Entropy from extra dimensions

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Heating may result from the collapse of the extra compact dimension in Kaluza-Klein cosmologies. This interesting phenomenon was pointed out in a recent letter by Alvarez and Gavela. Here we correct and clarify the physical interpretation of this effect.

Recently there has been interest in the cosmology of universes with more than four dimensions.[1] In particular it has been pointed out that these Kaluza-Klein[2] cosmologies can have an "inflationary" phase, and thus resolve, perhaps, well-known cosmological conundrums such as the horizon problem.[3] We wish to comment here on a letter by Alvarez and Gavela[4] in which the authors raise the interesting question of heating due to the collapse of the extra dimensions in these Kaluza-Klein cosmologies. Our main purpose is to make clear some of the physics which we believe is made obscure by the viewpoint of those authors. We will briefly sketch their line of approach and then our own.

The $(4+D)$-dimensional universe is assumed to be described by an interval of the form[5]

$$\begin{aligned} ds^2 &= g_{AB}d\xi^A d\xi^B \\ &= -dt^2 + R(t)^2 g_{kl}dx^k dx^l + B(t)^2 g_{ab}dy^a dy^b \ . \end{aligned} \tag{1}$$

The first two terms represent the familiar four-dimensional Robertson-Walker space-time with scale factor $R(t)$. The last term gives the interval in the compact, extra D-dimensional space. It is assumed that this additional "ball" can be represented by a time-independent metric for a homogeneous space $g_{ab}(y)$, and an overall, time-dependent scale factor $B(t)$. Although at very early times both $R(t)$ and $B(t)$ may be small and expanding, it is assumed that after a short time $B(t)$ begins to contract while $R(t)$ continues to expand. It is also assumed that the reaction times are short enough (compared to the expansion and contraction times) to maintain thermal equilibrium in the matter system so that it may be characterized by a single temperature T, and moreover, the entropy in any comoving volume remains constant. Finally, it is assumed that the matter is sufficiently hot to be described by free particles. It is natural within the context of the Kaluza-Klein theory to require that these particles be massless in the extended $(4+D)$-dimensional space-time. Since the extra D-dimensional ball is compact, the energy E of a particle with three-space momentum $\vec{p}$ is given by

$$E^2 = \vec{p}^{\,2} + B(t)^{-2}\lambda \ , \tag{2}$$

where λ is an eigenvalue of a Laplace operator built on the metric g_{ab}. Note that $\vec{p}$ is the momentum measured in a local inertial frame.[6]

The authors of Ref. 4 consider bosons which are described by the Planck distribution function $f=(e^{E/T}-1)^{-1}$ in the $(4+D)$-dimensional space-time. They then define what they call an equivalent "four-dimensional distribution function" $f_4(\vec{p},t)$, by averaging f over the phase space of the extra D dimensions. The resulting distribution is no longer of the Planckian form. Nevertheless, the authors define an "effective four-dimensional temperature" T_{eff} by fitting f_4 to a four-dimensional Planckian distribution. That is, they require that the four-dimensional energy density given by f_4 be the same as that given by a true four-dimensional Planckian distribution at temperature T_{eff}. Further, they compute the "effective four-dimensional entropy" density S_4 from T_{eff}, and express its dependence on R and B. This is claimed as the main result of their paper.

Our main concern in this comment is to emphasize that these "effective" quantities f_4, T_{eff}, and S_4 are not the sensible quantities to study from the physical point of view, as they hide the true physics of what is going on, which we will now describe.

From a $(4+D)$-dimensional vantage point, one has (by assumption) a gas of massless Bose or Fermi particles—one species of particle (with various polarization states). However, from a four-dimensional point of view, one has *many* species of particles of different masses. The various normal-mode excitations of the matter field in the compact extra dimensions appear as fields of definite charge and mass in the four-dimensional world, as is well known.[7] That is, $M_\lambda{}^2 = B(t)^{-2}\lambda$ appears as a (mass)^2 in Eq. (2). If $T \ll B^{-1}$, none of these modes are excited save for those with $\lambda=0$, which give a gas of massless four-dimensional particles. If $T \gg B^{-1}$, many of these modes are excited, and in four dimensions it appears that one has a gas consisting of a mixture of *many species* of various masses. Thus, it is wrong to treat the effective four-dimensional situation as though it were a massless gas at temperature T_{eff}, as do the authors of Ref. 4. It is really a gas of many particle types of various masses at temperature T. The true four-dimensional effective temperature is the same as the true $(4+D)$-dimensional temperature, namely, T. It is not the temperature which looks different in the dimensionally reduced world, but the particle spectrum. Indeed, T_{eff} is not really the temperature of anything. A gas of one species of particle at temperature T_{eff} would (by the definition of T_{eff}) have the correct energy density. However, it would differ from the true situation in other macroscopic, thermodynamic quantities such as pressure, entropy density, and so forth.

