quad (2.3)$$

we found values for λ^2 of approximately $\lambda^2\approx 0.3$ (GeV)^2, which leads to a radius $R=1/\lambda$ of approximately $R\approx\frac{1}{3}\times 10^{-13}$ cm [as already reported in Eq. (I 3.31)]. But we also noticed a slight dependence of λ^2 upon s. This is depicted in Fig. 1 where we have drawn $m^2/s(s+1)$ versus s and in Fig. 2 where we have drawn $m^2/s(s+1)$ versus $s(s+1)$.[16] Figure 2 is the analog of Fig. 3 for the rotational bands in nuclei and of Fig. 4 for the rotational bands in molecules, in which $(E(J)-E_0)/J(J+1)$ versus $J(J+1)$ is drawn. In molecular and nuclear physics the rotational energy levels are conventionally parametrized by

$$E(J)-E_0=J(J+1)[B-DJ(J+1)] \quad (2.4a)$$

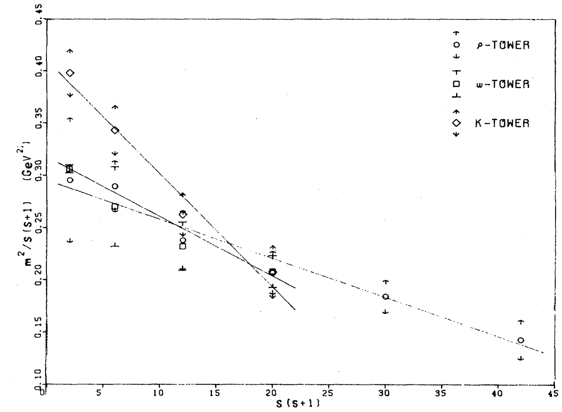

FIG. 2. Rotational bands of the ρ, ω, and K meson towers.

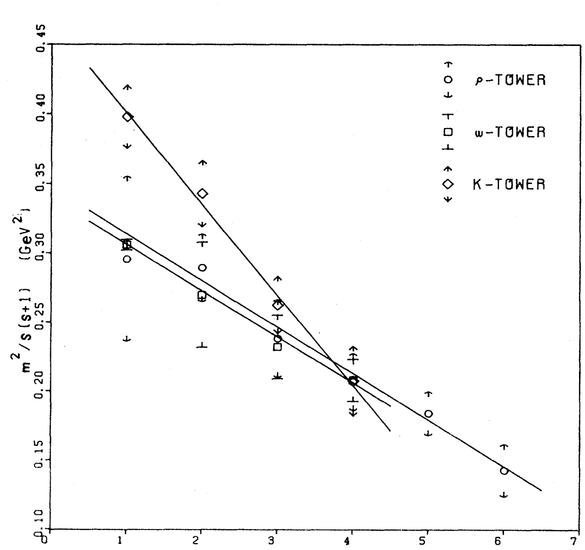

FIG. 1. Rotational bands of the ρ, ω, and K meson towers.

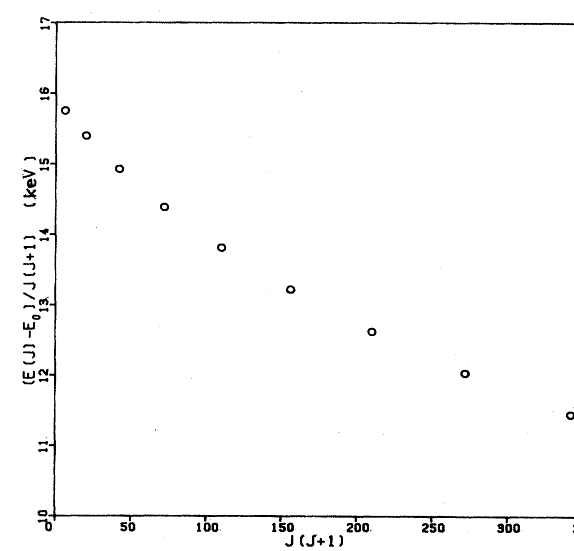

FIG. 3. Ground-state rotational band of ^{172}Hf [plot following A. Bohr and B. Mottelson (Ref. 13); data from F. S. Stephens *et al.*, Nucl. Phys. 63, 82 (1965)].

