Ground-state wave function of linearized gravity

James B. Hartle
Department of Physics, University of California, Santa Barbara, California 93106
(Received 9 December 1983)

The ground-state wave function for linearized gravity is calculated from the Euclidean functional-integral prescription. The result is identical to that obtained from the canonical theory of quantum gravity.

I. INTRODUCTION

A natural prescription for the ground-state wave function of quantum gravity can be given in terms of Euclidean functional integrals.[1-3] Schematically, the amplitude for a three-geometry to occur in this ground state is

$$\Psi_0[\text{three-geometry}]=\sum_{\text{four-geometries}}\exp(-I[g])\ . \tag{1.1}$$

Here, I is the Euclidean gravitational action and the sum is over all Euclidean four-geometries which (1) have a boundary on which the induced three-geometry is the argument of the wave function and (2) satisfy appropriate conditions to specify the ground state.

In the case of spatially closed three-geometries the appropriate condition is that the Euclidean four-geometries be compact. Implemented in simple closed minisuperspace models this prescription (1.1) gives a wave function which may be reasonably interpreted as the ground state in the sense of a state of minimum excitation.[1] In models with more complicated matter interactions it begins to approximate the features of the present universe.[2,3] In the case of asymptotically flat three-geometries the appropriate class of Euclidean four-geometries for defining the ground state are those which are asymptotically flat. Thus one would write

$$\Psi_0[{}^3g_{ij},t]=\int \delta g\, e^{-I[g]}\ , \tag{1.2}$$

where the integral is now over all Euclidean four-geometries which are asymptotically Euclidean (flat) and bounded by an asymptotically flat hypersurface labeled by t on which the induced three-metric is ${}^3g_{ij}$.

It would be desirable to compare the Euclidean functional-integral prescription for the ground-state wave function with the results of the canonical theory for this quantity. There appears to be only one result of this type. This is the calculation by Kuchar[4] of the ground-state wave function for linearized gravity. He finds for the wave function on the physical configuration space

$$\psi_0[h_{ij}^{\rm TT},t]=N\exp\left[-\frac{1}{4l^2}\int d^3k\,\omega_{\vec{k}}h_{ij}^{\rm TT*}(\vec{k})h^{{\rm TT}ij}(\vec{k})\right]\ . \tag{1.3}$$

Here, $h_{ij}^{\rm TT}(\vec{k})$ is a Fourier component of the transverse-traceless part of the deviation of the three-metric from the flat three-metric in rectangular coordinates, $\omega_{\vec{k}}=|\vec{k}|$, N is a normalization factor, and $l=(16\pi G)^{1/2}$ is the Planck length in the units $\hbar=c=1$ we use throughout this paper.

In Sec. III, we shall show that the Euclidean functional-integral prescription of Eq. (1.2) made precise and applied to linearized gravity does yield the ground-state wave function of the canonical theory (1.3). This is not a surprising result. Linearized gravity is the theory of a spin-two field on a flat background and in ordinary flat-space field theory the corresponding functional integral is well known to give the ground-state wave function. Still, it seems useful to set forth the construction in some detail both to demonstrate the congruence of the result with that of the canonical theory and also to indicate how various technical issues arising in the general theory such as the necessity of conformal rotations and the elimination of redundant degrees of freedom are treated in its linearized version. Because of the close connection with flat-space field theory we shall begin (Sec. II) with a brief review of the Euclidean functional-integral construction of the free electromagnetic ground state as a guide and comparison for linearized gravity.

The application of the Euclidean functional-integral prescription to calculate the ground-state wave function for asymptotically flat spacetimes in the full nonlinear Einstein theory is a problem presenting greater challenges than that for the linearized theory. Linearized gravity is the theory whose action is that of Einstein's theory expanded to quadratic order in deviations from flat space. Since the action is quadratic the functional integral for the wave function can be evaluated exactly. We cannot hope to evaluate the functional integral (1.2) for the wave function exactly in the nonlinear theory but one can obtain an approximation using semiclassical methods. In Sec. IV, we consider the semiclassical approximation to the ground-state wave function of the full theory evaluated for three-geometries which are only slightly curved and satisfy the linearization of the Hamiltonian constraint of the full theory. As one might expect this approximate wave function of the full theory coincides with the exact ground-state wave function of the approximate theory; that is, with the ground-state wave function of linearized gravity.