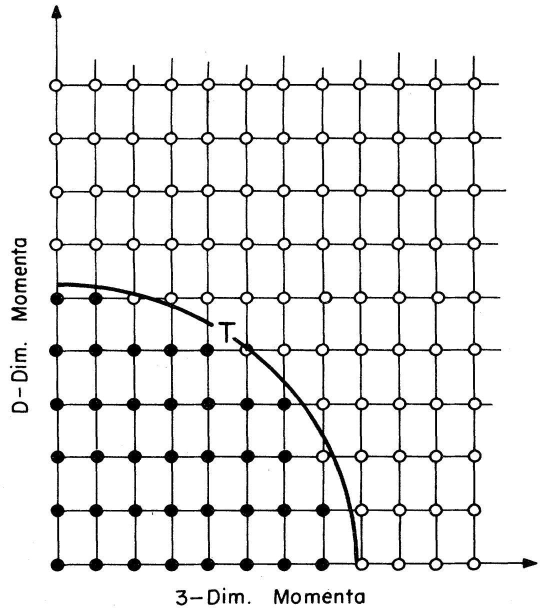

FIG. 1. A schematic representation of the energy level of the $(D+3)$-dimensional field $E=[\vec{p}^{\,2}+\lambda/B(t)^2]^{1/2}$. The levels of $(\vec{p}^{\,2})^{1/2}$ are plotted on the horizontal axis. The levels of $[\lambda/B(t)^2]^{1/2}$ are plotted vertically. The energy is the distance to the origin. The circular arc is the temperature. Solid and empty circles denote filled and empty energy levels, respectively.

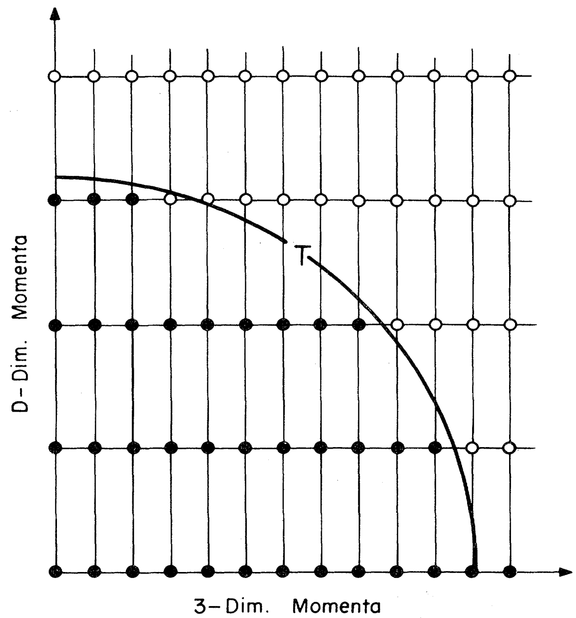

FIG. 2. Because of the contraction of the compact dimensions (B decreases) the level spacings have become larger in the vertical direction relative to Fig. 1. Conservation of entropy implies that T has increased. Note that the higher modes of the D-dimensional kinetic energy are being frozen out.