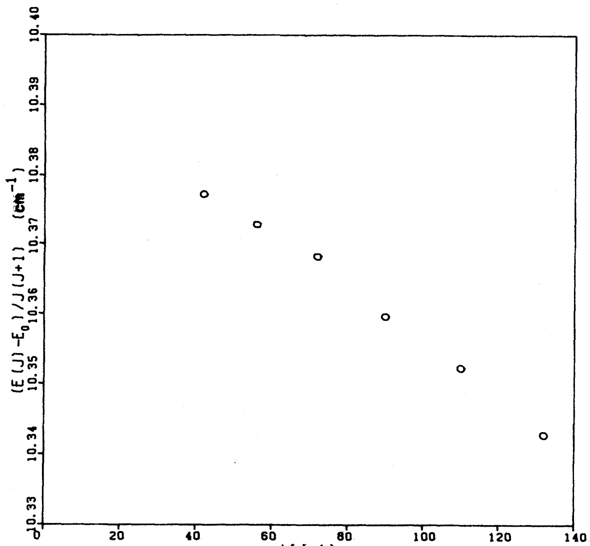

FIG. 4. Ground-state rotational band of HCl [data from M. Czerny, Z. Phys. 34, 227 (1925)].

and the dependence of the moment of inertia upon J,

$$\frac{1}{2I}=B-DJ(J+1)\ , \tag{2.4b}$$

is explained as the effect of centrifugal stretching.[11,12] The nonrigidity, which is expressed by the ratio D/B,[17] for molecular rotators is $D/B\approx 10^{-4}$ and for nuclear rotators is $D/B\approx 10^{-3}$. In analogy to these nonrelativistic cases we would expect a slight dependence of λ^2 (or $1/R^2=\lambda^2$) upon s of the form

$$\lambda^2=\lambda_1{}^2-\lambda_2{}^2 s(s+1)\ , \tag{2.5}$$

where $\lambda_2{}^2$ describes the effect of the centrifugal stretching and is expected to be a few orders of magnitude smaller than $\lambda_1{}^2$. The curves in Fig. 2 represent a fit of Eq. (2.3) with Eq. (2.5) to the experimental data of the three towers. The values of the parameters $\hat{\alpha}^2$, $\lambda_1{}^2$, and $\lambda_2{}^2$ for the three towers are given in the upper left-hand part of Table V. We also give the values of the masses predicted by this fit for the various s^P states. All the fits are good, as can be seen from the value of χ^2/n_D. Since $\hat{\alpha}^2=\alpha^2-\frac{9}{4}$ in all the fits is consistent with zero, we have fixed it at zero in order to reduce the number of fitted parameters to two. The results of these fits are given in the lower left-hand part of Table V. Since χ^2 increases only minimally, the confidence level for these fits is even better than for the fits with $\hat{\alpha}^2$ as a parameter. The nonrigidity for the hadronic rotational bands is $\lambda_2{}^2/\lambda_1{}^2\approx 10^{-2}$, i.e., it is an order of magnitude larger than the nonrigidity D/B for the nuclear bands and two orders of magnitude larger than for the molecular bands.

In Fig. 1 we have assumed that the s dependence of λ^2 is given by

$$\lambda^2=\lambda_1{}^2-\lambda_2{}^2 s\ . \tag{2.6}$$

This gives an even better fit to the experimental data as is displayed in the right-hand part of Table V. We do not know of any theoretical arguments that would discriminate between Eqs. (2.5) and (2.6), but taking the fine-structure effects into account[15] leads to a formula for λ^2 where Eqs. (2.5) and (2.6) are the two external cases. In order to check whether the $J(J+1)$ dependence of $1/I$ is essential for the nuclear rotators, in Fig. 5 we have redrawn $(E(J)-E_0)/J(J+1)$ versus J using the same data that was used for Fig. 3. One can see that

$$\frac{1}{2I}=B-DJ$$

TABLE V. Fits of the quantum relativistic rotator mass formula to the experimental data for three meson towers with normal s^P and positive C_nP. Fits A include predicted masses in GeV.

Fit A	$m=[\hat{\alpha}^2+s(s+1)]^{1/2}[\lambda_1{}^2-\lambda_2{}^2 s(s+1)]^{1/2}$			$m=[\hat{\alpha}^2+s(s+1)]^{1/2}(\lambda_1{}^2-\lambda_2{}^2 s)^{1/2}$		
s^P	ρ tower	ω tower	K tower	ρ tower	ω tower	K tower
1^-	0.79	0.783	0.893	0.78	0.783	0.893
2^+	1.28	1.26	1.413	1.30	1.27	1.421
3^-	1.72	1.68	1.81	1.72	1.69	1.800
4^+	2.08	2.03	2.006	2.06	2.04	2.012
5^-	2.34	(2.28)		2.32	(2.31)	
6^+	2.44	(2.38)		2.46	(2.50)	
χ^2/n_D	0.84/3	0.06/1	0.50/1	0.49/3	0.29/1	0.05/1
$\hat{\alpha}^2$	0.30±0.59	0.34±0.50	0.26±0.34	−0.09±0.52	0.075±1.036	−0.042±0.657
$\lambda_1{}^2$	0.28±0.03	0.27±0.06	0.37±0.05	0.35±0.05	0.32±0.19	0.47±0.16
$\lambda_2{}^2$	0.003±0.001	0.003±0.003	0.009±0.003	0.03±0.01	0.029±0.050	0.068±0.048