II. GROUND-STATE WAVE FUNCTION FOR ELECTROMAGNETISM

In its simplest and most directly interpretable form the wave function is a function on the configuration space associated with the physical degrees of freedom of a system and of the time. In the case of electromagnetism the physical configuration space might be taken to be spanned by the two transverse components of the three-vector potential $A_i^T(\vec{x})$, at each spatial point. (The range of latin indices is 1 to 3; that of greek indices 0 to 4.) Thus we would write

$$\psi=\psi[A_i^T(\vec{x}),t] . \tag{2.1}$$

This wave function is unconstrained and the scalar product giving the probability interpretation could be written schematically as

$$(\psi_1,\psi_2)=\int \delta A_i^T \psi_1^*[A_i^T(\vec{x}),t]\psi_2[A_i^T(\vec{x}),t] , \tag{2.2}$$

where the functional integral is over the two physical degrees of freedom at each space point.

For some purposes it is convenient to consider the wave function as a function of all three-vector potentials, and write

$$\Psi=\Psi[A_i(\vec{x}),t] . \tag{2.3}$$

Wave functions on this extended configuration space, which now includes the longitudinal part of the vector potential $A_i^L(\vec{x})$, would necessarily satisfy the operator form of the constraint Maxwell's equation

$$(\vec{\nabla}\cdot\vec{E})\Psi[A_i(\vec{x}),t]=0 . \tag{2.4}$$

The scalar product expressed as an integral over the A_i will take a more complicated form. Consideration of wave functions on extended spaces of variables may be useful in theories such as general relativity where it is not as easy to identify the physical degrees of freedom as it is in electrodynamics.

The Euclidean prescription for the electromagnetic ground-state wave function would read as follows:[5]

$$\psi_0[A_i^T,t]=\int \delta A_\mu (\det C^s)^{1/2}\delta[C^s] \times(\det C)^{1/2}\delta[C]\exp(-I[A]) . \tag{2.5}$$

Here, I is the Euclidean electrodynamic action

$$I[A]=\tfrac{1}{4}\int d^4x\, F_{\alpha\beta}F^{\alpha\beta} , \tag{2.6}$$

with

$$F_{\alpha\beta}=\nabla_\alpha A_\beta-\nabla_\beta A_\alpha . \tag{2.7}$$

The integration is over all four-vector potentials which match the prescribed $A_i^T(\vec{x})$ on the boundary surface and which vanish sufficiently fast at Euclidean infinity as discussed more precisely below. Included in the integration are integrals over the components $A_0(\vec{x})$ and $A_i^L(\vec{x})$ on the boundary surface which are unrestricted by the argument of the wave function. $C=0$ is the volume gauge condition to be enforced over the whole of the region to the past of the boundary surface. For definiteness one might consider the Lorentz gauge where

$$C=\nabla_\alpha A^\alpha . \tag{2.8}$$

A condition like (2.8), however, does not completely fix the gauge. For example, $\nabla_\alpha A^\alpha=0$ is preserved by gauge transformations generated by gauge functions Λ which satisfy $\nabla^2\Lambda=0$. This can be used to impose a condition on A_α on a single surface. For example, we might choose

$$C^s=\vec{\nabla}\cdot\vec{A}=0 \tag{2.9}$$

on the boundary surface. Such a condition is needed on the boundary surface as otherwise the integration over A_i^L would be infinite. There is therefore a gauge δ function $\delta[C^s]$ in Eq. (2.5) acting only on the surface to enforce the condition $C^s=0$ there and make the integration finite.

The measure for the functional integral can be made precise by defining the integral as the limit of a multiple integral over the Fourier components $A_\alpha(t,\vec{k})$ of the vector potential on discrete time slices. The measure can be thought of as

$$\text{const}\times\prod_\tau\prod_{\vec{k}}\prod_\alpha dA_\alpha(\tau,\vec{k}) . \tag{2.10}$$

Certainly only configurations whose asymptotic behavior is consistent with a finite action contribute to the functional integral (2.5). In rectangular coordinates this requires the components of A_μ to fall off faster than $1/r$ at infinity. In fact, we shall find it sufficient to impose stronger conditions so that Fourier transforms of the A_μ exist and the familiar formal manipulations of functional integrals can be performed. These conditions will restrict the falloff of the prescribed $A_i^T(\vec{x})$ on the boundary surface but not in any essential way as it would be physically reasonable to restrict these radiative degrees of freedom to have compact support.

