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New perturbative approach to renormalizable field theories

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A new method for obtaining perturbative predictions in quantum field theory is developed. Our method gives finite predictions, which are free from scheme ambiguities, for any quantity of interest (like a cross section or a Green's function) starting directly from the bare regularized Lagrangian. The central idea in our approach is to incorporate directly the consequences of dimensional transmutation for the predictions of the theory. We thus completely bypass the conventional renormalization procedure and the ambiguities associated with it. The case of massless theories with a single dimensionless coupling constant is treated in detail to illustrate our approach.

I. INTRODUCTION

In quantum field theory the bare perturbation series for any quantity of interest $\mathscr{R}$, which may be a cross section, Green's function, etc., is not well defined since the coefficients of expansion are infinite. To extract finite results the usual procedure, in a renormalizable theory, is to absorb the infinities in the bare parameters (coupling constants, masses, etc.) and fields present in the Lagrangian. The definitions of the renormalized parameters and fields in terms of the corresponding bare quantities are, however, not unique because of the possibility of finite renormalizations. Consequently, finite-order predictions for $\mathscr{R}$ in the renormalized theory depend on the renormalization scheme (RS) used. The conventional renormalization procedure, therefore, gives predictions for $\mathscr{R}$ which, although finite, are still ambiguous.

The fact that predictions for $\mathscr{R}$ are not well defined in the bare perturbation theory appears to suggest that $\mathscr{R}$ itself is not directly computable in the theory. The RS dependence of the predictions in the renormalized theory is also perhaps an indication of the same thing. If this is really so, then one may naturally ask how and in what form does the theory determine $\mathscr{R}$. In the answer to this basic question lies the key to the approach which we develop here. This approach leads to a new perturbative method for obtaining finite predictions which have the additional feature that they are free from RS ambiguities.

To understand better and sharpen the question posed above, let us focus our attention on a renormalizable massless field theory which has only one dimensionless coupling constant g_0. Consider now a dimensionless physical quantity $R(Q)$, which depends on only one external energy scale Q, in such a theory. Since there is no mass scale present in the Lagrangian, simple dimensional analysis tells us that if R is unambiguously calculable in the theory, then it must be a finite constant independent of Q. This physically uninteresting result would not follow if R is not directly computable or uniquely specified by the theory. In that case, R can have a nontrivial dependence on Q if instead the theory specifies uniquely the derivative of R with respect to Q as a function of R. That is,

$$Q\frac{dR(Q)}{dQ}\equiv R'(Q)=F(R(Q)) . \quad (1.1)$$

Actually, as we shall show in Sec. II, this is exactly what happens in the theory under consideration.

Equation (1.1) requires the knowledge of R at some $Q=Q_0$ (which may be obtained from experiments) to predict it at any other Q. This boundary condition on Eq. (1.1) provides the necessary scale Λ_R for R to have a nontrivial dependence on Q. The dependence of R on the parameter Λ_R (undetermined by the theory) is consistent with the fact that the theory already contains one free parameter, namely, the unrenormalized coupling constant g_0. What has actually happened is that this one parameter dependence of R on g_0 has now appeared, by "dimensional transmutation,"[1] through Λ_R.

From a practical point of view, the key question is how to identify and compute functions like $F(R)$. We address ourselves to this question in the following sections. In Sec. II, we show that for any physical quantity R, $F(R)$ can be obtained as a series in R with finite coefficients, starting directly from the regularized unrenormalized perturbation series for R (in terms of g_0). The renormalizability of the theory guarantees the finiteness of the coefficients of series expansion of $F(R)$ in R. One can use $F(R)$ calculated to a certain order in R in Eq. (1.1) to obtain a new kind of "perturbative" approximations for R. These perturbative predictions for R are clearly free from RS ambiguities associated with the conventional renormalization procedure. It is this method of obtaining perturbative predictions in renormalizable field theories that we wish to advocate here. In Sec. III, we extend our method to Green's functions. Recognizing that the normalization of a Green's function is also not computable, we find that in this case it is the second derivative with respect to an external momentum which is well defined and computable. We give the example of the gluon propagator in massless QCD to illustrate our method for obtaining perturbative predictions for Green's functions. Section IV contains some general remarks and our conclusions.

