# Universality and quantum gravity

Bruce L. Nelson* and Prakash Panangaden†
*Physics Department, University of Utah, Salt Lake City, Utah 84112*
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The gravitational coupling constants associated with several renormalized quantum field theories in curved spacetime all exhibit similar high-energy scaling behavior. This suggests a universal structure of high-energy semiclassical gravity.

## I. INTRODUCTION

When a free quantum field is coupled to a classical gravitational field, it is necessary to renormalize the gravitational coupling constants in order to render the matter field's stress tensor finite. Because of the renormalization, these constants display a nontrival scaling behavior, even in the free-field case.[1] In Sec. II we discuss the scaling of the gravitational coupling constants for the scalar, spin-$\frac{1}{2}$, and photon fields. Our main result is that the high-energy behavior is the same for all these theories. In Sec. III we show that in an asymptotically free theory the high-energy scaling is essentially the same as in the free theory and, hence, there seems to be a universal structure of high-energy semiclassical gravity.

## II. THE SCALING OF THE GRAVITATIONAL COUPLING CONSTANTS AND SPECIFIC RESULTS FOR FREE-FIELD THEORIES

The generating function for the Green's functions of a matter field can be written as the functional integral

$$Z[J]\equiv \int [d\phi]\exp\left[i\int \sqrt{g}\, d^n x\, L_B^{(\text{matter})} + \int \sqrt{g}\, d^n x\, J_B\phi_B\right] . \tag{2.1}$$

The bare matter Lagrangian contains the counterterms necessary for mass and coupling-constant renormalization. The bare external field $J_B$ is used to bring in the wave-function renormalization:

$$J_B = Z_\phi{}^{-1/2} J_R . \tag{2.2}$$

The expectation value of the stress tensor is defined by

$$\langle T_{\mu\nu}\rangle_B = \frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}} Z[J=0] \tag{2.3}$$

and is found to be infinite. It is renormalized by adding counter terms proportional to 1, $R$, and terms quadratic in the spacetime curvature. Thus,

$$\langle T_{\mu\nu}\rangle_R = \langle T_{\mu\nu}\rangle_B - \text{pole terms} , \tag{2.4}$$

where the pole terms involve only invariant metric quantities. Since these terms involve geometric quantities, they can be interpreted as renormalizing the gravitational action, and we write

$$Z = \int [d\phi] e^{i\int L_B} , \tag{2.5}$$

where

$$L_B = L_B^{(\text{matter})} + L_B^{(\text{geometry})} . \tag{2.6}$$

$L_B^{(\text{geometry})}$ now has terms quadratic in the curvature tensor. Taking the functional derivative of this new $Z$ with respect to $g^{\mu\nu}$ gives the renormalized modified Einstein equation; i.e.,

$$\frac{2}{\sqrt{g}}\frac{\delta Z}{\delta g^{\mu\nu}} = 0$$

gives

$$\langle T_{\mu\nu}\rangle_{\text{renormalized}} = \mathsf{G}_{\mu\nu}^{\text{renormalized}} , \tag{2.7}$$

where $\mathsf{G}$ is the modified Einstein tensor containing quadratic curvature terms.

We choose the particular form of $L^{(\text{geometry})}$ to be[2]

$$L^{(\text{geometry})} = \Lambda + \kappa R + a\mathscr{G} + b\mathscr{H} + c\mathscr{I} , \tag{2.8}$$

where $\mathscr{G}$ is the Gauss-Bonet form in four dimensions, $\mathscr{H}$ is the square of the Weyl tensor, and $\mathscr{I}$ is simply $R^2$. $\mathscr{G}$ and $\mathscr{H}$ are scale invariant in four dimensions, $\mathscr{I}$ is not. Specifically,

$$\mathscr{G} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 , \tag{2.9a}$$

$$\mathscr{H} = {}^{(n)}C_{\alpha\beta\gamma\delta}\,{}^{(n)}C^{\alpha\beta\gamma\delta} , \tag{2.9b}$$

and

$$\mathscr{I} = R^2 . \tag{2.9c}$$

$\mathsf{G}$ is then

$$\mathsf{G}_{\mu\nu} = -\Lambda g_{\mu\nu} + 2\kappa G_{\mu\nu} - a\mathscr{G}_{\mu\nu} - b\mathscr{H}_{\mu\nu} - c\mathscr{I}_{\mu\nu} , \tag{2.10}$$