The wave function $\Psi[A_i,t]$ on the extended configuration space of three-vector potentials is given by a functional integral of the same form as (2.5) but without the surface gauge-fixing δ function, and integration over A_i^L on the boundary surface. The wave function ψ can thus be recovered from Ψ by integrating over A_i^L with an appropriate δ function to fix the gauge.

The functional integral which gives the wave function is easily evaluated by translating the integration variables to ones centered about an extremum of the action which matches the argument of the wave function on the boundary surface and which vanishes at infinity. To see this, it is sufficient to consider gauge conditions which are linear in the A_μ such as Eq. (2.8). Under translations, these are unchanged, the measure [Eq. (2.10)] is invariant, and the quadratic action is a sum of the action for the extremum configuration and that for the translated variables. The entire dependence on the argument of the wave function then comes from the action of the extremum configuration which is easily evaluated.

To be specific in the electrodynamic case let us take the Lorentz condition (2.8) for C and (2.9) for C^s. A configuration which extremizes the action, satisfies the required conditions at the boundary surface and at infinity, and satisfies the gauge conditions is, in spatial Fourier components,

$$\hat{A}_0(\tau,\vec{k})=0,\quad \hat{A}^L_i(\tau,\vec{k})=0\ ,$$
$$\hat{A}^T_i(\tau,\vec{k})=A^T_i(\vec{k})e^{\omega_{\vec{k}}(\tau-t)}\ . \tag{2.11}$$

Here $A^T_i(\vec{k})$ is the Fourier component of the argument of the wave function at time t and $\omega_{\vec{k}}=|\vec{k}|$. Introducing new integration variables a_μ by

$$\hat{A}_\mu=A_\mu+a_\mu\ , \tag{2.12}$$

we find that the a_μ vanish both on the boundary surface and at infinity. The translated functional integral is $\exp(-I[\hat{A}])$ times an integral of the form (2.5) over the a_μ. The latter contributes only a constant factor N independent of A^T_i. Evaluating the action for the extremum configuration the result for the ground-state wave functional is

$$\psi_0[A^T_i(\vec{x}),t]=N\exp\left[-\tfrac{1}{2}\int d^3k\,\omega_{\vec{k}}A^{T*}_i(\vec{k})A^{Ti}(\vec{k})\right]\ . \tag{2.13}$$

This is the correct and classic result. Each normal mode of the electromagnetic field is in a harmonic-oscillator ground state.

A parallel calculation of the wave function on the extended configuration space would yield the same result up to normalization factor. One thereby sees directly that the constraint

$$(\nabla\cdot\vec{E})\Psi_0=i\nabla\frac{\partial\Psi_0}{\delta A_i}=i\nabla_i\frac{\delta\Psi_0}{\delta A^L_i}=0 \tag{2.14}$$

is satisfied since Ψ_0 is independent of $A^L_i(\vec{x})$. Further, the integral connecting Ψ_0 and ψ_0 can be explicitly evaluated. The integral over $A^L_i(\vec{x})$ of a Ψ_0 of the form (2.11) times a gauge-fixing δ function gives a result of the same form as (2.11) up to normalizing factor.

III. GROUND-STATE WAVE FUNCTION FOR LINEARIZED GRAVITY

Linearized gravity is the field theory whose action is the Einstein action expanded to quadratic order in deviations of the metric from flat space. In contrast to Einstein's theory the physical degrees of freedom of the linearized theory are readily identified.[6] They are the two independent, transverse-traceless (TT) components of the three-metric of a nearly flat spacelike surface. More explicitly, consider a spacelike surface which becomes a flat surface of constant Minkowski time t when the metric perturbations vanish. Write the three-metric on this surface as[7]

$$^3g_{ij}=\delta_{ij}+h_{ij}(\vec{x}) \tag{3.1}$$

and decompose the deviations h_{ij} as

$$h_{ij}(\vec{x})=h^{TT}_{ij}(\vec{x})+h^T_{ij}(\vec{x})+h^L_{ij}(\vec{x})\ , \tag{3.2}$$

where

$$\nabla^i h^{TT}_{ij}=0,\quad \nabla^i h^T_{ij}=0,\quad h^{TTi}{}_i=0\ . \tag{3.3}$$