II. PHYSICAL QUANTITIES

We shall be interested here in renormalizable massless field theories which have only one dimensionless coupling constant g_0 (e.g., massless quantum chromodynamics). For simplicity first consider a physical quantity which depends on only one external energy scale Q. Corresponding to it one can always construct a dimensionless measurable quantity R such that its regularized unrenormalized perturbation expansion is of the form

$$R=a_0+r_{10}a_0^2+r_{20}a_0^3+r_{30}a_0^4+\cdots \quad (2.1)$$

Here $a_0\equiv g_0^2/4\pi^2$ and the subscript 0 denotes unrenormalized quantities. The coefficients r_{n0} depend on Q through the regularizing scale (e.g., an ultraviolet cutoff). Since a_0 does not depend on Q, differentiating both sides of Eq. (2.1) with respect to Q, we get

$$Q\frac{dR}{dQ}\equiv R'=r'_{10}a_0^2+r'_{20}a_0^3+r'_{30}a_0^4+\cdots , \quad (2.2)$$

where

$$r'_{n0}\equiv Q\frac{dr_{n0}}{dQ} .$$

For small R, Eq. (2.2) suggests the following power-series expansion for R' in terms of R:

$$R'=F(R)=-fR^2(1+f_1R+f_2R^2+\cdots) . \quad (2.3)$$

Substituting for R from Eq. (2.1) in the above equation, we get

$$-R'=fa_0^2+f(2r_{10}+f_1)a_0^3+f(r_{10}^2+2r_{20}+3r_{10}f_1+f_2)a_0^4+\cdots . \quad (2.4)$$

Comparing Eqs. (2.2) and (2.4), we see that

$$-f=r'_{10} , \quad (2.5a)$$

$$-ff_1=r'_{20}-2r'_{10}r_{10} , \quad (2.5b)$$

$$-ff_2=r'_{30}-3r'_{20}r_{10}-2r'_{10}r_{20}+5r'_{10}r_{10}^2 , \quad (2.5c)$$

etc.

To show that the coefficients f,f_1,f_2, etc., obtained above are finite in the limit in which the ultraviolet cutoff is removed, we use the fact that the theory is renormalizable. This implies that one can define a renormalized coupling constant g by

$$a_0=a+z_1a^2+z_2a^3+z_3a^4+\cdots , \quad (2.6)$$

where $a\equiv g^2/4\pi^2$, such that R can be written as a series in terms of a with finite (but RS-dependent) coefficients r_n:

$$R=a+r_1a^2+r_2a^3+r_3a^4+\cdots . \quad (2.7)$$

From Eqs. (2.1), (2.6), and (2.7), we obtain the following relations among the unrenormalized coefficients r_{n0} and the renormalized coefficients r_n:

$$r_1=r_{10}+z_1 , \quad (2.8a)$$

$$r_2=r_{20}+z_2+2z_1r_{10} , \quad (2.8b)$$

$$r_3=r_{30}+z_3+3z_1r_{20}+(z_1^2+2z_2)r_{10} , \quad (2.8c)$$

etc. Using now the fact that the z_n are mere constants, independent of Q, it can be readily verified that

$$-f=r'_{10}=r'_1 , \quad (2.9a)$$

$$-ff_1=r'_{20}-2r'_{10}r_{10}=r'_2r_1-2r'_1r_1 , \quad (2.9b)$$

$$-ff_2=r'_{30}-3r'_{20}r_{10}-2r'_{10}r_{20}+5r'_{10}r_{10}^2=r'_3-3r'_2r_1-2r'_1r_2+5r'_1r_1^2 , \quad (2.9c)$$

etc. Equations (2.9) imply that $f,f_1,f_2,\ldots$ are finite *and* RS independent. We emphasize here that the above proof of finiteness of $f,f_1,f_2,\ldots$ only uses the property of renormalizability of the theory. No specific renormalization scheme has been used. In other words, had we started with the regularized Lagrangian and computed the r_{n0}, we would have found their combinations in Eqs. (2.5) automatically finite.

