Charged particles with electromagnetic interactions and U(1)-gauge theory: Hamiltonian and Lagrangian formalisms

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The motion of charged particles in external electromagnetic fields is reviewed with the purpose of determining the *whole* set of constants of motion. The Johnson-Lippmann results concerning the interaction with a constant magnetic field are taken as the starting point of the study. Our results are obtained through simple group-theoretical arguments based essentially on extended Lie algebras associated with the kinematical group of the (constant) electromagnetic field involved in the interaction. Nonrelativistic Schrödinger (or Pauli) and relativistic Dirac Hamiltonians are considered. The corresponding Lagrangian densities are then studied when the charged particles move in arbitrary electromagnetic fields. Through Noether's theorem, we get the constants of motion when coordinate *and* gauge transformations are combined. These results complete the U(1)-gauge theory and relate the works of Bacry, Combe, and Richard and of Jackiw and Manton when external gauge fields are considered. These developments enhance the minimal-coupling principle, the U(1)-gauge theory, and Noether's theorem.

I. INTRODUCTION

Nonrelativistic and (or) relativistic charged particles moving in classical external electromagnetic fields constitute the simplest quantum systems with interaction. These systems are treated using the Maxwell theory for the electromagnetic fields together with the Schrödinger equation (or the Pauli one) in the nonrelativistic case, or the Dirac (for spin-$\frac{1}{2}$ particles) equations in the relativistic case. Consequences of these treatments are the well-known principle of minimal electromagnetic coupling[1] and the first characteristics of the U(1)-gauge theory.[1] In the specialized literature, one of the most interesting papers is that of Johnson and Lippman (JL),[2] published 35 years ago. It deals with the motion of a charged particle in a constant magnetic field $\vec{B}$, treated in both nonrelativistic and relativistic quantum theories.

In view of the recent impact in particle physics of gauge theories[3] [among these, the U(1)-gauge theory has played a pioneer and prominent role], the numerous group-theoretical developments[4–10] applied to minimal-coupling schemes, and finally the current interest[11,12] in symmetries and associated conserved physical quantities (or constants of motion) through basic tools like the Noether theorem,[13] we plan to reconsider here the JL contribution. Among other things, our purpose is to generalize JL's results to the case of motion in a constant electromagnetic field. More precisely, we want to extend the *Hamiltonian approach* used extensively by JL and to consider a *Lagrangian approach* to invariances and conserved quantities based on Noether's theorem and recent contributions.[12] These approaches lead in complementary ways to the conservation laws which occur in the U(1)-gauge theory, and may be generalized to *arbitrary* gauge theories.[14]

Reconsideration of the JL paper[2] is also motivated by the results of Bargmann[15] and Levy-Leblond[16] on group extensions and invariance principles, respectively. More precisely, if classical electromagnetic interactions enter, we effectively deal with symmetry groups of potentials[4,10] and we speak about the extension by $\mathbb{R}$ of the kinematical group[17,18] G_F of the associated electromagnetic field F.

Let us summarize the contents of this paper. In Sec. II, we briefly survey the JL results in the Schrödinger and Dirac theories in order to establish notation and to introduce such studies within group theory. In Sec. III, we show how to complete the JL results by taking the cases of *free* Schrödinger and Dirac theories (Sec. III A) and by extending our considerations to the interacting case (Sec. III B), i.e., for charged particles interacting with a constant electromagnetic field $F\equiv(\vec{E},\vec{B})$. We also present some remarks on the nonrelativistic limit of our Dirac results and on the derivation of wave equations with classical interactions using Hoogland's construction.[8] Then, we return to the JL restricted interaction ($\vec{E}=0$) and we make some general comments (Sec. III C). In Sec. IV, we point out some peculiarities of the Jackiw-Manton[12] and Bacry-Combe-Richard[17] approaches and their applications of Noether's theorem in the framework of U(1)-gauge theory. We recall the more general form of Noether's theorem and exploit (Sec. IV A) the U(1)-gauge theory when the Dirac field is coupled to an *arbitrary* electromagnetic field and when space-time coordinate transformations are combined with *local* gauge transformations. Then, we emphasize the connection of the preceding results with the elements of the extended Lie algebra of the kinematical group of a constant electromagnetic field (Sec. IV B) which leads to a clear relation between the present results and those of Sec. III in the relativistic spin-$\frac{1}{2}$ case, for example. Finally, in Sec. V some more comments and conclusions are presented.

Our notation is conventional and we have chosen to use natural units ($\hbar=1, c=1$) in both relativisic and nonrelativistic contexts. We refer to Minkowskian space [$G_M=\mathrm{diag}(1,-1,-1,-1)$] with space-time events $x\equiv\{x^\mu\}\equiv\{x^0,x^i\}\equiv(t,\vec{r})$. Greek indices run from 0 to 3 and Latin ones from 1 to 3. The summation convention on repeated indices is used. The parameters m and e stand for the mass and the charge of a particle. Moreover, we shall make use of a few notions from differential geometry,[19] such as Lie derivatives for studying infinitesimal motions. These are discussed for example, in the papers by Forgacs and Manton[11] and by Jackiw and Manton.[12]