The physical configuration space is spanned by the two independent components of $h^{TT}_{ij}(\vec{x})$. A wave function on the physical configuration space is thus written

$$\psi=\psi[h^{TT}_{ij}(\vec{x}),t]\ . \tag{3.4}$$

The Euclidean functional-integral prescription for the ground-state wave function $\psi_0(h^{TT}_{ij},t)$ for linearized gravity is to sum $\exp[-(\text{Euclidean action})]$ over those nearly flat Euclidean four-geometries which match the given h^{TT}_{ij} on the boundary hypersurface labeled by t and which are asymptotically flat at Euclidean infinity to the past of this surface. To make this construction explicit we need first to consider the Euclidean action for linearized gravity. This is constructed from the Euclidean action for Einstein's theory[8]

$$l^2I[g]=-2\int_{\partial M}K(^3g)^{1/2}d^3x-\int_M R(g)^{1/2}d^4x \tag{3.5}$$

by writing the Euclidean four-metric as

$$g_{\alpha\beta}(x)=\delta_{\alpha\beta}+h_{\alpha\beta}(x) \tag{3.6}$$

and expanding (3.5) to quadratic order in $h_{\alpha\beta}$. For constructing the ground-state wave function we take ∂M to be a flat slice in flat Euclidean space and M to be the region of Euclidean flat space to the past of this surface. The $h_{\alpha\beta}$ of interest are those which vanish sufficiently rapidly at infinity so that $g_{\alpha\beta}$ becomes asymptotically flat and so that the action is finite. With these restrictions the Euclidean action for linearized gravity is

$$l^2I_2[h]=\tfrac{1}{2}\int_{\partial M}d^3x\,h_{ij}\dot{\pi}^{ij}+\tfrac{1}{2}\int_M d^4x\,h_{\alpha\beta}\dot{G}^{\alpha\beta}+C_\infty\ . \tag{3.7}$$

The constituents of this expression are as follows: $\dot{G}_{\alpha\beta}$ is the linearized Einstein tensor

$$\dot{G}_{\alpha\beta}=\tfrac{1}{2}(-\nabla^2\bar{h}_{\alpha\beta}-\delta_{\alpha\beta}\nabla_\gamma\nabla_\delta\bar{h}^{\gamma\delta}+\nabla_\alpha\nabla_\gamma\bar{h}^\gamma{}_\beta+\nabla_\beta\nabla_\gamma\bar{h}^\gamma{}_\alpha)\ , \tag{3.8}$$

where

$$\bar{h}_{\alpha\beta}=h_{\alpha\beta}-\tfrac{1}{2}\delta_{\alpha\beta}h \tag{3.9}$$

and the usual conventions are followed that indices are raised and lowered with the flat metric and $h=h^\gamma{}_\gamma$. $\dot{\pi}^{ij}$ is the linearized "Euclidean momentum conjugate to h_{ij}." It is

$$\dot{\pi}_{ij}=\dot{K}_{ij}-\delta_{ij}\dot{K}\ , \tag{3.10}$$

where $\dot{K}_{ij}$ is the perturbation of the extrinsic curvature of the boundary surface and $\dot{K}$ is its trace. If we use rectangular coordinates where the boundary surface is labeled by $\tau=t=\text{const}$,

$$\dot{K}_{ij}=\tfrac{1}{2}(\nabla_\tau h_{ij}-\nabla_i h_{\tau j}-\nabla_j h_{\tau i})\ . \tag{3.11}$$

The constant C_∞ is the contribution, if any, of the surface term in (3.5) at Euclidean infinity. It is quadratic in $h_{\alpha\beta}$, but its exact form will not be important for us.