The fact that $f,f_1,f_2,\ldots$ are finite and RS independent suggests that they are somehow connected with the RS invariants[2,3] $b,\rho_1,\rho_2,\ldots$. It can be easily shown (see the Appendix) that in fact these two sets are equal,

$$f=b,\ \ f_1=\rho_1,\ \ f_2=\rho_2,\ldots . \quad (2.10)$$

An important consequence of this result is that the first two coefficients, f and f_1, in Eq. (2.3) are also process independent. This fact can actually be demonstrated directly using Eqs. (2.8) and (2.9). Consider the physical quantities R and $\tilde{R}$, for two different processes, which have expansions of the form of Eq. (2.1). Then, since the z_n do not depend on the process, we have from Eqs. (2.8)

$$\tilde{r}_1-r_1=\tilde{r}_{10}-r_{10} \quad (2.11a)$$

and

$$(\tilde{r}_2-\tilde{r}_1^2)-(r_2-r_1^2)=(\tilde{r}_{20}-\tilde{r}_{10}^2)-(r_{20}-r_{10}^2) . \quad (2.11b)$$

Clearly the combinations on the right-hand side of Eqs. (2.11) are RS independent and, by virtue of the left-hand side, finite. Hence they are also independent of Q. It follows therefore that

$$\tilde{r}'_{10}=r'_{10} \quad (2.12a)$$

and

$$\tilde{r}'_{20}-2\tilde{r}'_{10}\tilde{r}_{10}=r'_{20}-2r'_{10}r_{10} . \quad (2.12b)$$

This completes the proof of the process independence of f and f_1. Following this method, it can also be shown that the other coefficients $f_2,f_3,\ldots$ *depend* on the process.

The above results can be summarized in the equation[4]

$$Q\frac{dR}{dQ}=b\rho(R) , \quad (2.13)$$

where the function $\rho(R)$ is unambiguously calculable in perturbation theory as a series in R with finite coefficients ρ_n:

$$\rho(R)=-R^2(1+\rho_1R+\rho_2R^2+\cdots) . \quad (2.14)$$

The coefficients $b,\rho_1,\rho_2,\ldots$ can be computed directly from the unrenormalized perturbation series for R. The first two coefficients b and ρ_1 are universal in the sense that they are the same for any R. In fact, they are also the first two coefficients of the β function [see Eq. (A1)]. An important consequence of this is that for an asymptotically (infrared) free theory like QCD (QED) for which $b>0$ ($b<0$), $R(Q)\to 0$ as $Q\to\infty$ ($Q\to 0$) for any process R.

We shall make an important use of this fact below.

The general solution of Eq. (2.13), as can be verified by direct differentiation, can be written as

$$b\ln\frac{Q}{\Lambda_R}=\frac{1}{R}-\rho_1\ln\left[1+\frac{1}{\rho_1 R}\right]+\int_0^R dx\left[\frac{1}{\rho(x)}+\frac{1}{x^2(1+\rho_1 x)}\right]. \tag{2.15}$$

This equation was earlier obtained in Ref. 2. Here Λ_R is the constant of integration. It is the characteristic scale for the physical quantity R. In principle, it can be fixed by giving the value of R at some $Q=Q_0$. Equation (2.15) then determines R implicitly as a function of Q. In the present approach, different physical quantities $R,\widetilde{R},\ldots$ will automatically have scales $\Lambda_R,\Lambda_{\widetilde{R}},\ldots,$ which are specific to them. Does this mean that the theory has many independent scales? The answer to this question is no. In fact, we will now show that the knowledge of the scale Λ_R for R is sufficient to determine the scale $\Lambda_{\widetilde{R}}$ for $\widetilde{R}$. From expansions like Eq. (2.1) for R and $\widetilde{R}$, one has

$$\frac{1}{R(Q)}-\frac{1}{\widetilde{R}(Q)}=(\widetilde{r}_{10}-r_{10})+[(\widetilde{r}_{20}-\widetilde{r}_{10}{}^2)-(r_{20}-r_{10}{}^2)]\Gamma(Q) \tag{2.16a}$$

and

$$R(Q)/\widetilde{R}(Q)=1-(\widetilde{r}_{10}-r_{10})\widetilde{\Gamma}(Q)\ . \tag{2.16b}$$

Here $\Gamma(Q)$ annd $\widetilde{\Gamma}(Q)$ are physical quantities having expansions of the form of Eq. (2.1). Moreover, as shown in Eq. (2.11), their coefficients in Eqs. (2.16) are finite. Now, from Eq. (2.15) for R and $\widetilde{R}$ one has

$$b\ln\frac{\Lambda_{\widetilde{R}}}{\Lambda_R}=\left[\frac{1}{\widetilde{R}(Q)}-\frac{1}{R(Q)}\right]+\text{other terms}\ . \tag{2.17}$$