where

$$g_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^n x\sqrt{g} , \tag{2.11a}$$

$$-2G_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^n x\sqrt{g}R , \tag{2.11b}$$

$$\mathscr{G}_{\mu\nu} \equiv \frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^n x\sqrt{g}\,\mathscr{G} , \tag{2.11c}$$

$$\mathscr{H}_{\mu\nu} \equiv \frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^n x\sqrt{g}\,\mathscr{H} , \tag{2.11d}$$

and

$$\mathscr{I}^{\mu\nu} \equiv \frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{g}\, \mathscr{I} \ . \tag{2.11e}$$

We are interested in the high-energy behavior of the semiclassical field equation (2.7). However, in a generic curved spacetime there is no covariant meaning to energy scaling. In a flat spacetime, however, the scaling of the momentum $p$ to $\alpha p$ is equivalent to a conformal transformation of the metric from $\eta_{\mu\nu}$ to $\eta_{\mu\nu}/\alpha^2$. We take this last approach as defining scaling in a curved spacetime.[1,3] The scaled field equations are then[1]

$$\left\langle T_\mu^\nu \left[ \lambda_R, m_R, \mu, \frac{g_{\mu\nu}}{\alpha} \right] \right\rangle_R = \mathrm{G}_{\mu R}^{\,\nu} \left[ \Lambda_R, K_R, a_R, b_R, c_R, \frac{g_{\mu\nu}}{\alpha} \right] . \tag{2.12}$$

Here the subscript $R$ denotes renormalized quantities, $\lambda$ is a generic coupling constant, and $\mu$ is the mass scale introduced in dimensional regularization. It can be shown that this scaled-up field equation is equivalent to a scaled-up version of a field equation at normal energies ($\alpha=1$), but with all the coupling constants replaced by effective coupling constants. Thus (2.12) is equivalent to the equation

$$\alpha^4 \langle T_\mu^\nu(\lambda(\alpha), m(\alpha), \mu, g_{\mu\nu}) \rangle_R = \alpha^4 \mathrm{G}_\mu^{\,\nu}(\Lambda(\alpha), \kappa(\alpha), a(\alpha), b(\alpha), c(\alpha), g_{\mu\nu}) \ . \tag{2.13}$$

The effective $\lambda$ and $m$ are the usual ones encountered in studying the scaling of the Green's functions.[3,4] For matter fields with dimensionless coupling constants, the effective gravitational constants are given by[1,2]

$$\Lambda(\alpha) = \Lambda_R + \int_1^\alpha \frac{d\alpha'}{\alpha'} \frac{m_R^{\,4}(\lambda, m, \alpha')}{(\alpha')^4} \beta_\Lambda(\lambda_R(\lambda, \alpha')) \ , \tag{2.14}$$

$$\kappa(\alpha) = \kappa_R + \int_1^\alpha \frac{d\alpha'}{\alpha'} \frac{m_R^{\,2}(\lambda, m, \alpha')}{(\alpha')^2} \times \beta_\kappa(\lambda_R(\lambda, \alpha'), \xi_R(\lambda, \xi, \alpha')) \ , \tag{2.15}$$

$$a(\alpha) = a_R + \int_1^\alpha \frac{d\alpha'}{\alpha'} \beta_a \ ,$$
$$b(\alpha) = b_R + \int_1^{\alpha'} \frac{d\alpha'}{\alpha'} \beta_b \ , \tag{2.16}$$
$$c(\alpha) = c_R + \int_1^{\alpha'} \frac{d\alpha'}{\alpha'} \beta_c \ .$$

Here $\xi$ denotes the coupling constant appearing in couplings of the field directly to the curvature, such as $\xi R \phi^2$. The $\beta$ functions are given in derivatives of the pole terms in $\langle T^{\mu\nu} \rangle_B$ with respect to $\mu$.[1,2]