The action (3.7) is left unchanged by the gauge transformation

$$h_{\alpha\beta}\rightarrow h_{\alpha\beta}+\nabla_{(\alpha}\xi_{\beta)}\ . \tag{3.12}$$

In fact, the form of the action is determined by the requirements that it give the correct linearized field equations, contain no higher than first derivatives, and be invariant under gauge transformations including those which do not vanish on the boundary surface. An expression equivalent to (3.7) with a more compact volume term is[9]

$$l^2 I_2[h]=\tfrac{1}{2}\int d^4x\left(\tfrac{1}{2}\nabla_\alpha \bar{h}_{\beta\gamma}\nabla^\alpha h^{\beta\gamma}-\nabla_\alpha \bar{h}^{\alpha\beta}\nabla_\gamma \bar{h}^{\gamma}{}_{\beta}\right)+(\text{surface terms})\ . \tag{3.13}$$

The ground-state wave function for linearized gravity may be constructed from a Euclidean functional integral which has the form

$$\psi_0[h_{ij}^{\rm TT},t]=\int \delta h_{\alpha\beta}\left[\prod_\alpha (\det C_s^\alpha)^{1/2}\delta[C_s^\alpha]\right]\times\left[\prod_\beta (\det C^\beta)^{1/2}\delta[C^\beta]\right]\exp(-I_2[h])\ . \tag{3.14}$$

The ingredients of this expression are as follows: I_2 is the action for linearized gravity discussed above. C^α and C_s^α are four-volume and four-surface gauge-fixing conditions, respectively. A frequently convenient choice for the C^α are the four conditions

$$C^\alpha(h)=\nabla_\alpha \bar{h}^{\alpha\beta}=0\ . \tag{3.15}$$

The four surface conditions are necessary to remove the gauge freedom left unfixed by the conditions $C^\alpha=0$. For example, the condition (3.15) is left unchanged by gauge transformations of the form (3.12) with

$$\nabla^2\xi^\beta=0\ . \tag{3.16}$$

If we consider only gauge transformations which vanish at Euclidean infinity, thereby preserving the vanishing of the $h_{\alpha\beta}$ at infinity, we can use this freedom to impose four conditions on the $h_{\alpha\beta}$ on the boundary surface. Together with the four conditions C^α these reduce the ten $h_{\alpha\beta}$ to two on the boundary surface—the correct number of specifiable degrees of freedom on the boundary surface. The integral in (3.14) is over all linearized field configurations $h_{\alpha\beta}$ which match the prescribed $h_{ij}^{\rm TT}$ on the boundary surface and which vanish at Euclidean infinity to the past of this surface sufficiently fast to yield finite action. The measure may be made concrete by dividing the Euclidean time up into discrete slices of constant τ and considering the integral as the limit of a multiple integral of the spatial Fourier components of $h_{\alpha\beta}$ on each slice. For the measure one would write

$$\text{const}\times\prod_\tau\prod_{\vec{k}}\prod_{\alpha,\beta} dh_{\alpha\beta}(\tau,\vec{k})\ . \tag{3.17}$$

There are gauge-fixing δ functions and associated determinants for each mode and for each slice including the boundary surface. On the boundary surface only $h_{ij}^{\rm TT}$ is fixed by the argument of the wave function; all other components of $h_{\alpha\beta}$ on the boundary surface are integrated over.

The construction of ψ_0 given above is parallel to that for electromagnetism and would be completely so were it not for the fact that the Euclidean action for linearized gravity, like that for general relativity, is not positive definite. One can easily verify this by calculating the action in the form (3.13) for the particular perturbation $h_{\alpha\beta}=\frac{1}{4}\delta_{\alpha\beta}h$ with h of compact support away from the boundary surface. This is not a difficulty to be resolved by choice of gauge because the action is gauge invariant. As in the full theory, one can construct a well-defined functional integral by distorting the contours corresponding to integrations over the conformal components of the metric to complex values. This was discussed for the full theory by Gibbons, Hawking, and Perry[10] and for the linearized theory by Gibbons and Perry.[9] We shall follow their procedure here. From a Hamiltonian perspective this procedure is simply a manipulation of the integrations of the unphysical degrees of freedom necessary to obtain a convergent functional integral.[11]