Fit B	$m=[s(s+1)]^{1/2}[\lambda_1{}^2-\lambda_2{}^2 s(s+1)]^{1/2}$			$m=[s(s+1)]^{1/2}(\lambda_1{}^2-\lambda_2{}^2 s)^{1/2}$		
	ρ tower	ω tower	K tower	ρ tower	ω tower	K tower
χ^2/n_D ($\hat{\alpha}^2=0$)	1.19/4	0.73/2	1.19/2	0.53/4	0.12/2	0.25/2
$\lambda_1{}^2$	0.30±0.02	0.32±0.01	0.41±0.02	0.35±0.03	0.34±0.01	0.47±0.03
$\lambda_2{}^2$	0.004±0.001	0.006±0.001	0.011±0.002	0.034±0.006	0.033±0.005	0.065±0.011

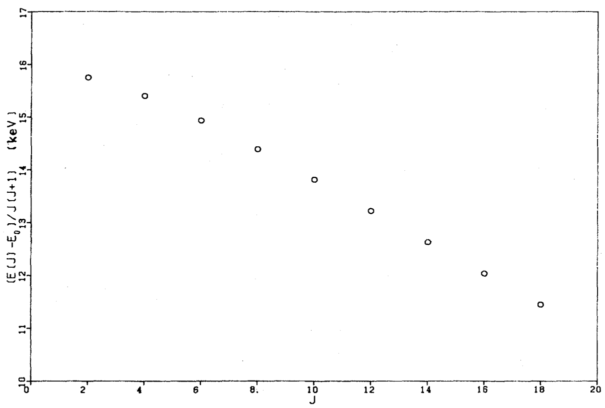

FIG. 5. Ground-state rotational band of ^{172}Hf [data from F. S. Stephens *et al.*, Nucl. Phys. 63, 82 (1965)].

gives even better agreement with the experimental data than the conventional ansatz of Eq. (2.4).

From the results of the fits we see that λ_1^2 and λ_2^2 take on nearly the same value for the ρ and the ω towers. For the quadratic correction term, Eq. (2.5), $\lambda_1^2 \approx 0.28$ GeV^2 and for the linear correction term, Eq. (2.6), $\lambda_1^2 \approx 0.34$ GeV^2. For the ρ and ω towers, $\hat{\alpha}^2$ is comparable to zero and fixing it at zero (i.e., taking for α^2 the lowest group theoretically allowed value of $\frac{9}{4}$) increases the χ^2 only minimally and therewith makes the fit significantly better. For the K tower λ_1^2 is slightly larger than for the nonstrange towers, but within the errors (one standard deviation from the minimum) all three values of λ_1^2 agree with each other for both the quadratic and the linear correction term. If λ_1^2 turns out to be larger for the K tower, it may be explained as coming from the different constituents of the diquark dumbbell.

From our fits in Table V and Figs. 1 and 2, we conclude that the normal s^P and positive C_nP mesons form rotational bands similar to the rotational states of molecules and nuclei. The relativistic rotator, like the molecular and nuclear rotators, is also not completely rigid but stretches with increasing angular momentum. The similarity between the rotator properties of molecules and nuclei is well known.[12] That this similarity also extends from the nonrelativistic to the relativistic domain, as demonstrated by comparing Fig. 2 with Figs. 3 and 4 (or Fig. 1 with Fig. 5), is a fascinating display of unity in physics.

In conclusion, we now summarize the basic assumptions and results of this three-part series of papers. The quantum relativistic rotator is defined by a relativistic Hamiltonian and other quantum observables to which the rules of constrained Hamiltonian mechanics (with Dirac-Poisson brackets replaced with commutators) are applied.[18] The Hamiltonian is conjectured to be given in terms of the second-order Casimir operator of the "dynamical group" $SO(4,1)_{B_\mu, J_{\mu\nu}}$ (Ref. 19) [see Eq. (I 2.38)] which contains one parameter, λ, if the rotator is rigid or two parameters, λ_1 and λ_2 [which are related to λ by Eqs. (2.5) or (2.6)], if centrifugal stretching is considered. To justify the name quantum relativistic rotator, we have shown in Sec. I that this model corresponds to the quantum relativistic mass point in the elementary limit and to the nonrelativistic rotator, with moment of inertia given by Eq. (1.45), in the nonrelativistic limit.