The physical situation may be clarified by contemplating Fig. 1. The horizontal axis denotes schematically the three-dimensional momentum space, while the vertical axis denotes the D-dimensional momentum space of the extra dimensions. The circular arc represents the true temperature T. The vertical and horizontal lines represent energy levels (we represent the three-dimensional levels as being discrete for clarity). The levels where $E<T$ are mostly filled (denoted by black circles) while those for which $T>E$ are mostly empty (denoted by open circles). From the four-dimensional view, each row of states is the set of states of a specific species of particle; the higher the row, the larger is the mass of that species. As the size $B(t)$ of the compact dimension shrinks, the level spacings in the vertical directions get larger. Conservation of entropy tells us that the number of filled levels remains roughly constant. So if, for example, $R(t)$ remains fixed while $B(t)$ shrinks, $T(t)$ will increase as shown in Fig. 2. But $T(t)$ will not increase as fast as the level spacing $B(t)^{-1}$, and so the more massive modes will be progressively frozen out. Eventually when $B(t)^{-1}$ exceeds $T(t)$, very few massive particles will remain, and we are left with a hot gas of massless particles in the four-dimensional space-time.

This whole process is exactly analogous to what happens in standard cosmology when the temperature drops below the mass M of some particle species. When $T\gg M_p$, there are many protons and antiprotons about. As the temperature drops below M_p, the protons and antiprotons freeze out. They annihilate and dump their energy into lighter species, thus causing those lighter species to heat up. As T falls below M_K, M_π, M_μ, M_e, the kaons, pions, muons, and electrons and their antiparticles freeze out as well and dump their energy into photons, the photons heat up. It is the same physics in the Kaluza-Klein case.[8]

Let us now give a quantitative analysis of the physics which we have just described. A basic assumption is that we have a conserved entropy current s^A in $4+D$ dimensions:

$$s^A{}_{;A}=(-\det g_{CD})^{-1/2}\frac{\partial}{\partial \xi_A}[(-\det g_{CD})^{1/2}s^A]=0\ . \quad (3)$$

Since the three-dimensional space and the extra D-dimensional ball are both isotropic, s^A has only a nonvanishing time component s^0. We integrate over the extra D dimensions to obtain the entropy density observable in four-dimensional space-time,

$$s_4(t)=\int (dy)B(t)^D(\det g_{ab})^{1/2}s^0\ . \quad (4)$$

The conservation law (3) tells us that

$$R(t)^3 s_4(t)=C \quad (5)$$

is a constant.

We assume that at early times the temperature T_i is sufficiently large that $T_i\gg B^{-1}$. In this epoch the discrete spectra $B(t)^{-2}\lambda$ can be replaced by $(p^a)^2$, where p^a is a continuous momentum variable defined in a local inertial frame in the ball. Thus, in the early epoch

$$E=[\vec{p}^{\,2}+(p^a)^2]^{1/2}\ , \quad (6)$$

and the entropy density in the $(3+D)$-dimensional space is given by the usual, flat-space formula

$$s^0=\frac{\partial}{\partial T}T\int\frac{(d^{3+D}p)}{(2\pi)^{3+D}}\omega(E/T)\ , \tag{7}$$

in which

$$\omega(E/T)=-b_p\ln(1-e^{-E/T})+f_p\ln(1+e^{-E/T})\ , \tag{8}$$

where b_p and f_p are the number of bosonic and fermionic degrees of polarization. Expanding the logarithms and performing the integration term by term produces a series that defines the Riemann ζ function with overall coefficient involving Γ functions. Thus, one easily computes

$$s^0=\left[b_p+f_p\left[1-\frac{1}{2^{3+D}}\right]\right]\times\frac{2\Gamma(3+D/2)}{\pi^{2+D/2}}\zeta(4+D)T^{3+D}\ . \tag{9}$$

Therefore the constant C is identified as

$$C=R(t)^3B(t)^DV_D\left[b_p+f_p\left[1-\frac{1}{2^{3+D}}\right]\right]\times\frac{2\Gamma(3+D/2)}{\pi^{2+D/2}}\zeta(4+D)T^{3+D}\ , \tag{10}$$

in which

$$V_D=\int(dy)(\det g_{ab})^{1/2} \tag{11}$$

is the geometrical, invariant volume of the extra D-dimensional ball.

Given the two scale factors $R(t)$ and $B(t)$, Eq. (10) defines the temperature T. This determination is valid over a range of early times, where the matter can be treated as free, massless particles. However, as the ball shrinks, the product $B(t)T$ must decrease since the opposite behavior leads to a contradiction: If $B(t)T$ were to remain constant or increase, then Eq. (10) requires that $R(t)T$ remain constant or decrease, which, with $R(t)$ increasing, requires that T, and with it $B(t)T$, decrease.