We first decompose the metric perturbations into conformal equivalence classes by writing

$$h_{\alpha\beta}(x)=\varphi_{\alpha\beta}(x)+2\delta_{\alpha\beta}\chi(x)\ . \tag{3.18}$$

To fix the decomposition an essentially arbitrary condition must be imposed on $\varphi_{\alpha\beta}$. Following the guide of the full theory we require the gauge-invariant condition that the linearized scalar curvature constructed from $\varphi_{\alpha\beta}$ vanish:

$$\dot{R}(\varphi_{\alpha\beta})=\nabla^\alpha\nabla^\beta\varphi_{\alpha\beta}-\nabla^2\varphi=0\ . \tag{3.19}$$

To construct the appropriate χ for given $h_{\alpha\beta}$, one would solve

$$\nabla^2\chi=-\tfrac{1}{6}(\nabla^\alpha\nabla^\beta h_{\alpha\beta}-\nabla^2 h)\ . \tag{3.20}$$

A useful set of boundary conditions to make the decomposition unique is to require that χ vanish at infinity and on the boundary surface. From the electrostatic analogy such a χ always exists.

As a result of the decomposition (3.19) and the boundary condition for χ the action splits into two parts:

$$l^2 I_2[h_{\alpha\beta}]=l^2 I_2[\varphi_{\alpha\beta}]-6\int d^4x(\nabla_\alpha\chi)(\nabla^\alpha\chi)\ . \tag{3.21}$$

The second term is negative definite. The first is positive semidefinite on square-integrable tensors $\varphi_{\alpha\beta}$ which satisfy $\dot{R}(\varphi_{\alpha\beta})=0$ [Eq. (3.19)] and which vanish on the boundary surface. In fact, when the Fourier transform of $\varphi_{\alpha\beta}$ exists

$$l^2 I_2[\varphi_{\alpha\beta}]=\tfrac{1}{4}\int\frac{d^4k}{(2\pi)^4}k^2\tilde{\varphi}^*_{\alpha\beta}(k)\tilde{\varphi}^{\alpha\beta}(k)\ , \tag{3.22}$$

where

$$\tilde{\varphi}_{\alpha\beta}(k)=P_\alpha{}^\gamma\varphi_{\gamma\delta}P^\delta{}_\beta \tag{3.23}$$

and

$$P_{\alpha\beta}=\delta_{\alpha\beta}-k_\alpha k_\beta/k^2\ . \tag{3.24}$$

The form (3.22) is positive semidefinite. Put differently but equivalently, the operator $\dot{G}_{\alpha\beta}$ has positive or zero eigenvalues on the $\varphi_{\alpha\beta}$ satisfying the above conditions. Although the asymptotic behavior needed for this result is stronger than that required simply by finite action, this will be sufficient to construct the functional integral as we shall now show.

Corresponding to the decomposition (3.18) we can decompose the integration in Eq. (3.14) over the $h_{\alpha\beta}$ into an integration over χ and an integration over $\varphi_{\alpha\beta}$ satisfying $\dot{R}(\varphi_{\alpha\beta})=0$. Since decomposition is linear the Jacobian of this transformation of variables is a constant and we can write formally

$$\delta h_{\alpha\beta}=\text{const}\times\delta\varphi_{\alpha\beta}\delta\chi\delta[\dot{R}(\varphi_{\alpha\beta})] \tag{3.25}$$

and integrate over unconstrained $\varphi_{\alpha\beta}$ and χ. To make the functional integral well defined we rotate the χ contour of integration to purely imaginary values. The action (3.21) is then positive and the functional integral (3.14) convergent.