We now discuss the explicit forms of these effective couplings for the free scalar, spin-$\frac{1}{2}$, and photon fields. For a free field, $\lambda$, $m$, and $\xi$ have simple scaling:

$$\lambda(\alpha) = \lambda_R = 0 \ , \tag{2.17a}$$

$$m(\alpha) = \alpha^{-1} m_R \ , \tag{2.17b}$$

$$\xi(\alpha) = \xi_R \ . \tag{2.17c}$$

Although the $\beta$ functions may be functions of $\xi$, even for a field theory, they are not explicit functions of $\alpha$. Thus the renormalization-group (RG) equations for $\Lambda$, $\kappa$, $a$, $b$, and $c$ can be integrated immediately to give

$$\Lambda(\alpha) = \Lambda_R + \beta_\Lambda m_R^{\,4} \frac{\ln\alpha}{\alpha^4} \ , \tag{2.18a}$$

$$\kappa(\alpha) = \kappa_R + \beta_\kappa(\xi_R) m_R^{\,2} \frac{\ln\alpha}{\alpha^2} \ , \tag{2.18b}$$

$$a(\alpha) = a_R + \beta_a \ln\alpha \ , \tag{2.18c}$$

$$b(\alpha) = b_R + \beta_b \ln\alpha \ , \tag{2.18d}$$

$$c(\alpha) = c_R + \beta_c(\xi_R) \ln\alpha \ . \tag{2.18e}$$

Table I contains the values of the $\beta$ functions for the three free theories under consideration.[5] In the table $e_1$ and $f_1$ are constants that have not, to our knowledge, been computed for the massive spin-$\frac{1}{2}$ theory. The remainder is obtained from Dowker and Critchley.[5] The important results are the following.

First, for all three theories, we find that in the high-energy limit of $\alpha \to \infty$,

$$\Lambda(\alpha) \to \Lambda_R$$

and

$$\kappa(\alpha) \to \kappa_R \ .$$

Thus at high energies, the effective standard Einstein couplings are finite and approach their renormalized values.

Second, if the massless version of the theory is conformally invariant (i.e., if $\xi = \frac{1}{6}$ in the scalar case), then the quadratic curvature term $R^2$ does not scale:

$$c(\alpha) = c_R \ .$$

TABLE I. Values of the $\beta$ functions for the free scalar, spin-$\frac{1}{2}$, and photon fields.

| Theory | Lagrangian | $\beta_\Lambda$ | $\beta_\kappa$ | $\beta_a$ | $\beta_b$ | $\beta_c$ |
|---|---|---|---|---|---|---|
| Free scalar | $-\frac{1}{2}(\partial_\mu\phi\partial_\nu g^{\mu\nu} + m^2\phi^2 + \xi R\phi^2)$ | $\frac{-1}{2\times 16\pi^2}$ | $\frac{-(\xi-\frac{1}{6})}{16\pi^2}$ | $\frac{1}{360\times 16\pi^2}$ | $\frac{-1}{120\times 16\pi^2}$ | $\frac{-(\xi-\frac{1}{6})^2}{2\times 16\pi^2}$ |
| Free spin-$\frac{1}{2}$ | $-\frac{1}{2}\bar{\psi}(\gamma^\mu\partial_\mu + m)\psi + \text{H.c.}$ | $-e_1(\text{spin-}\frac{1}{2})$ | $-f_1(\text{spin-}\frac{1}{2})$ | $\frac{11}{360\times 16\pi^2}$ | $\frac{-1}{20\times 16\pi^2}$ | 0 |
| Free photon | $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ | 0 | 0 | $\frac{31}{180\times 16\pi^3}$ | $\frac{-10}{16\pi^2}$ | 0 |

In particular, if $c_R=0$, there does not arise a scale-breaking, effective $R^2$ coupling at high energies. Again this fact is true for all three theories.

Finally, at high energies, the quadratic curvature terms that are conformally invariant in four dimensions become dominant. In fact, we find that for all three theories, as $\alpha\to\infty$,

$$a\to+\infty \text{ as } \ln\alpha$$

and

$$b\to-\infty \text{ as } \ln\alpha \ .$$

Because of its similarity to the photon field, we expect that the free part of the Yang-Mills field will induce the same high-energy behavior of the gravitational coupling constants. We discuss the modifications caused by interaction terms in the matter Lagrangian in the next section.

## III. ASYMPTOTICALLY FREE THEORIES

In this section we consider two interacting but asymptotically free theories. We show that the high-energy behavior of the effective gravitational coupling constants is essentially the same as for the free-field theories considered in the previous section.