II. A SHORT SURVEY OF THE JL CONSTANTS OF MOTION

The Schrödinger equation for a nonrelativistic particle interacting with a *constant* magnetic field $\vec{B}$ along the third axis [$\vec{B}=(0,0,B)$] is given by

$$i\frac{\partial\psi}{\partial t}=i\partial_t\psi=H_S\psi \tag{2.1}$$

with the Hamiltonian

$$H_S=\frac{1}{2m}\vec{\Pi}\cdot\vec{\Pi}\ , \tag{2.2}$$

where

$$\vec{\Pi}=\vec{p}+e\vec{A}=-i\vec{\nabla}+e\vec{A}\ . \tag{2.3}$$

The vector potential $\vec{A}$ associated with the field $\vec{B}$ ($=\vec{\nabla}\times\vec{A}$) is chosen in the form of the gauge symmetrical potential

$$\vec{A}=\tfrac{1}{2}\vec{B}\times\vec{r}=\tfrac{1}{2}(-By,Bx,0)\ . \tag{2.4}$$

The JL constants of motion are[2]

$$\begin{aligned}\pi^1&=\Pi^1+eBy=p^1-eA^1\ ,\\ \pi^2&=\Pi^2-eBx=p^2-eA^2\ ,\\ \pi^3&=\Pi^3=p^3\ ,\end{aligned} \tag{2.5}$$

and

$$L^3=(\vec{r}\times\vec{p}\,)^3=xp^2-yp^1\ . \tag{2.6}$$

These quantities are not explicitly time dependent: they commute with the Hamiltonian (2.2). Among themselves, they obey the following commutation relations:

$$\begin{aligned}&[\pi^1,\pi^2]=ieB,\ \ [\pi^1,\pi^3]=0,\ \ [\pi^2,\pi^3]=0\ ,\\ &[L^3,\pi^1]=i\pi^2,\ \ [L^3,\pi^2]=-i\pi^1,\ \ [L^3,\pi^3]=0\ .\end{aligned} \tag{2.7}$$

For a relativistic spin-$\frac{1}{2}$ particle interacting with the same field $\vec{B}$, the motion is described by the Dirac equation

$$i\partial_t\psi=H_D\psi \tag{2.8}$$

with the Hamiltonian

$$H_D=\vec{\alpha}\cdot\vec{\Pi}+\beta m\ , \tag{2.9}$$

where $\vec{\alpha}$ and β are the well-known Dirac matrices.

Here, the constants of the motion are still given by the quantities (2.5), but the third component of the orbital angular momentum L^3 is replaced by the third component of the total angular momentum

$$J^3=L^3+\Sigma^3=(\vec{r}\times\vec{p}\,)^3-\frac{i}{2}\alpha^1\alpha^2\ . \tag{2.10}$$

J^3 commutes with the Hamiltonian (2.9) and satisfies the same commutation relations as L^3 in (2.7).

Now, the constants of the motion (2.5) and (2.6) in the Schrödinger context or (2.5) and (2.10) in the Dirac one, taken together with the commutation relations (2.7), can be identified with the generators of a Lie algebra. This algebra is a *subalgebra* of a larger one associated with the "physical kinematical group"[17,18] of the field $\vec{B}$. Our aim is to show that *all* the generators of such physical kinematical algebras are interpreted as constants of motion in this Hamiltonian formalism. From this point of view, there are some missing constants of motion in the JL approach.

III. CONSTANTS OF MOTION AND HAMILTONIAN FORMALISM

Within the Hamiltonian formalism, an arbitrary operator C is a constant of motion if the following condition holds:[1,20]

$$\dot{C}=\frac{dC}{dt}=\frac{\partial C}{\partial t}+i[H,C]=0\ . \tag{3.1}$$

We shall consider the cases of free (Sec. III A) and interacting (Sec. III B) particles, and shall also return to the JL situation and make some general comments (Sec. III C).

A. Free particles

The free nonrelativistic Hamiltonian for a particle of mass m is given by

$$H_S^0=\frac{1}{2m}\vec{p}^{\,2}\ . \tag{3.2}$$

The constants of motion are known to be the momentum $\vec{p}$, the angular momentum $\vec{L}=\vec{r}\times\vec{p}$, and the operators

$$\vec{K}=t\vec{p}-m\vec{r}\ . \tag{3.3}$$

Let us point out that $\vec{p}$ and $\vec{L}$ are time independent, so that they do commute with the Hamiltonian (3.2) while $\vec{K}$ does not.

These well-known results[16] follow from the fact that the *extended* Galilean symmetry group is the invariance group of the free Schrödinger equation. We notice that the operators $P^0=H_S^0$, $\vec{P}=\vec{p}$, $\vec{J}=\vec{L}$, and $\vec{K}$ satisfy the usual commutation relations of the Galilean Lie algebra apart from the commutators between the K^i and P^j:

$$[K^i,P^j]=-im\delta^{ij}\ , \tag{3.4}$$

which characterize *nontrivial* extensions[15] by R of the

Galilean algebra. There is, in fact, a one-dimensional vector space of extensions characterized by the mass m. The operators P^0 and $\vec{P}$ are the well-known time and space translation generators, the J^i ($i=1,2,3$) are the generators of spatial rotations, and the K^i ($i=1,2,3$) are associated with pure Galilean transformations inside the extension by R of the Galilean group. Let us recall that the existence of such an extension is intimately connected with the existence of *projective*[15] irreducible unitary representations of the Galilean group.