Since Λ_R and $\Lambda_{\widetilde{R}}$ are independent of Q, we can take the limit $Q\to\infty$ (for asymptotically free theories) or $Q\to 0$ (for infrared-free theories). In this limit, $R(Q)$, $\widetilde{R}(Q)$, $\Gamma(Q)$, and $\widetilde{\Gamma}(Q)$ vanish and the difference

$$[1/\widetilde{R}(Q)-1/R(Q)]\to(\widetilde{r}_{10}-r_{10})\ .$$

The "other terms" in Eq. (2.17) therefore vanish and so we get

$$\Lambda_{\widetilde{R}}/\Lambda_R=\exp\left[\frac{1}{b}(\widetilde{r}_{10}-r_{10})\right]. \tag{2.18}$$

Equation (2.18) shows, as expected, that there is only one independent free parameter (a mass scale) in the theory corresponding to the fact that to begin with there was only one free parameter in the Lagrangian, namely, the bare coupling constant g_0.

For practical applications, $\rho(R)$ may be approximated by the first few terms on the right-hand side of Eq. (2.14) for small R. If we define the nth-order approximation to $\rho(R)$ by

$$\rho^{(n)}(R)\equiv -R^2(1+\rho_1 R+\rho_2 R^2+\cdots+\rho_{n-1}R^{n-1})\ , \tag{2.19}$$

then the corresponding approximation $R^{(n)}$ to R satisfies, according to Eq. (2.15), the transcendental equation

$$b\ln\frac{Q}{\Lambda_R}=\frac{1}{R^{(n)}}-\rho_1\ln\left[1+\frac{1}{\rho_1 R^{(n)}}\right]+\int_0^{R^{(n)}}dx\left[\frac{1}{\rho^{(n)}(x)}+\frac{1}{x^2(1+\rho_1 x)}\right]. \tag{2.20}$$

An important point about Eq. (2.20) is that it is *not* a perturbation expansion for $R^{(n)}$ in the conventional sense since there is no coupling constant or expansion parameter present in it. The convergence of the successive approximations $R^{(2)},R^{(3)},\ldots$ to R is now controlled by the magnitude of R itself. One way of testing theoretical predictions in the present approach would be to fit values of $\Lambda_R,\Lambda_{\widetilde{R}},\ldots$ to data on different processes using approximate predictions and then compare the ratio $(\Lambda_{\widetilde{R}})_{\text{expt}}/(\Lambda_R)_{\text{expt}}$ so obtained with the exact theoretical prediction Eq. (2.18). There are many other ways of testing predictions in the present approach. One can also give a precise criterion for good convergence of successive approximations defined in Eq. (2.20). These important practical questions and several applications to QCD processes have been discussed in detail in Ref. 5.

Extension of the above discussion to the case of a physical quantity which depends on more than one external energy variable $Q,Q_1,Q_2,\ldots$ is straightforward. One simply takes the derivative with respect to Q in Eq. (2.2) holding the ratios $x_1=Q_1/Q$, $x_2=Q_2/Q,\ldots$ fixed. The rest of the arguments then go through without any change except that Λ_R and $\rho_2,\rho_3,\ldots$ may now depend on the dimensionless variables $x_1,x_2,\ldots$.

To summarize the discussion of this section, we have shown that in a renormalizable massless field theory with a single dimensionless coupling constant, *only* the derivative $\rho(R)$ of a physical quantity R with respect to an external energy scale is well defined and unambiguously calculable. That is, $\rho(R)$ can be obtained as a series in R with *finite and RS-independent* coefficients. Meaningful successive approximation to $\rho(R)$ can now be defined [see Eq. (2.14)] and the corresponding approximations to R obtained [see Eq. (2.20)]. It is this new procedure of obtaining approximate predictions for any physical quantity that we have called a new perturbative approach to renormalizable field theories.

III. GREEN'S FUNCTIONS

We now extend our approach, developed so far for physical quantities, to Green's functions. There is an additional complication here because Green's functions get explicitly renormalized, unlike physical quantities. That is to say, a renormalized Green's function G is obtained from the unrenormalized Green's function G_0 by multiplying the latter by an infinite constant Z_G, the renormalization constant. Thus

$$G=Z_G G_0\ . \tag{3.1}$$

Consequently, to be in a position to apply the discussion of Sec. II here also, we must first construct an object out of G (analogous to R) which does not get explicitly renormalized and is therefore independent of any RS. As we shall see below, this implies that the second derivative of G (with respect to an external momentum) is well defined and unambiguously calculable in perturbation theory. We illustrate this procedure for the gluon propagator in massless QCD.

