Energy-momentum, angular momentum, and supercharge in 2 + 1 supergravity

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It is pointed out that there is no way to define linear momentum and supercharge for a generic solution to 2 + 1 supergravity, whose asymptotic symmetries only include time translations and spatial rotations. A Hamiltonian expression for the energy and the angular momentum is also given.

It has been shown recently that the energy and the angular momentum of a gravitating system in $2+1$ dimensions are defined by the global geometrical properties of spacetime.[1,2] These results are extended here to the case of $2+1$ supergravity. We point out that there is no way to define a meaningful supercharge in general, just as there is no way to talk about linear momentum. This follows from the fact that the asymptotic symmetries of a generic solution do not include all the generators of the super-Poincaré algebra, but only the time translations and the space rotations.

We also show that the canonical generators of the remaining asymptotic symmetries, whose values reduce to surface integrals when the constraint equations hold, are numerically equal to the energy and the angular momentum as defined in Ref. 1.

In $2+1$ dimensions, the gauge fields of supergravity (i.e., the tetrads $h_{(\lambda)\alpha}$ and the spinor field ψ_λ) are pure gauge outside matter (which we will always assume to be localized in space); supergravity has no dynamical degrees of freedom of its own. This means that one can locally bring the metric and the spinor field to the form $g_{\alpha\beta}=\eta_{\alpha\beta}$, $\psi_\lambda=0$ by appropriate gauge transformations.

It turns out, however, that these transformations cannot be extended in general to the whole region surrounding the sources. Hence, it is too restrictive to assume that spacetime is asymptotically Minkowskian, contrary to what is usually done in four dimensions.

We will assume instead that the metric can be made to approach asymptotically the metric of a conic spacetime. This is the generic behavior in pure gravity[1,3,4] and we will argue below that it does not rule out interesting possibilities in supergravity. We will also take the spinor field to vanish at infinity, which is compatible with the field equations in the gauge $g_{\alpha\beta}=g^{\text{cone}}_{\alpha\beta}$ [$T_{\alpha\beta}(\psi_\lambda)$ must vanish at infinity because $R_{\alpha\beta}\to 0$].

Our first task now is to determine the gauge transformations that leave the asymptotic form of the fields invariant since their generators define the conserved charges, as in any gauge theory. The symmetry equations read, in terms of the asymptotic fields,

$$\xi_{\alpha;\beta}+\xi_{\beta;\alpha}=0 \quad (\delta g^{\text{cone}}_{\alpha\beta}=0) \tag{1}$$

and

$$\nabla_\mu\epsilon=0 \quad (\delta\psi_\mu=0)\ . \tag{2}$$

In Minkowski space, the solutions to the linear system (1), (2) form an eight-dimensional manifold (six Killing vectors and two Killing spinors). In a conic spacetime, however, there are only two linearly independent solutions because of nontrivial global matching conditions.

To prove that assertion, it is necessary to recall some geometric properties of a flat conic spacetime, which is obtained by removing the region limited by two half-planes in Minkowski space, say $\phi=2\pi-\beta$ $(0<\beta<2\pi)$ and $\phi=0$, and identifying the points $(t,r,2\pi-\beta)$, $(t',r',0)$ according to the rule $t'=t+A$, $r'=r$. As is well known, the only Killing vectors possessed by this spacetime are $\partial/\partial t$ and $\partial/\partial\phi$. The other Killing vectors of Minkowski space are not globally defined on the cone because they fail to satisfy the matching conditions along the seams. This allows us to concentrate only on the equation $\nabla_\mu\epsilon=0$.

When A vanishes, the trajectories generated by $\partial/\partial\phi$ are closed. When $A\neq 0$, they are not, but those generated by $\xi\equiv\partial/\partial\phi-[A/(2\pi-\beta)]\partial/\partial t$ are. One can take as coordinates $t'\equiv t+A\phi/(2\pi-\beta)$, r, and ϕ. The coordinate t' does not have the jump defect at $\phi=0$ and the boundary at infinity of the surface $t'=\text{const}$ is a circle, not a helix. In this coordinate system, the metric reads[1]

$$ds^2=-\left[dt'-\frac{A}{2\pi-\beta}d\phi\right]^2+dr^2+r^2d\phi^2\ ,$$
$$0\le\phi\le 2\pi-\beta\ . \tag{3}$$

Only its form for large r is of interest to us, since it is not valid inside the sources. The cross term $dt'd\phi$ in (3) reflects the nonorthogonality of ξ and $\partial/\partial t$.