The free relativistic (Dirac) Hamiltonian for a spin-$\frac{1}{2}$ particle is given by

$$H_D^0=\vec{\alpha}\cdot\vec{p}+\beta m \ . \tag{3.5}$$

The constants of motion are the momentum $\vec{p}$, the total angular momentum

$$\vec{J}=\vec{r}\times\vec{p}+\vec{\Sigma}=\vec{L}+\vec{\Sigma} \ , \tag{3.6}$$

where

$$\Sigma=(\Sigma^1,\Sigma^2,\Sigma^3)=-\frac{i}{2}(\alpha^2\alpha^3,\alpha^3\alpha^1,\alpha^1\alpha^2) \tag{3.7}$$

are the so-called spin matrices of the Dirac algebra, and the quantities

$$\vec{K}=t\vec{p}-\vec{r}H_D^0+i\frac{\vec{\alpha}}{2} \ . \tag{3.8}$$

The operators $P^0=H_D^0$, $\vec{P}=\vec{p}$, $\vec{J}=\vec{L}+\vec{\Sigma}\equiv(3.6)$, and $\vec{K}\equiv(3.8)$ are the generators of the Poincaré Lie algebra,[21] in a specific realization taking into account the presence of the spin operators. This corresponds to the invariance of the free Dirac equations under the Poincaré group.

Here, we have to notice that the Poincaré group admits only *true* irreducible unitary representations, so that we do not have to refer to an extended symmetry group. We speak about *trivial* extensions of the Poincaré Lie algebra.

B. Interacting particles

As already mentioned in the Introduction, by "interacting particles" we mean that the charged particles interact with an external *constant* electromagnetic field $F\equiv(\vec{E},\vec{B})$. Let us immediately fix F as[17] a "parallel" F ($F_{\parallel}$) along the third axis:

$$F_{\parallel}\equiv\{\vec{E}=(0,0,E),\vec{B}=(0,0,B\} \ , \tag{3.9}$$

E and B being arbitrary constants. $F_{\parallel}$ derives from the so-called gauge-symmetrical potentials explicitly given by

$$V=-\tfrac{1}{2}Ez, \ \ \vec{A}=\tfrac{1}{2}(-By,Bx,-Et) \ , \tag{3.10}$$

or a gauge transformation of these. The vector potential (2.4) is a particular case of Eq. (3.10) if the electric field is zero.

Let us determine the constants of motion for charged particles interacting with $F_{\parallel}$. The Schrödinger Hamiltonian is

$$H_S=\frac{1}{2m}\vec{\Pi}^{\,2}-eV, \ \ \vec{\Pi}\equiv(2.3). \tag{3.11}$$

The constants of motion are not difficult to find if we notice that, with Eqs. (3.10) and (3.11), we have

$$\dot{\vec{p}}=-\frac{e}{2}\left[\frac{B}{m}\Pi^2,-\frac{B}{m}\Pi^1,E\right], \ \ \dot{\vec{r}}=\frac{1}{m}\vec{\Pi} \tag{3.12}$$

and

$$\dot{H}_S=\partial_t H_S=-\frac{eE}{2m}\Pi^3 \ . \tag{3.13}$$

Then, the constants of motion are

$$\left.\begin{aligned}\pi^1&=p^1-eA^1=\Pi^1+eBy \ ,\\ \pi^2&=p^2-eA^2=\Pi^2-eBx \ ,\end{aligned}\right. \tag{3.14a}$$

$$\pi^3=p^3-eA^3=\Pi^3+eEt \ , \tag{3.14b}$$

$$\pi^0=H_S-eV \ , \tag{3.14c}$$

$$J^3=xp^2-yp^1=L^3, \ \ K^3=tp^3-mz \ . \tag{3.14d}$$

The only nonzero commutators between these operators are the following ones:

$$\begin{aligned}&[\pi^1,\pi^2]=ieB, \ \ [\pi^0,\pi^3]=ieE \ ,\\ &[J^3,\pi^1]=i\pi^2, \ \ [J^3,\pi^2]=-i\pi^1 \ ,\\ &[K^3,\pi^0]=-i\pi^3, \ \ [K^3,\pi^3]=-im \ .\end{aligned} \tag{3.15}$$

Let us now make some comments. We get only *six* constants of motion in this interacting case while there were ten in the free case. Moreover, in comparison with the free case, we recover among them only two unchanged conserved quantities (i.e., J^3 and K^3), and we notice that the other four modified quantities explicitly depend on the interaction through the electromagnetic potentials (3.10). These properties can easily be explained by group-theoretical arguments. We see that in the explicit realization (3.14), intimately connected with the choice (3.10), the only unchanged operators are J^3 and K^3, because these generate the two-dimensional Lie algebra associated with the symmetry group of the chosen potentials V and $\vec{A}$.[9,10] Moreover, the algebra (3.15) corresponds to an *extended* Lie algebra associated with an extension by R of the nonrelativistic kinematical group[18] $G_{F_{\parallel}}$ if we recall that, for $F_{\parallel}\equiv(3.9)$, we deal with[10,18] the algebra

$$\mathsf{G}_{F_{\parallel}}\equiv\{P^0,\vec{P},J^3,K^3\} \ . \tag{3.16}$$

This algebra belongs to a two-dimensional vector space of extensions characterized by the mass m and the charge e.