For simplicity, we shall assume covariant quantization in the Landau gauge. A nonzero value of the gauge parameter introduces unnecessary complications which are not relevant for the following discussion. The full unrenormalized gluon propagator can then be written as

$$[D^{ab}_{\mu\nu}(p)]_0=d^{ab}_{\mu\nu}(p)/[1+\pi_0(p^2)]\ , \tag{3.2}$$

$$d^{ab}_{\mu\nu}(p)=-\frac{i\delta^{ab}}{p^2}\left[g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right] . \tag{3.3}$$

Here a,b are color indices and, as before, the subscript 0 denotes unrenormalized quantities. The dimensionless function $\pi_0(p^2)$ can be obtained from the proper two-point function for the gluon field. The renormalized gluon propagator is given by

$$D^{ab}_{\mu\nu}(p)=d^{ab}_{\mu\nu}(p)/[1+\pi(p^2)]\ , \tag{3.4}$$

where

$$[1+\pi(p^2)]^{-1}=Z_3[1+\pi_0(p^2)]^{-1} \tag{3.5}$$

and Z_3 is the gluon wave-function renormalization constant. Since Z_3 is independent of p^2, we obtain the following simple but important relation:

$$\begin{aligned}\tilde{R}_\pi &\equiv p^2\frac{d}{dp^2}\ln[1+\pi(p^2)]\\ &=p^2\frac{d}{dp^2}\ln[1+\pi_0(p^2)]\ .\end{aligned} \tag{3.6}$$

Obviously, $\tilde{R}_\pi$ does not get explicitly renormalized and therefore is the analog of the physical quantity R of Sec. II. It can be obtained directly from $\pi_0(p^2)$ and has a perturbation expansion in a_0 which is of the form of Eq. (2.1):

$$\tilde{R}_\pi\equiv cR_\pi\ , \tag{3.7a}$$

$$R_\pi=a_0+\pi_{10}a_0{}^2+\pi_{20}a_0{}^3+\pi_{30}a_0{}^4+\cdots\ . \tag{3.7b}$$

The finite constant c, which can be obtained from ordinary one-loop calculation of $\pi_0(p^2)$, is essentially the one-loop gluon-field anomalous dimension. For QCD, $c=(39-4N_f)/24$, where N_f is the number of quark flavors. Since R_π is a quantity of the same nature as R, the entire earlier discussion can be applied to it. Thus, we have

$$p^2\frac{dR_\pi}{dp^2}=\tfrac{1}{2}b^{(\pi)}\rho_\pi(R_\pi)\ , \tag{3.8a}$$

$$\rho_\pi(R_\pi)=-R_\pi{}^2(1+\rho_1^{(\pi)}R_\pi+\rho_2^{(\pi)}R_\pi{}^2+\cdots)\ . \tag{3.8b}$$

The factor $\frac{1}{2}$ on the right-hand side of Eq. (3.8a) is present because of the derivative with respect to $\ln p^2$ [cf. $\ln Q$ in Eq. (2.13)] on the left-hand side. The function $\rho_\pi(R_\pi)$ is well defined and computable as a series in R_π. Following the arguments of the previous section, one can show that $b^{(\pi)}=b$ and $\rho_1^{(\pi)}=\rho_1$; these two coefficients are universal. Further, it is clear that $b^{(\pi)},\rho_1^{(\pi)},\rho_2^{(\pi)},\ldots$ will satisfy relations identical to Eqs. (2.10) and (2.5) (with r_{n0} replaced by π_{n0}). Approximations to $\rho_\pi(R_\pi)$ can therefore be made to obtain approximate predictions for the gluon propagator.