It is now easy to analyze the equations $\nabla_\lambda\epsilon=0$. In the orthonormal frame

$$(-dt'+[A/(2\pi-\beta)]d\phi\ ,dr\ ,rd\phi)\ ,$$

the nonzero spin connection coefficients are given by

$$\omega_{(r)(\hat\phi)\phi}=1=-\omega_{(\hat\phi)(r)\phi}\ . \tag{4}$$

The equations $\nabla_0\epsilon=\nabla_r\epsilon=0$ simply imply $\epsilon=\epsilon(\phi)$. The critical equation is $\nabla_\phi\epsilon=0$, from which one infers

$$\epsilon(\phi)=\left[\cos\frac{\phi}{2}I+\sin\frac{\phi}{2}\gamma_{(0)}\right]\epsilon(0)\ . \tag{5}$$

Because $\phi=0$ and $\phi=2\pi-\beta$ define the same point, $\epsilon(\phi)$

must come back to minus its original value $\epsilon(0)$ at $\phi=2\pi-\beta$ (minus, because the frame dr, $r\,d\phi$ makes a full revolution with respect to the standard frames when one goes around the sources). This leads to the condition

$$-\epsilon(0)=\left[-\cos\frac{\beta}{2}I+\sin\frac{\beta}{2}\gamma_0\right]\epsilon(0) \tag{6}$$

which constraints $\epsilon(0)$ to vanish ($\beta\neq 0$).[5]

Since the only asymptotic symmetries of the fields $(h_{(\lambda)\alpha},\psi_\lambda)$ are the time translations and the space rotations, one can only define energy and angular momentum.[6] In the Hamiltonian formalism, these quantities are the generators of the corresponding symmetries and reduce to surface integrals on-shell, as we now discuss.

Because the charges are given by global invariants,[1] the choice of the coordinate system is less critical than in four dimensions: these charges are integrals of spatial densities [see (18b) below]. We will work here either with the form (3) of the line element, rewritten as

$$ds^2=-(dt'-k\,d\bar\phi)^2+dr^2+\chi^2r^2d\bar\phi^2\ , \tag{7}$$

$$k=\frac{A}{2\pi}\ ,\quad \chi=\frac{2\pi-\beta}{2\pi}<1\ ,\ \chi\bar\phi=\phi\ , \tag{8}$$

or with the form

$$ds^2=-(dt')^2+2k\,dt'd\bar\phi+f^2(\rho)(dx^2+dy^2)\ , \tag{9}$$

$$f(\rho)=\frac{|k|\sinh\chi\ln\rho}{\rho}\ ,\quad r=\frac{|k|}{\chi}\cosh\chi\ln\rho\ ,$$
$$x=\rho\cos\bar\phi\ ,\quad y=\rho\sin\bar\phi\ . \tag{10}$$

(If $k=0$, one has $r=\rho^\chi, f=\chi\rho^{\chi-1}$.)

The momenta π^{ij} are given (asymptotically) by

$$\pi^{r\phi}=-\frac{k\chi^2r}{(\chi^2r^2-k^2)^{3/2}[1+k/(\chi^2r^2-k^2)]^{1/2}} \tag{11}$$

in the coordinate system (8) since

$$N=\left[1+\frac{k}{\chi^2r^2-k^2}\right]^{1/2}\ ,\quad N_\phi=k\ , \tag{12}$$

whereas the spinor field ψ_λ vanishes as $r\to\infty$. These asymptotic values are attained as quickly as needed since, as we have shown under natural boundary conditions, there cannot be a "topological supercharge" that prevents ψ_λ from being gauge related to $\psi_\lambda=0$ outside the sources.

Having studied the asymptotic behavior of the fields, we can now evaluate the charges. The canonical generators of the coordinate transformations that vanish at infinity are given by

$$H[\eta^\perp,\eta^k]=\int d^2x(\eta^\perp\mathscr{H}+\eta^k\mathscr{H}_k) \tag{13}$$

with

$$\mathscr{H}=G_{rsmn}(\pi^{rs}-\mathscr{P}^{rs})(\pi^{mn}-\mathscr{P}^{mn})-R\sqrt{g}+\mathscr{H}^I\ , \tag{14}$$

$$\mathscr{H}_k=-2\pi_k{}^s{}_{|s}+\mathscr{H}_k{}^{3/2}\ . \tag{15}$$

Here, $\mathscr{H}^I$ and $\mathscr{P}^{rs}$ are expressions with undifferentiated fields of fourth and second orders in the spinors. Their explicit form will not concern us here. Similarly, we will not need $\mathscr{H}_k{}^{3/2}$ (these can be found in Ref. 7).