As a final remark in the nonrelativistic context, let us relate these considerations with a study by Hoogland[8] of minimal electromagnetic coupling in wave equations. The Hamiltonian (3.11) can directly be written in terms of the quantities (3.14) as

$$H_S=\frac{1}{2m}(\vec{\pi}\cdot\vec{\pi}+2eBJ^3-2eEK^3)+eV \ . \tag{3.17}$$

Then the Schrödinger equation in terms of the constants of motion is

$$\pi^0\psi=\frac{1}{2m}(\vec{\pi}^2+2eBJ^3-2eEK^3)\psi \ . \tag{3.18}$$

Such an equation has been obtained by Hoogland from the derivation of the Casimir operators associated with the extension of $G_{F_{\|}}$ by R. We refer to the original paper[8] for specific details on this interesting approach.

The Dirac Hamiltonian becomes in the interacting case

$$H_D=\vec{\alpha}\cdot\vec{\Pi}+\beta m-eV \tag{3.19}$$

and the constants of motion are determined as

$$\pi^1=p^1-eA^1,\quad \pi^2=p^2-eA^2 \ , \tag{3.20a}$$

$$\pi^3=p^3-eA^3,\quad \pi^0=H_D-eV \ , \tag{3.20b}$$

$$J^3=xp^2-yp^1+\Sigma^3=L^3+\Sigma^3 \ , \tag{3.20c}$$

and

$$K^3=tp^3-zH_D+\frac{i}{2}\alpha^3 \ . \tag{3.20d}$$

These six operators lead to the nonzero commutators

$$\begin{aligned}&[\pi^1,\pi^2]=ieB,\quad [\pi^0,\pi^3]=ieE \ ,\\ &[J^3,\pi^1]=i\pi^2,\quad [J^3,\pi^2]=-i\pi^1 \ ,\\ &[K^3,\pi^0]=-i\pi^3,\quad [K^3,\pi^3]=-i\pi^0 \ ,\end{aligned} \tag{3.21}$$

giving once again an extended algebra of $\mathsf{G}_{F_{\|}}\equiv(3.16)$ and corresponding to a one-dimensional vector space of extensions characterized by only the charge of the particles.

Now, let us point out that, with the Hamiltonian operator (3.19), we can get a second-order equation which is explicitly written in terms of the constants of motion,

$$(\pi^0)^2\psi=(\vec{\pi}^2+m^2+2eBJ^3-2eEK^3)\psi \ , \tag{3.22}$$

where we have used the property

$$(H_D+eV)^2=(\vec{\alpha}\cdot\vec{\Pi}+\beta m)^2=\vec{\Pi}^2+m^2+2eB\Sigma^3 \ . \tag{3.23}$$

Equation (3.22) takes also the equivalent form

$$\left[\Pi^\mu\Pi_\mu-m^2+2e\left[\frac{iE}{2}\alpha^3-B\Sigma^3\right]\right]\psi=0 \ , \tag{3.24}$$

where

$$2e\left[\frac{iE}{2}\alpha^3-B\Sigma^3\right]=-\frac{ie}{2}\gamma_\mu\gamma_\nu F^{\mu\nu}$$

with $F\equiv\{F^{\mu\nu}\}$ given by (3.9). Such considerations are typical of the Feynman—Gell-Mann[22] developments leading to their well-known equation

$$[\Pi^\mu\Pi_\mu-m^2-e\vec{\sigma}\cdot(\vec{B}+i\vec{E})]\phi=0 \tag{3.25}$$

in terms of a two-component wave function.

Equation (3.22) is a new result in the framework of Hoogland's approach[8] applied here to spin-$\frac{1}{2}$ particles interacting with a constant $F_{\|}$.

As a last comment, let us recall the nonrelativistic limit of the Dirac theory. The nonrelativistic interacting or Pauli Hamiltonian obtained from (3.19) reads

$$H_P=\frac{1}{2m}\vec{\Pi}\cdot\vec{\Pi}+\frac{eB}{2m}\sigma^3-eV \ , \tag{3.26}$$

where we recognize the usual spin term leading to the magnetic moment, the Landé factor, etc. This leads to a specific realization of the (six) conserved quantities

$$\pi^1=p^1-eA^1,\quad \pi^2=p^2-eA^2,\quad \pi^3=p^3-eA^3 \ , \tag{3.27a}$$

$$\pi^0=H_P-eV \ , \tag{3.27b}$$

$$J^3=xp^2-yp^1+\tfrac{1}{2}\sigma^3,\quad K^3=tp^3-mz \ . \tag{3.27c}$$

Such a realization can be obtained from Eqs. (3.19) and (3.20) through usual considerations such as, e.g., the Foldy-Wouthysen[23] transformation. Remember that the $\vec{\Sigma}$ and $\vec{\alpha}$ matrices are, respectively, even and odd matrices in the standard Dirac representation, so that results [(3.20c), (3.20d)]$\rightarrow$(3.27c) are obvious when we restrict ourselves to the large components.