The first theory we consider is the scalar theory with a $\lambda\phi^4$ self-interaction with a negative value for $\lambda$. Although this is not a realistic field theory because the vacuum is unstable, it is a simple asymptotically free model theory and does serve to illustrate the high-energy behavior of gravity coupled to an asymptotically free theory.

In this theory zero is a stable asymptotic fixed point of the coupling constant $\lambda$, and the effective mass approaches zero as the energy scale goes to infinity. If one introduces a $\xi R\phi^2$ term in the Langrangian, then $\xi$ approaches $\frac{1}{6}$ (the conformally invariant value) as the energy is scaled up.

The Lagrangian we consider is

$$L=-\tfrac{1}{2}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+m^2\phi^2+\xi R\phi^2-\frac{|\lambda|}{12}\phi^4\right] . \quad (3.1)$$

For this theory also, the effective gravitational coupling constants are given by Eqs. (2.14)–(2.16). We restrict ourselves to the case $\xi_R=\frac{1}{6}$ so that, aside from the mass term, the theory is conformally invariant. The $\beta$ functions in Eqs. (2.14)–(2.16) are calculated as power series in $\lambda$, e.g.,

$$\beta_\Lambda=\beta_\Lambda^{(0)}+\beta_\Lambda^{(1)}\lambda+\beta_\Lambda^{(2)}\lambda^2+\cdots \ . \quad (3.2)$$

The coefficients have been calculated to sixth order by Hathrell.[6] The dominant scaling behavior is given by the zero-order terms in these expressions. The only way the effect of the interaction could manifest itself is in the nontrivial scaling of the parameters of the field theory, i.e., $m$, $\lambda$, and $\xi$.

The scaling behavior of these parameters is given by the standard RG equations[4]

$$\frac{\alpha\, d\lambda(\alpha)}{d\alpha}=\beta(\lambda(\alpha)) \text{ with } \lambda(1)=\lambda_R \quad (3.3)$$

and

$$\frac{\alpha\, dm(\alpha)}{d\alpha}=-[1+\gamma_m(\gamma(\alpha))]_m(\alpha) \text{ with } m(1)=m_R \ . \quad (3.4)$$

The quantities $\beta$ and $\gamma_m$ are calculated in perturbation theory and are given by Collins[7] as

$$\beta(\lambda)=\frac{3\lambda^2}{16\pi^2}-\frac{17\lambda^3}{3(16\pi^2)^2}+\cdots \quad (3.5)$$

and

$$\gamma_m(\lambda)=\frac{-\lambda}{16\pi^2}+\frac{5\lambda^2}{6(16\pi^2)^2}+\cdots \ . \quad (3.6)$$

These RG equations can be integrated to give $\lambda$ and $m$ as functions of $\alpha$, $\lambda_R$, and $m_R$. The resulting equations can be inverted to give

$$\lambda_R=\lambda_R(\alpha,\lambda) \ , \qquad m_R=m_R(\alpha,\lambda,m) \ . \quad (3.7)$$

When $\xi_R=\frac{1}{6}$ the high-energy behavior of $k$ and $c$ are trivial,[1]

$$\kappa(\alpha)\to\kappa_R$$

and (3.8)

$$c(\alpha)\to c_R \text{ as } \alpha\to\infty \ .$$

The other three gravitational constants exhibit the following high-energy behavior:

$$\Lambda(\alpha)\sim\Lambda_R-\frac{m_R{}^4}{32\pi^2}\frac{\ln(\alpha)}{\alpha^4} \to\Lambda_R \ , \quad (3.9)$$

$$a(\alpha)\sim a_R+\frac{1}{360\times16\pi^2}\ln\alpha+O(\ln\lambda/\lambda_R)\to+\infty \ , \quad (3.10)$$

$$b(\alpha)\sim b_R-\frac{1}{120\times16\pi^2}\ln\alpha+O(\ln\lambda/\lambda_R)\to-\infty \ . \quad (3.11)$$

Thus, we see that in the interacting case the high-energy behavior is essentially the same as in the free case.