When the vector fields $\eta^\perp,\eta^k$ do not vanish at infinity but approach asymptotic Killing vectors, one cannot use H as a generator because it has ill-defined Poisson brackets with the fields.[8] One has to supplement H with appropriate boundary terms, so designed as to cancel the surface terms resulting from partial integration in δH.

These boundary terms are easy to find here, for two reasons. (i) The only place where spatial derivatives occur in the super-Hamiltonian is through the term $R\sqrt{g}$, which is a divergence in two dimensions, $R\sqrt{g}=\partial_i v^i$. (ii) The spinor derivatives in $\mathscr{H}_k{}^{3/2}$ do not pose any problem. Hence, the total generator of these more general coordinate transformations—or better, "surface deformations"—is given by

$$H^T[\eta^\perp,\eta^k]=\int d^2x(\eta^\perp\mathscr{H}+\eta^k\mathscr{H}_k)+\int d\Sigma_k(\eta^\perp_\infty v^k+2\tilde\eta_\infty\xi^s\pi_s{}^k) \tag{16}$$

with

$$\eta^\perp\underset{r\to\infty}{\longrightarrow}\eta^\perp_\infty\ ,\quad \eta^k\underset{r\to\infty}{\longrightarrow}\tilde\eta_\infty\xi^k\ ,\quad \xi=\frac{\partial}{\partial\phi}\ , \tag{17}$$

where $\eta^\perp_\infty$ and $\tilde\eta_\infty$ are constant. One easily checks, using the form (3) of the metric, that this generator has well-defined functional derivatives within the allowed class of g_{ij} ($\delta g_{ij}\sim\rho^{-2M}\ln\rho\,\delta_{ij}\delta M$) and π^{ij}.

When the equations of motion hold, H^T does not vanish but reduces to the boundary terms

$$H^T[\eta^\perp,\eta^k]\approx\int d\Sigma_k(\eta^\perp_\infty v^k+2\tilde\eta_\infty\xi^s\pi_s{}^k) \tag{18a}$$

$$\approx\int d^2x[\eta^\perp_\infty R\sqrt{g}+2\tilde\eta_\infty(\xi^s\pi_s{}^k)_{|k}]\ . \tag{18b}$$

Our purpose is to evaluate (18) for $\partial/\partial t$ (energy) and $\partial/\partial\bar\phi-k\partial/\partial t$ (angular momentum—the generator is normalized in such a way that a full revolution corresponds to 2π).

The surface term related to the curvature is most easily computed in the coordinate system (9), where

$$R\sqrt{g}=-(\partial^2\ln f/\partial x^2+\partial^2\ln f/\partial y^2)\ .$$

One finds

$$H^T\left[\frac{\partial}{\partial t}\right]\equiv E=\beta \tag{19}$$

and

$$H^T(\xi)\equiv J=-2A\chi \tag{20}$$

since $\partial/\partial t=Nn+N^ke_k$. These values coincide with those of Ref. 1. One of the advantages of the Hamiltonian formalism is that one can define the Poisson brackets of the charges.[8,9] In our case, they simply commute.

Let us now come back to our assumption that the asymptotic geometry is given by $g_{\alpha\beta}=g^{\text{cone}}_{\alpha\beta}$ with $\psi_\lambda\to 0$. One might wonder whether a different asymptotic behavior of the fields would not enable one to define a topological supercharge Q_A. If this were the case, however, energy-momentum would also have to be meaningful, because of the relation $[Q_A,Q_B]=(\gamma_{(0)}\gamma^\mu)_{AB}P_\mu$ with

$[P_\mu, P_\nu]=0$. Spacetime would thus possess three asymptotic commuting Killing vectors. It would accordingly be asymptotically Minkowskian (three commuting Killing vectors imply all Poincaré generators). From this, one should be able to infer that the fields can continuously be related by a gauge transformation to $g_{\alpha\beta}=0, \psi_\lambda=0$ outside the sources—since the local "superflatness" criterion is satisfied and spacetime obeys the correct boundary conditions—and hence, that the conserved charges are all trivial and equal to zero.[10,11]

As a final comment, we note that the proof of the positivity of the energy in general relativity based on supergravity $[E=\frac{1}{2}\sum Q_A{}^2$ (Ref. 12)] fails in 2 + 1 dimensions since there is no meaningful supercharge. Positivity of the energy and stability of Minkowski space are guaranteed by a much simpler mechanism: on a maximal slice ($\pi=0$)—which we assume to exist—the scalar curvature R must be positive by the constraint equations.[13]

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1S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) 152, 220 (1984).