C. The JL case and some general comments

If the electric field $\vec{E}$ in Eqs. (3.9) is zero, the formulas of Sec. III B reduce to those of Sec. II. Compared to the JL results, we get *all* the constants of motion, and, in particular, π^0 and K^3, which have not been previously mentioned. Another result consists in noticing that the electromagnetic field $F_{\|}\equiv(3.9)$ and the magnetic field $\vec{B}\equiv(0,0,B)$ admit the *same* kinematical group *but different* extended Lie algebras leading to different sets of constants of motion. Moreover, for nonrelativistic particles, we point out that the algebra (2.7) is a subalgebra of (3.15) and we notice the physical interest of the supplementary constant of motion K^3 in connection with the mass of the particles.

Let us now return to the interaction with the constant field $F_{\|}\equiv(3.9)$ in order to note that the constants of motion (3.14) and (3.20) do depend on the choice (3.10) for V and $\vec{A}$. But we know that the potentials leading to such an $F_{\|}$ field fall into equivalence classes through gauge transformations, so that we can reconsider our problem with other explicit forms of V and $\vec{A}$ (leading to the *same* $F_{\|}$). Recalling our recent results,[10] let us point out that the dimension of the symmetry group of (3.10) is 2 and that this is not the *maximal* dimension in the relativistic context. We found[10] $n_{\|}=3$. So, as already mentioned in Sec. III B, the number of unchanged (with respect to the free case) constants of motion is equal to 2 with the potentials (3.10), but can be equal to 3 *if* we choose a potential with maximal symmetry as follows:

$$\begin{aligned}&V=\tfrac{1}{2}E(t-z) \ ,\\ &\vec{A}=\tfrac{1}{2}(-By,Bx,-E(t-z)) \ .\end{aligned} \tag{3.28}$$

In fact, if the potentials (3.10) admit the symmetry $\{J^3,K^3\}$, the ones given by (3.28) admit $\{J^3,K^3,P^0-P^3\}$. The corresponding constants of motion are given (in the Dirac case) by (3.20) in connection with (3.10) while they become, with (3.28),

$$\pi^0 = H_D + \frac{eE}{2}(t+z) ,$$
$$\pi^3 = p^3 + \frac{eE}{2}(t+z) , \tag{3.29}$$

and (3.20a), (3.20c), and (3.20d) leading to unchanged constants $\pi^0 - \pi^3$, J^3, and K^3. Let us also notice that, if the interaction is purely magnetic, the potential (2.2) admits a four-dimensional symmetry leading to π^0, π^3, J^3, and K^3 as unchanged constants of motion.

Other types of electromagnetic interactions can evidently be considered. In particular, for *constant* electromagnetic fields in the relativistic case, there exists besides $F_{\parallel}$'s also $F_{\perp}$'s. We know that the kinematical group $G_{F_\perp}$ of $F_\perp$ is not isomorphic to $G_{F_{\parallel}}$, but is of the same dimension. Similar considerations to those of Sec. III B can then be developed with a fixed $F_\perp$ chosen as

$$F_\perp \equiv \{\vec{E} = (E,0,0), \vec{B} = (0,E,0)\} , \tag{3.30}$$

which derives from gauge-symmetrical potentials given by

$$V = -\tfrac{1}{2}Ex, \quad \vec{A} = \tfrac{1}{2}E(z-t,0,-x) . \tag{3.31}$$

The associated kinematical algebra[17] is

$$G_{F_\perp} \equiv \{P^0, \vec{P}, J^1 + K^2, J^2 - K^1\} . \tag{3.32}$$

Once again, there are *six* constants of motion in the description of spinor particles which are also associated with the generators of an extension by R of $G_{F_\perp}$. They can be easily realized and we leave their determination as an exercise for the reader. In the *nonrelativistic* context, Bacry, Combe, and Richard[18] have shown that such an orthogonal $F_\perp$ leads to a meaningful situation when it reduces to the magnetic field $\vec{B}$ alone. In this case, we also recover the JL case.[2] As a last remark, these nonrelativistic considerations are restricted to the magnetic limit of Galilean electromagnetism as discussed by Le Bellac and Levy-Leblond.[24]

If *arbitrary* electromagnetic fields are considered in either the relativistic or nonrelativistic cases, the determination of the constants of motion can be achieved by finding the extended Lie algebra associated with the symmetry group of such a field. Knowledge of the potentials and of their symmetries yields information about the constants of motion which are unchanged with respect to the free case.