As an illustration we shall compute the gluon propagator in the second-order approximation: $\rho_\pi\simeq-R_\pi{}^2\times(1+\rho_1R_\pi)$. Equation (3.8a) is then easily integrated to give

$$R_\pi(p^2)=1/\rho_1[t+H(t)]\ , \tag{3.9a}$$

where the function $H(t)$ satisfies the transcendental equation

$$e^{H(t)}-H(t)-1=t \tag{3.9b}$$

and

$$t\equiv\frac{b}{2\rho_1}\ln(-p^2/\Lambda_\pi{}^2),\ \ p^2<0\ . \tag{3.9c}$$

The constant of integration appears as Λ_π and provides the appropriate mass scale for the gluon propagator. It can be related to Λ_R by a relation similar to Eq. (2.19). Equation (3.9a) can be integrated once more to obtain $[1+\pi(p^2)]$. The result is

$$\begin{aligned}[1+\pi(p^2)]&=[1+\pi(p_0{}^2)]\exp\left[\frac{2c}{b}[H(t)-H(t_0)]\right]\\ &\equiv N\exp\left[\frac{2c}{b}H(t)\right]\ ,\end{aligned} \tag{3.10}$$

where t_0 is defined as in Eq. (3.9c) with p^2 replaced by $p_0{}^2$. The constant of integration now appears as the finite normalization constant N in Eq. (3.10). It is clearly not calculable in the theory. As defined in this equation, it is, however, free from any RS ambiguities associated with the conventional renormalization procedure. N may be fixed, as usual, by imposing a normalization condition on the gluon propagator. To this approximation the gluon propagator is then given by

$$D^{ab}_{\mu\nu}(p)=d^{ab}_{\mu\nu}(p)\left[N\exp\left[\frac{2c}{b}H(t)\right]\right]^{-1} . \tag{3.11}$$

The example given above illustrates our general approach for obtaining perturbative approximations to any Green's function. One first constructs a quantity like R_π which does not get renormalized. The corresponding $\rho_\pi(R_\pi)$ is then obtained as a power series in R_π. Approximations to it are well defined and free from RS ambiguities. The corresponding approximation to the Green's function can then be obtained by integrating back. The main point which we have tried to emphasize, in this and the previous section, is that only quantities like $\rho(R)$ and $\rho_\pi(R_\pi)$ are well defined and unambiguously calculable in perturbation theory; they can be obtained directly from the corresponding regularized unrenormalized perturbation expansions. We thus completely bypass the conventional renormalization procedure and the RS ambiguities associated with it.

IV. SUMMARY AND CONCLUDING REMARKS

Observing that in quantum field theory finite-order perturbative predictions for any quantity of interest $\mathscr{R}$ (cross sections, Green's functions, etc.) are not well defined in the bare theory and ambiguous in the renormalized theory, we were led to suggest that $\mathscr{R}$ is not directly calculable in the theory. This then naturally raised the question as to how and in what form does the theory determine $\mathscr{R}$. As we have seen, the answer to this question lies in a new perturbative approach to renormalizable field theories which, right from the start, incorporates the consequences of dimensional transmutation for the predictions of the theory. We have shown how this approach works for a renormalizable massless field theory with only one coupling constant. In this case, we have shown that the theory at best specifies uniquely (i) the first derivative of a physical quantity with respect to some external energy scale and (ii) the second derivative of a Green's function with respect to some external momentum.

From the practical point of view, a new kind of "perturbation theory" has emerged from the present work. Since functions like $F(R)$ can be obtained as a series in R, systematic order-by-order (in R) approximations to $F(R)$ can be made and the equation $R'=F(R)$ integrated to get the corresponding approximate predictions for R. These predictions are clearly free from any RS ambiguities associated with the conventional renormalization procedure. An important point about these predictions is that the coupling constant (bare or renormalized) is completely eliminated from them. Convergence of perturbative approximations for R is now controlled by the magnitude of R itself.[5]

An important problem, not attempted here, is the extension of the present approach to theories with masses. The main difficulty here is that perturbation theory is not analytic in masses, so we cannot expand around zero mass. A possible approach, at least for nonconfining theories like QED, would be to define the physical mass of a particle as the pole in the corresponding propagator and then eliminate the bare mass from all quantities of interest in favor of the physical mass and the bare coupling constant. The resulting problem can then perhaps be tackled using the methods presented in this paper. The next step would be to extend the present approach to massive theories with many coupling constants. This is important for applications to electroweak and grand unified theories. Work in this direction is in progress.

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APPENDIX

For the sake of completeness we first briefly outline the method for obtaining the RS invariants in terms of the perturbation series coefficients and the β-function coefficients.[5] After this we prove Eq. (2.10).