We assumed that in the initial epoch $B(t)T$ was large so that the quantum modes in the ball could be well approximated by a momentum integral. If this is not the case, we must make the replacement

$$\int(dy)B(t)^D(\det g_{ab})^{1/2}\int\frac{(d^Dp)}{(2\pi)^D}\to\sum_\lambda\ , \tag{12}$$

and use Eq. (2) for the energy. Scaling the remaining three-dimensional momenta by the temperature so as to obtain dimensionless integration variables $\vec{q}=\vec{p}/T$, we see that this integration involves

$$E/T=\left[\vec{q}^{\,2}+\frac{\lambda}{[B(t)T]^2}\right]^{1/2}\ . \tag{13}$$

Thus as $B(t)T$ becomes very small, all the masses become relatively very large, and these modes cannot be thermally excited; they are frozen out of equilibrium. The zero modes with $\lambda=0$ remain, of course. Hence, in the later epoch which we have just described

$$s_4=\frac{\partial}{\partial T}T\int\frac{(d^3p)}{(2\pi)^3}\omega_4(p/T)\ , \tag{14}$$

where

$$\omega_4(p/T)=-b_0\ln(1-e^{-p/T})+f_0\ln(1+e^{-p/T})\ , \tag{15}$$

with b_0 and f_0 the number of bosonic and fermionic zero modes. We now obtain

$$C=r(t)^3s_4=R(t)^3(b_0+\tfrac{7}{8}f_0)\frac{4}{\pi^2}\zeta(4)T^3\ . \tag{16}$$

Comparing Eq. (16) with Eq. (10), we conclude that the temperature T_f in the late epoch is related to the temperature T_i in the early epoch by

$$T_f=T_i\frac{R(t_i)}{R(t_f)}[B(t_i)T_i]^{D/3}\times\text{const.} \tag{17}$$

Although (by assumption) $B(t_i)T_i$ is very large, $R(t_i)/R(t_f)$ is very small. Therefore, the final temperature can be either larger or smaller than the initial temperature.

ACKNOWLEDGMENT

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[1]A. Chodos and S. Detweiler, Phys. Rev. D 21, 2167 (1980); P. G. O. Freund, Nucl. Phys. B209, 146 (1982); E. Kolb and R. Slansky, Phys. Lett. 135B, 378 (1984); S. Randjbar, A. Salam, and J. Strathdee, *ibid.* 135B, 388 (1984); D. Sahdev, *ibid.* 137B, 155 (1984). See Sahdev for other references.

[2]T. Kaluza, Sitzungsber. Preuss, Akad. Wiss. Phys. Math. K1, 966 (1921); O. Klein, Z. Phys. 37, 875 (1926); Nature (London) 118, 516 (1926), and many recent papers. See E. Cremmer and J. Scherk, Nucl. Phys. B103, 399 (1976).

[3]See A. Guth, Phys. Rev. D 23, 347 (1981). For a discussion of a Kaluza-Klein inflationary picture, see Sahdev (Ref. 1).

[4]E. Alvarez and M. Belen Gavela, Phys. Rev. Lett. 51, 931 (1983).

[5]This interval is a more general form than that used in Ref. 4. Our notation also differs somewhat from that used in Ref. 4. In particular, we choose the extra coordinates y^a to be dimensionless and of order unity. The scale factor $B(t)$ thus gives

the size of the extra "ball," the size denoted by AL in Ref. 4.

[6]If q^{μ} is the covariant four-momentum which transforms like the velocity $\dot{Z}^{\mu}$, then $p^k = e^k{}_l q^l$, where $e^k{}_l$ is a dreibein decomposition of the spatial metric $R(t)^2 g_{kl} = e^m{}_k e^m{}_l$.

[7]See A. Salam and J. Strathdee, Ann. Phys. (N.Y.) 141, 316 (1982).

[8]We tacitly assume that there are no quantum numbers that forbid the decay of any of the higher Kaluza-Klein modes. See Kolb and Slansky (Ref. 1).