IV. CONSTANTS OF MOTION AND LAGRANGIAN FORMALISM

Noether's theorem[13] is one of the most powerful tools in modern physics. Recently, it has been used extensively in connection with arbitrary gauge theories,[3] as in the work of Jackiw and Manton[12] dealing with space-time and gauge-transformation properties of Lagrangian densities. Let us recall the Noether theorem in its *generalized* form as given by Bacry, Combe, and Richard;[17]

"If, under infinitesimal transformations on space-time events $x = \{x^\mu, \mu = 0,1,2,3\}$

$$x \to x' : \; x^\mu \to x'^\mu = x^\mu - \xi^\mu , \tag{4.1}$$

such that

$$\partial_\mu \xi^\mu(x) = \xi^\mu{}_{,\mu}(x) = 0 , \tag{4.2}$$

the wave functions $\psi_\alpha(x)$ $(\alpha = 1, \ldots, n)$ and their first derivatives $\psi_{\alpha,\mu}(x)$ transform according to

$$\psi'_\alpha(x') = \psi_\alpha(x) + \delta\psi_\alpha(x) ,$$
$$\psi'_{\alpha,\mu'}(x') = \psi_{\alpha,\mu}(x) + \delta\psi_{\alpha,\mu}(x) , \tag{4.3}$$

and if the Lagrangian density

$$\mathscr{L}(\psi_\alpha(x), \psi_{\alpha,\mu}(x), x) \equiv \mathscr{L} \tag{4.4}$$

transforms as

$$\mathscr{L}(\psi'_\alpha(x'), \psi'_{\alpha,\mu'}(x')x') = \mathscr{L} + \partial_\mu Z^\mu(\psi_\alpha(x), x) , \tag{4.5}$$

then there exists a conserved current $J \equiv \{J^\mu\}$ given by

$$J^\mu = \frac{\partial \mathscr{L}}{\partial \psi_{\alpha,\mu}(x)} \delta\psi_\alpha(x) + \left[\frac{\partial \mathscr{L}}{\partial \psi_{\alpha,\mu}(x)} \psi_{\alpha,\nu}(x) - \mathscr{L}\delta^\mu{}_\nu \right] \xi^\nu - Z^\mu \tag{4.6}$$

such that

$$\partial_\mu J^\mu = 0 , \tag{4.7}$$

which leads to the associated constant of motion

$$C = \int d\vec{r}\, J^0 ." \tag{4.8}$$

Here we want to apply this theorem to the coupling of Dirac and Maxwell theories. This coupling leads to the well-known Abelian U(1)-gauge theory as far as local gauge transformations on Dirac wave functions and on electromagnetic potentials are concerned. Gauge invariance in quantum electrodynamics then appears as a dynamical principle.[3]

If coordinate transformations are combined with local gauge transformations, the covariance of the theory and its gauge invariance have to be examined simultaneously and some current developments suggest that special care is necessary. Jackiw and Manton[12] have considered such a problem in the case of spin-$\frac{1}{2}$ particles coupled to arbitrary gauge fields. Here we apply their considerations in the U(1)-theory. First, we take the interacting case with an arbitrary electromagnetic field (Sec. IV A) and, later, we specialize to the case of the constant "parallel" $F \equiv (3.9)$ (Sec. IV B), in order to relate the Hamiltonian and Lagrangian approaches. In fact, we shall be particularly interested in external[12] gauge fields, i.e., when the potential A can only vary from point to point.

A. Dirac particles in an arbitrary electromagnetic field

Quantum electrodynamics is characterized by a Lagrangian density expressed in terms of the covariant derivatives $D_\mu = \partial_\mu - ieA_\mu$:

$$L_T = \frac{i}{2}(\overline{D_\mu \psi}\gamma^\mu \psi - \bar{\psi}\gamma^\mu D_\mu \psi) + m\bar{\psi}\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$= L_D + L_M , \tag{4.9}$$

where $\overline{\psi}$ is the adjoint wave function defined by $\overline{\psi}=\psi^\dagger\gamma^0$ ($\gamma^0\equiv\beta$).

Within the *coordinate* Poincaré transformations (4.1) i.e., with

$$\xi^\mu=-a^\mu-\omega^\mu{}_\nu x^\nu , \tag{4.10}$$

a^μ and $\omega^{\mu\nu}$ being the infinitesimal translational and Lorentz parameters, respectively, let us consider the transformations (4.3) with

$$\delta\psi(x)=\Omega\psi(x),\ \ \overline{\delta\psi}(x)=-\overline{\psi}(x)\Omega \tag{4.11}$$

and

$$\delta A_\mu(x)=\omega_\mu{}^\nu A_\nu(x)=(\partial_\mu\xi^\nu)A_\nu(x) , \tag{4.12}$$

where Ω stands for the homogeneous part:

$$\Omega=\tfrac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu . \tag{4.13}$$

The Lagrangian density (4.9) is strictly invariant with respect to these transformations (4.11) and (4.12). Then, from Noether's theorem (with $\mathscr{L}=L_T$, $Z^\mu\equiv 0$), we get, through Eq. (4.6), the total conserved current:[12]

$$J^\mu_T=J^\mu_{0,D}+J^\mu_M , \tag{4.14}$$

where

$$J^\mu_{0,D}=-\frac{i}{2}\overline{\psi}\gamma^\mu(\xi^\nu\partial_\nu\psi+\Omega\psi)+\text{H.c.} \tag{4.15}$$

and

$$J^\mu_M=T^\mu{}_\rho\xi^\rho+F^{\mu\nu}\partial_\nu(\xi^\rho A_\rho) . \tag{4.16}$$

Equation (4.15) is the conserved current associated with the *free* Dirac theory and Eq. (4.16) is the conserved current associated with the *free* Maxwell theory. Here let us recall that $T\equiv\{T^{\mu\nu}\}$ is the (symmetric) energy-momentum tensor defined[25] by

$$T^{\mu\nu}=\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g^{\mu\nu}+F^{\mu\sigma}F_\sigma{}^\nu .$$