The couplant a in Eq. (2.7) satisfies the β-function equation

$$\mu\frac{\partial a}{\partial\mu}=\beta(a)=-ba^2(1+c_1a+c_2a^2+\cdots)\,, \qquad \text{(A1)}$$

where μ denotes the renormalization point. In a different RS characterized by the couplant $\bar{a}$, where $\bar{a}$ and a are related by the equation

$$a=\bar{a}+v_1\bar{a}^2+v_2\bar{a}^3+\cdots\,, \qquad \text{(A2)}$$

the physical quantity R has the expansion

$$R=\bar{a}+\bar{r}_1\bar{a}^2+\bar{r}_2\bar{a}^3+\cdots\,. \qquad \text{(A3)}$$

The couplant $\bar{a}$ satisfies the β-function equation

$$\mu\frac{\partial\bar{a}}{\partial\mu}=\bar{\beta}(\bar{a})=-b\bar{a}^2(1+\bar{c}_1\bar{a}+\bar{c}_2\bar{a}^2+\cdots)\,. \qquad \text{(A4)}$$

Moreover, $\bar{\beta}(\bar{a})$ and $\beta(a)$ are related by the equation

$$\beta(a)=\frac{\partial a}{\partial\bar{a}}\bar{\beta}(\bar{a})\,. \qquad \text{(A5)}$$

The prescription for obtaining the invariants is simply this. Equations (2.7), (A2), and (A3) give $\bar{r}_i$ in terms of r_i and v_i; and Eqs. (A1) and (A2) and (A4) and (A5) give $\bar{c}_i$ in terms of c_i and v_i. To any given order n these constitute two sets of n equations from which $v_1,v_2,\ldots,v_n$ can be eliminated. In each of the resulting n relations there are no terms which mix the barred and unbarred quantities. The separation occurs in such a way that a polynomial function of the former is equal to the same function of the latter and is therefore RS invariant. We now list some of the invariants obtained in this way:

$$\rho_1=c_1\,, \qquad \text{(A6)}$$

$$\rho_2=c_2+r_2-r_1\rho_1-r_1^2\,, \qquad \text{(A7)}$$

$$\rho_3=c_3+2r_3-4r_2r_1-2r_1\rho_2-r_1^2\rho_1+2r_1^3\,, \qquad \text{(A8)}$$

etc. There is another invariant which can be obtained using the well-known relation[6] between QCD scale parameters Λ and $\bar{\Lambda}$ in two different renormalization schemes: $b\ln(\Lambda/\bar{\Lambda})=v_1$. Also, we have $\bar{r}_1-r_1=v_1$. Therefore we find that

$$\rho_0=b\ln\frac{\mu}{\Lambda}-r_1 \qquad \text{(A9)}$$

is also a scheme invariant.

The proof of Eq. (2.10) now proceeds as follows. Since R is dimensionless and depends on only one energy scale Q, the r_i must be functions of the dimensionless variable μ/Q only. Now since $\rho_1,\rho_2,\ldots$ are RS invariant, it follows that they must be independent of μ and hence also of Q. Thus $\rho_1,\rho_2,\ldots$ are constants independent of Q. Moreover, $b,c_1,c_2,\ldots$ are also constants independent of Q. Therefore, differentiating Eqs. (A6)—(A8), we get

$$r_1'\rho_1=r_2'-2r_1'r_1\,, \qquad \text{(A10)}$$

$$r_1'\rho_2=r_3'-2r_2r_1'-3r_1r_2'+5r_1^2r_1'\,, \qquad \text{(A11)}$$

etc. Further, Eq. (A9) implies that $r_1 = b\ln\mu/Q + \text{const}$. Therefore, we have

$$r_1' = -b\ . \tag{A12}$$

For QCD, $b = (33 - 2N_f)/6$ and

$$\rho_1 = (153 - 18N_f)/2(33 - 2N_f)\ ,$$

where N_f is the number of quark flavors.

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[2]A. Dhar, Phys. Lett. 128B, 407 (1983).

[3]P. M. Stevenson, Phys. Rev. D 23, 2916 (1981).

[4]This equation was obtained earlier by G. Grunberg [Phys. Lett. 95B, 70 (1980)] using the so-called fastest-apparent-convergence (FAC) scheme. However, our motivation and the underlying philosophy of our approach, as discussed in the Introduction, is very different from his. Our derivation of this equation, starting directly from the unrenormalized expansion for R, makes no reference to any scheme whatsoever. See also footnote 3 in Ref. 5.

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