A single relativistic rotator, characterized by the value of the parameter α in the Hamiltonian, consists of a tower of spin levels $s = 1,2,3,\ldots,$ with masses given by Eq. (2.3). Thus for every rotator there corresponds three parameters if centrifugal stretching is included where α is the analog of the ground-state energy for the nonrelativistic rotator and where λ_1 and λ_2 describe the relativistic analog of the moment of inertia with elasticity. If two rotators happen to have identical "moments of inertia," as may be the case for all mesons with normal s^P and positive C_nP and should be the case for all nonstrange diquarks of this particular kind, then the parameters λ and λ_2 will have the same values for each "rotator." Also, if two rotators have the same "ground-state mass," then α should take on the same values for each.

We have fitted the hadron masses of the ρ tower (with six known masses), the ω tower (with four known masses), and the K tower (with four known masses) to the mass formula Eq. (2.3) predicted by our Hamiltonian. The results in Table V show that both nonstrange meson towers (presently containing ten hadron masses) can indeed be fitted with the two structure parameters: $\lambda_1^2 \approx 0.31$ (GeV)^2 and $\lambda_2^2 \approx 0.005$ (GeV)^2 (where one and the same value $\hat{\alpha}^2 = 0$ has been taken for the ground state). For the strange meson tower the moment of inertia appears to be slightly different, although a fit of all 18 known mesons with normal s^P and positive C_nP to Eq. (2.3) (with $\hat{\alpha} = 0$), does give an acceptable fit.

We have also applied the mass formula, Eq. (2.3), to nucleon and hyperon resonances. Here one does not have a very clear selection criteria which assigns a set of hadrons to a particular tower so that some of the baryon resonances have to be excluded arbitrarily. But, the existing baryon data are consistent with Eq. (2.3).

As mentioned at the beginning of Sec. I of this paper the transition matrix elements, which are another test for the rotational bands in molecular and nuclear physics, cannot be tested for the relativistic rotator since the required data (radiative transitions and photoproduction or other production scattering between higher spin hadrons) do not yet exist. Therefore, for the time being, the mass-spin spectrum remains the only testing ground for the QRR and here the evidence appears to be favorable.

ACKNOWLEDGMENTS

Support from the Department of Energy is gratefully acknowledged. Part of this work was done while one of the authors (A.B.) was a guest of the Max Planck Institute, Munich, and the Institute of Physics, University of Wurzburg. He would like to express his gratitude to the host institutions and the Alexander von Humboldt Foundation for their hospitality and to W. Dreschler for valuable discussions on the subject of this paper.

*On leave from Institute for Theoretical Physics, Warsaw University, Warsaw, Poland and Centro de Investigacion y de Estudios Avanzados del Instituto Politécnico Nacional, Mexico 14, D.F., Mexico.

[1]R. R. Aldinger, A. Bohm, P. Kielanowski, M. Loewe, P. Magnollay, N. Mukunda, W. Dreschsler, and S. R. Komy, Phys. Rev. D. 28, 3020 (1983). We shall refer to this reference as paper I and to equations in it as the particular equation number preceded by a I, e.g., Eq. (I 2.38).

[2]A. Bohm, M. Loewe, L. C. Biedenharn, and H. van Dam, Phys. Rev. D 28, 3032 (1983). We shall refer to this reference as paper II and to equations in it as the particular equation number preceded by a II, e.g., Eq. (II 1.2).

[3]E. P. Wigner, Ann. Math. 40, 149 (1939); in *Group Theoretical Concepts and Methods in Elementary Particle Physics,* edited by F. Gürsey (Gordon and Breach, New York, 1964), p. 37.

[4]A. Bohm, *Quantum Mechanics* (Springer, New York, 1979), pp. 95–112.

[5]E. Inönü and E. P. Wigner, Proc. Natl. Acad. Sci. USA 39, 510 (1953).