Moreover, under *local gauge* transformations characterized by infinitesimal modifications,

$$\delta\psi=ie\alpha(x)\psi,\ \ \overline{\delta\psi}=-ie\alpha(x)\overline{\psi},\ \ \delta A_\mu=\partial_\mu\alpha(x) , \tag{4.17}$$

the Lagrangian density (4.9) is also strictly invariant.[3] From Noether's theorem, we get, from Eqs. (4.6) (with $\xi^\mu=0$, $Z^\mu\equiv 0$) and (4.17), the gauge-conserved current

$$J^\mu_{\text{gauge}}=-e\overline{\psi}\gamma^\mu\psi\alpha(x)+F^{\mu\nu}\partial_\nu\alpha(x) . \tag{4.18}$$

Under combined coordinate *and* gauge transformations, that is,

$$\psi'(x')=\psi(x)+\Omega\psi(x)+ie\alpha(x)\psi(x) , \tag{4.19}$$

$$\overline{\psi}'(x')=\overline{\psi}(x)-\overline{\psi}(x)\Omega-ie\alpha(x)\overline{\psi}(x) , \tag{4.20}$$

and

$$A'_\mu(x')=A_\mu(x)+(\partial_\mu\xi^\nu)A_\nu(x)+\partial_\mu\alpha(x) , \tag{4.21}$$

we get the conserved current

$$J^\mu=J^\mu_T+J^\mu_{\text{gauge}} . \tag{4.22}$$

Now, if the gauge field is external,[12] we must be able to combine coordinate and gauge transformations in such a way that

$$A'_\mu(x)=A_\mu(x) \tag{4.23}$$

in order to get a conservation law. This relation is equivalent to

$$L_\xi A_\mu(x)=-\partial_\mu\alpha(x) , \tag{4.24}$$

where the Lie derivative with respect to the vector fields ξ is defined by

$$L_\xi A_\mu(x)=\xi^\nu\partial_\nu A_\mu(x)+(\partial_\mu\xi^\nu)A_\nu(x) .$$

The condition (4.24) is possible only if we limit ourselves to *particular* Poincaré transformations characterized in Eq. (4.1) by the ξ^μ_F leaving invariant the field F ($L_\xi F=0$) and to the compensating gauge transformations.[4] Let us denote

$$-\alpha(x)=W_{\xi_F}(x)\equiv W_F(x) , \tag{4.25}$$

where W_F depends on ξ_F.

From the formulas (4.19)—(4.21) with $\xi=\xi_F$, $\Omega=\Omega_F$, and $\alpha(x)=-W_F(x)$, the Lagrangian density (4.9) transforms in such a way that

$$L'_T=L_T-\partial_\mu(\xi^\mu_F L_M) \tag{4.26}$$

and Noether's theorem applies. We immediately get the conserved current

$$J^\mu_F(x)=-\frac{i}{2}\overline{\psi}\gamma^\mu(\Delta_F-ieW_F)\psi+\text{H.c.} , \tag{4.27}$$

where

$$\Delta_F\psi=\xi_F{}^\nu\partial_\nu\psi+\Omega_F\psi . \tag{4.28}$$

The constant of motion then takes the form

$$C_F=-\frac{i}{2}\int d\vec{r}\,\psi^\dagger(\Delta_F-ieW_F)\psi+\text{H.c.} . \tag{4.29}$$

B. Dirac particles in a constant electromagnetic field

The determination of the constants $C_F\equiv(4.29)$ requires explicit expressions for Δ_F and W_F when the electromagnetic field F is given. Let us take the case of the constant $F_{||}\equiv(3.9)$ characterized by the following ξ^μ_F:

$$\begin{gathered}\xi^0_F=vz-a^0 ,\\ \xi^1_F=\theta y-a^1,\ \ \xi^2_F=-(\theta x+a^2),\ \ \xi^3_F=vt-a^3\end{gathered} \tag{4.30}$$

if θ and v refer to rotations and boosts with respect to the third axis. If we choose the (particular) gauge-symmetrical potential (3.10) associated with the field $F_{||}$, and evaluate W_F from

$$L_\xi A_\mu=\partial_\mu W_F , \tag{4.31}$$

we find immediately[10,26]

$$W_F(x)=\tfrac{1}{2}B(a^1y-a^2x)-\tfrac{1}{2}E(a^0z-a^3t) \tag{4.32}$$

up to an additive constant.

With all these results, we finally obtain

$$(\Delta_F - ieW_F)\psi(x) = i(a_\mu \pi^\mu - \theta J^3 + vK^3)\psi(x) , \quad (4.33)$$

where

$$\pi^\mu = p^\mu - eA^\mu ,$$
$$J^3 = xp^2 - yp^1 - \frac{i}{2}\alpha^1\alpha^2 , \quad (4.34)$$
$$K^3 = tp^3 - zp^0 + \frac{i}{2}\alpha^3 .$$

Then, from (4.29), (4.33), and (4.34), we get the six constants of motion

$$\langle \pi^\mu \rangle = \int d\vec{r}\, \psi^\dagger \pi^\mu \psi ,$$
$$\langle J^3 \rangle = \int d\vec{r}\, \psi^\dagger J^3 \psi, \quad \langle K^3 \rangle = \int d\vec{r}\, \psi^\dagger K^3 \psi , \quad (4.35)$$

in correspondence with the invariances under space-time translations, rotations, and boosts around and along the third axis, respectively, all these transformations being combined with the gauge transformation $\alpha(x) = -W_F(x)$ according to Eq. (4.25).