2We use here the words "global geometrical properties" instead of "topological properties" to emphasize that the manifolds we are considering are topologically trivial (homeomorphic to R^3) and only differ from Minkowski space (in the regions where they are flat) by their global metric properties. The numbers which characterize these deviations can vary continuously and greatly differ from true topological invariants, which are discrete.

3A. Staruszkiewicz, Acta Phys. Pol. 24, 734 (1963).

4S. Giddings, J. Abbott, and K. Kuchař, University of Utah report, 1983 (unpublished).

5One can reach that same conclusion more directly by noticing that any Killing spinor defines a translation ξ_α by $\xi_\alpha=\bar{\epsilon}\gamma_\alpha\epsilon$ ($\xi_{\alpha;\beta}=0$). Since there is no spatial translation, $\bar{\epsilon}\gamma_1\epsilon$ and $\bar{\epsilon}\gamma_2\epsilon$ must vanish, which implies that ϵ itself is zero (ϵ is real). The method followed in the main text possesses the advantage that it requires a more detailed study of slices and frames on a conic spacetime, which is used later on.

6In order to be acceptable quantities, the total energy, momentum, angular momentum, and supercharge can only depend on the asymptotic location of the surface on which they are evaluated. This will be the case if they are given by appropriate line integrals at infinity. Now, the matter energy-momentum $\int T_M^{o\mu} d^2x$ certainly does not fulfill that condition ($T^{o\mu}_{M,\mu}\neq 0$). So, there must also be a contribution to the total energy-momentum from the gauge fields. This is also true of the supercharge. A correct, physically reasonable and unambiguous expression for the total conserved quantities only exists when the fields possess asymptotic symmetries. The "charges" are then the generators of these symmetries. This is why it is important to study first the algebra of asymptotic symmetries in 2 + 1 supergravity. The essential need for asymptotic Killing vectors (and spinors) in order to define constants of the motion in gravitation theory has been explicitly brought out in L. Abbott and S. Deser, Nucl. Phys. B195, 76 (1982).

7R. W. Tucker and P. S. Howe, J. Math. Phys. 19, 869 (1978); T. Dereli and S. Deser, J. Phys. A 11, L27 (1978); M. Pilati, Ph.D. thesis, Princeton University, 1980. The canonical decomposition of 3 + 1 supergravity can be found in S. Deser, J. Kay, and K. S. Stelle, Phys. Rev. D 16, 2448 (1977); E. S. Fradkin and M. A. Vasiliev, Phys. Lett. 72B, 70 (1977); M. Pilati, Nucl. Phys. B132, 138 (1978) and the above references.

8T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) 88, 276 (1974).

9That one can define a Poisson bracket structure in the space of the charges E, J shows again that these charges are quite different from ordinary topological charges, which cannot be viewed as generators of transformations in the space of the fields.

10It seems to be a general rule that only the commuting fields (actually, only their "body part") carry a nontrivial topological information.

11The analysis can be extended to the anti-de Sitter case, considered in S. Deser and R. Jackiw, Ann. Phys. (N.Y.) (to be published). No supercharge can be meaningfully defined when there is "energy." Otherwise, one would find a well-defined $\Lambda=0$ supercharge upon contraction (assuming continuity). Differently put, no nontrivial solution to the equations with $\Lambda<0$ can exist which has the anti-de Sitter superalgebra as asymptotic symmetry superalgebra. The asymptotic superalgebra must reduce to J_{03} and J_{12}, which simply commute [see J. D. Brown and M. Henneaux (unpublished)].

12S. Deser and C. Teitelboim, Phys. Rev. Lett. 39, 249 (1977).

13The conclusions of this paper are, strictly speaking, only valid for classical supergravity. Being related to asymptotic symmetries, they will remain true at the quantum level if one can prescribe the asymptotic behavior of the fields without violating the uncertainty principle.