[6]In Refs. 1 and 2, we have chosen the units such that $c=1$. If we want to study the nonrelativistic limit, we can no longer do so. Therefore, in this paper the physical quantities shall have nonrelativistic units (but we still keep $\hbar=1$). The nonrelativistic units can be restored in the previous papers by making the replacements $M\rightarrow Mc$ and $\hat{B}\rightarrow\hat{B}c$. The eigenvalue of the Casimir operator $P_\mu P^\mu$ of the Poincaré group is then called p^2 and the eigenvalue of the central element of the extended Galilei group is called m. The group contraction from the representation (p,s) of the Poincaré group to the representation (m,s, internal energy $U=$const) of the extended Galilei group is then accomplished in the usual way (Ref. 5) by taking the contraction constant $1/c$ to zero and going through a sequence of irreps such that $p\rightarrow\infty$, and $(1/c)p\rightarrow m$ remains finite. The contraction limit of the de Sitter group representations is similarly described by taking the contraction constant $\lambda=1/R$ to zero and going through a sequence of irreps such that $\alpha\rightarrow\infty$ and $\lambda\alpha\rightarrow p$ remains finite.

[7]For a derivation of all the unitary irreducible representations of SO(4,1) and the mathematical details concerning the relation between $SO(4,1)_{B_\mu,J_{\mu\nu}}$ and $\mathscr{P}_{P_\mu,J_{\mu\nu}}$ as well as the contraction process, see A. Bohm, in *Studies in Mathematical Physics,* edited by A. O. Barut (Reidel, Boston, 1973), p. 197, and references therein.

[8]Lévy Leblond, in *Group Theory and its Applications,* Vol. 2, edited by E. M. Loebl (Academic, New York, 1971), p. 221.

[9]In the Majorana representation, the eigenvalue ik_0c_N of the Casimir operator $-\frac{1}{8}\epsilon^{\mu\nu\rho\sigma}S_{\mu\nu}S_{\rho\sigma}=S_{i0}S_i$ is equal to zero because either $k_0=0$ (integer spin) or $c_N=0$ (half-integer spin).

[10]In the $1/c\rightarrow 0$ contraction from $SO(3,1)_{S_{\mu\nu}}$ to $E(3)_{g_i,S_i}$ we can choose two contraction limits for the representations. First, we can keep the representation (k_0,c_N) fixed. Then, by Eq. (1.17), g_i and, by Eq. (1.21), $d_i^{(\infty)}$ are represented by zero operators. Second, we can go through a series of representations (k_0,c_N) enlarging S_{i0} such that g_i and, by Eq. (1.21), d_i are represented by nonzero operators. Then the resulting E(3) representation is characterized by (k_0,ϵ), where k_0 is as above, and $\epsilon=\lim_{c\rightarrow\infty}(1/c)c_N$. For details, see Appendix V 3 of Ref. 4.

[11]G. Herzberg, *Molecular Spectra and Molecular Structure. I. Spectra of Diatomic Molecules* (Van Nostrand, New York, 1950); *Molecular Spectra and Molecular Structure. III. Electronic Structure of Polyatomic Molecules* (Van Nostrand, New York, 1966).

[12]A. Bohr and B. Mottelson, *Nuclear Structure* (Benjamin, New York, 1969), Vol. II.

[13]C_n stands for neutral charge parity and we use essentially the notation and nomenclature of Ref. 14 which is also the main source of our experimental numbers.

[14]Particle Data Group, Phys. Lett. 111B, 1 (1982).

[15]Fine-structure effects, which are a consequence of nonrigidity and centrifugal stretching, will be discussed in more detail in a forthcoming paper.

[16]It is not clear *a priori* whether the hadron tower is characterized by the eigenvalue $\hat{\alpha}^2$ of the Casimir operator of $SO(4,1)_{\hat{B}_\mu,J_{\mu\nu}}$ or by the dimensional eigenvalue $\lambda^2\hat{\alpha}^2$ of the Casimir operator of $SO(4,1)_{B_\mu,J_{\mu\nu}}$. For the ρ and ω towers, the phenomenological value of $\hat{\alpha}^2=\alpha^2-\frac{9}{4}$ obtained from the fits is essentially zero and for the K tower it is close to zero as can be seen from the results of our fit in the table. Thus there is not much of a difference in drawing $m^2/[\hat{\alpha}^2+s(s+1)]$ or $m^2/s(s+1)$.

[17]B and D are the symbols used in molecular physics where they are given in cm^{-1}. $E(J)$ stands for energy/$\hbar c$ and B is of the order of 1–10 cm^{-1} corresponding to $(2–20)\times10^{-14}$ GeV. In nuclear physics the standard symbols are A for our B and B for our D and they are given in keV. B $(=A)$ is of the order of 10 keV or 10^{-15} GeV. In particle physics λ is of the order of 0.5–1 GeV.

[18]P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, 1964); Can. J. Math. 2, 129 (1950).

[19]A. Bohm, Phys. Rev. 175, 1767 (1968).