We thus recover the values (3.20) of the constants of motion obtained from the Hamiltonian formalism. All the comments about the explicit form of such constants have already been discussed in Sec. III. Let us only add that, in connection with the remaining free constants J^3 and K^3, it follows from (4.32) that $W_F = 0$ for the corresponding rotations and boosts. According to (4.31), this implies strict invariance of the potential A with respect to these transformations.

V. COMMENTS AND CONCLUSIONS

Let us briefly consider the nonrelativistic case in the Lagrangian approach. Free Schrödinger particles are described by the Lagrangian density

$$L_{0,S} = \frac{1}{2m} \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi - \frac{i}{2}[\phi^*(\partial_t \phi) - (\partial_t \phi^*)\phi] . \quad (5.1)$$

Under infinitesimal Galilean transformations,[16]

$$t' = t + b, \quad \vec{x}' = \vec{x} + \vec{\theta} \times \vec{x} + \vec{v}t + \vec{a} = \vec{x} - \vec{\xi} , \quad (5.2)$$

the density (5.1) is invariant when the field ϕ transforms according to

$$\phi'(x') = \phi(x) + if(x)\phi(x) \quad (5.3)$$

with

$$f(x) = m\vec{v} \cdot \vec{x} . \quad (5.4)$$

Then, Noether's theorem applies with

$$\Delta\phi = -b\, \partial_t \phi + \vec{\xi} \cdot \vec{\nabla}\phi + if(x)\phi$$

and leads to the constant of motion

$$C_0 = -i \int d\vec{r}\, \phi^* \Delta\phi + \text{c.c.} \quad (5.5)$$

In the interacting case with an electromagnetic field F, the Lagrangian density contains covariant derivatives instead of the usual ones. It reads

$$L_S = \frac{1}{2m}(\vec{\Pi}\phi)^* \cdot (\vec{\Pi}\phi) - \tfrac{1}{2}[\phi^*(\Pi_0\phi) - (\Pi_0\phi)^*\phi] , \quad (5.6)$$

while the electromagnetic Lagrangian density in this nonrelativistic context corresponds to the *magnetic limit* discussed by Le Bellac and Levy-Leblond,[24]

$$L_F = -\tfrac{1}{2}\vec{B}^2 . \quad (5.7)$$

The total Lagrangian density

$$L_T = L_S + L_F \quad (5.8)$$

does not lead to the whole set of equations of motion,[27] but can be considered in order to get the constants of motion. In fact, $L_S \equiv (5.6)$ is invariant under the transformations (5.3) *and* under gauge transformations

$$\phi'(x) = \phi(x) + ie\alpha(x)\phi(x) \quad (5.9)$$

when the potentials transform like[24]

$$V'(x') = V(x) - \vec{v} \cdot \vec{A}(x) ,$$
$$\vec{A}'(x') = \vec{A}(x) + \vec{\theta} \times \vec{A}(x) \quad (5.10)$$

under coordinate transformations corresponding to (5.3) *and* like

$$V'(x) = V(x) + \partial_t \alpha(x) ,$$
$$\vec{A}'(x) = \vec{A}(x) - \vec{\nabla}\alpha(x) , \quad (5.11)$$

under gauge transformations corresponding to (5.9). By noticing that L_F is evidently invariant under (5.10) and (5.11), we may apply Noether's theorem to the total Lagrangian density (5.8). So, through Galilean coordinate transformations (associated with symmetries of the F field) combined with gauge transformations characterized by $\alpha(x) = -W_F(x)$, such that the conditions corresponding to Eq. (4.31) of the relativistic context are satisfied, we get the constant of motion

$$C_F = -i \int d\vec{r}\, \phi^*(\Delta_F - ieW_F)\phi + \text{c.c.}, \quad (5.12)$$

which may be compared with the value (5.5).

In conclusion, we see that, in both relativistic and nonrelativistic descriptions, we recover the constants of motion as associated with the generators of an extension by R of the symmetry group of the field F. These results have been demonstrated within the Hamiltonian approach in Sec. III and within the Lagrangian approach in Sec. IV. They are obtained not only for *constant* electromagnetic fields (Secs. III B and IV B) but also for *arbitrary* electromagnetic fields (Sec. IV A). In each approach, the JL results[2] follow in a particular case.

In connection with *compensating*[4] gauge transformations, let us also note that W_F is zero for symmetries of the field F if and only if the potential A is itself invariant under these F symmetries. Such a case cannot occur for constant electromagnetic fields but is conceivable for other fields leading to constants of motion directly associated with the generators of the field-symmetry algebra.

Finally, let us emphasize the importance of the role played in our considerations by the electromagnetic potential, the gauge field of this U(1)-gauge theory, and its physical consequences. Our results may be extended to arbitrary gauge field theories and may be understood in both the Hamiltonian *and* Lagrangian formalisms.

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