As a more realistic model of asymptotically free theories we now consider the Yang-Mills field. The Lagrangian in this case is

$$L=-\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a \ , \quad (3.12)$$

where the $F$'s are given in terms of the Yang-Mills potentials by

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+\lambda C^a_{bc}A^b_\mu A^c_\nu \ . \quad (3.13)$$

The $C$'s are the structure constants of the relevant group and $\lambda$ is a coupling constant. When $\lambda=0$ this reduces to the free-field case. This theory is conformally invariant in four dimensions. As a consequence, the counterterms needed should be conformally invariant;[8] and the only nonzero corrections will be to $\mathscr{G}$ and $\mathscr{H}$. In this theory then, for both the free and the interacting case,

$$\Lambda(\alpha)=\Lambda_R \ ,$$
$$\kappa(\alpha)=\kappa_R \ ,$$
$$c(\alpha)=c_R \ ;$$

nontrivial scaling is not observed for $a$ and $b$.

The $\beta$ functions for these constants can be written as power series in $\lambda$, thus

$$\beta_a=\beta_{a,0}+\beta_{a,1}\lambda+\beta_{a,2}\lambda^2+\cdots \ ,$$
$$\beta_b=\beta_{b,0}+\beta_{b,1}\lambda+\beta_{b,2}\lambda^2+\cdots \ .$$

Toms[9,10] has shown that

$$\beta_{a,0}>0 \ ,$$
$$\beta_{b,0}<0 \ .$$

Thus for the free theory, we find that as $\alpha\to 0$

$$a(\alpha)\to+\infty \text{ as } \ln\alpha$$

and

$$b(\alpha)\to-\infty \text{ as } \ln\alpha \ .$$

For the interacting case we use the analogs of Eqs. (3.10) and (3.11). We need to know the behavior of $\lambda(\alpha)$ which is obtained by solving

$$\frac{\alpha\, d\lambda(\alpha)}{d\alpha}=\beta(\lambda) \ .$$

For non-Abelian gauge fields the $\beta$ functions are known to be[11]

$$\beta(\lambda)=\frac{-\lambda^3}{(4\pi)^2}\left(\tfrac{11}{3}C-\tfrac{4}{3}T_f\right)+\frac{\lambda^5}{(4\pi)^4}\left(-\tfrac{34}{3}C^2+\tfrac{20}{3}CT_f+4C_fT_f\right)+O(\lambda^7) \ . \quad (3.14)$$

For an SU($N$) theory $C=N$ and $T_f$ is the number of fermion fields. For our purposes then

$$\beta(\lambda)=-\frac{\lambda^3}{(4\pi)^2}\left(\tfrac{11}{3}C\right)+\frac{\lambda^5}{(4\pi)^4}\left(-\tfrac{34}{3}C^2\right)+O(\lambda^7) \ . \quad (3.15)$$

Thus since the $\beta$ function begins at $\lambda^3$, the correction to the scaling of $a$ and $b$ due to the nonzero $\lambda$ occurs at order $1/\lambda$. Thus as $\alpha\to\infty$,

$$a(\alpha)-\beta_{a,0}\ln\alpha+O(1/\lambda(\alpha))\to+\infty \quad (3.16)$$

and for $b$

$$b(\alpha)\to\beta_{b,0}\ln\alpha+O(1/\lambda(\alpha))\to-\infty \ , \quad (3.17)$$

where $\lambda(\alpha)$ is given by

$$\ln\alpha=\frac{3(4\pi)^2}{22C}\left[\frac{1}{\lambda^2}-\frac{1}{{\lambda_R}^2}\right]+\tfrac{102}{121}\ln\lambda+O(\lambda^2) \ . \quad (3.18)$$

Thus, we see that the high-energy limit is the same as in the free-field case.

## IV. CONCLUSION

In the high-energy limit of the various theories studied here, the effective semiclassical gravitational action is dominated by the two quadratic curvature terms $\mathscr{G}$ and $\mathscr{H}$. This suggests that the appropriate action for quantum gravity contains only these terms and is asymptotically scale invariant.[1] In this case, the standard, low-energy Einstein terms would then arise an induced self-interaction term of the quantum gravitational field.[12] The fact that the coefficients of the $\ln\alpha$ terms in the effective $a$ and $b$ are different for the different couplings to the semiclassical gravity suggests restrictions on the basic Lagrangian for quantizing the combined gravitational and particle fields.

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*Now at AT&T Bell Laboratories, HO 2E-528, Holmdel, NJ 07733.

†Now at Computer Science Department, University of Utah, Salt Lake City, Utah 84112.

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