To evaluate the functional integral for the ground-state wave function explicitly we translate the integration over $\varphi_{\alpha\beta}$ about a classical solution of the linearized field equations which matches the prescribed h_{ij}^{TT} on the boundary surfaces, vanishes at Euclidean infinity to the past of this surface, and satisfies the gauge conditions $C^\alpha=0$ and $C_s^\alpha=0$. For the ensuing argument to work the gauge conditions must be linear and certainly chosen so the above requirements can be satisfied consistently. The conditions may be imposed on each conformal equivalence class (i.e., on the $\varphi_{\alpha\beta}$) since χ is a gauge invariant. For example, we might require the conditions (3.15) on $\varphi_{\alpha\beta}$ and in addition that φ and the transverse part of φ_{ij} vanish on the boundary surface. The unique solution of linearized field equations which satisfies these requirements, expressed in terms of the spatial Fourier components of h_{ij}^{TT}, is

$$\begin{aligned}\hat{\varphi}_{ij}(\tau,\vec{k})&=h_{ij}^{\text{TT}}(\vec{k})e^{\omega_k(\tau-t)}\ ,\\ \hat{\varphi}_{\tau\alpha}&=0\ .\end{aligned} \tag{3.26}$$

We now introduce new integration variables $f_{\alpha\beta}$ by the transformation

$$\varphi_{\alpha\beta}=\hat{\varphi}_{\alpha\beta}+f_{\alpha\beta}\ . \tag{3.27}$$

The measure (3.25) is invariant under this transformation as are the gauge conditions. The action decomposes into the sum of the action for $f_{\alpha\beta}$ and that for $\hat{\varphi}_{\alpha\beta}$. The functional integral in Eq. (3.14) is thus a factor $\exp(-I_2[\hat{\varphi}])$ times a functional integral of identical form over $f_{\alpha\beta}$ and χ. Since $f_{\alpha\beta}$ and χ vanish on the boundary surface this integral is independent of h_{ij}^{TT} and contributes only to the normalizing factor. In fact, since we have already verified that the action is positive on $f_{\alpha\beta}$ and χ which vanish on the boundary surface, which are constrained by Eq. (3.19) and which vanish sufficiently fast at infinity, the integral is convergent and could be carried out. Evaluating the action (3.7) on the solution (3.26) we find for the ground-state wave function of linearized gravity

$$\begin{aligned}\psi_0[h_{ij}^{\text{TT}}(\vec{x}),t]&\\ &=N\exp(-I_2[\hat{\varphi}])\\ &=N\exp\left[-\frac{1}{4l^2}\int d^3k\,\omega_k h_{ij}^{\text{TT}*}(\vec{k})h^{\text{TT}ij}(\vec{k})\right]\ .\end{aligned} \tag{3.28}$$

This is the wave function for the ground state found by Kuchar by Hamiltonian methods.[4]

One can also use the Euclidean functional integral to construct the ground-state wave function on an extended configuration space. For the full theory of general relativity the natural extended configuration space is a space of three-metrices on a spacelike surface. This is not the case for its linearized version. This is because the linearized version of the Hamiltonian constraint is not a relation constraining coordinates and momenta. Rather it is a condition on the configuration space itself:

$$^3\dot{R}(h_{ij})=\nabla_i\nabla_j h^{ij}-\vec{\nabla}^2 h^i{}_i=0\ . \tag{3.29}$$

As argued by Kuchar,[4] the natural extended configuration space for linearized gravity is not the space of all slightly nonflat three-metrics on a $t=\text{const}$ surface but rather only those which satisfy (3.29). We shall adopt this point of view in this paper. The Euclidean functional-integral construction of the ground-state wave function on this extended configuration space takes the same form as Eq. (3.14) with two exceptions. First, a three-metric h_{ij} satisfying (3.29) is fixed on the boundary rather than h_{ij}^{TT}. Second, as a consequence, the three gauge-fixing δ functions C_s^i which correspond to fixing the spatial coordinates in the boundary surface are omitted. There remain the four conditions C^α and the one surface condition to reduce the ten $h_{\alpha\beta}$ to five on the surface—the correct number of specifiable functions for a three-metric satisfying one constraint.

The functional integral can be made positive definite and evaluated by translation about a solution of the linearized field equations as before. For example, suppose one uses the gauge conditions (3.15) in each conformal equivalence class and $\varphi=0$ as the remaining surface gauge-fixing condition. The linearized field equations in the gauge (3.15) are

$$\nabla^2\varphi_{\alpha\beta}=0\ , \tag{3.30}$$

and the solution which matches the boundary conditions and satisfies the gauge conditions is

$$\begin{aligned}\hat{\varphi}_{ij}(\tau,\vec{k})&=h_{ij}(\vec{k})e^{\omega_{\vec{k}}(\tau-t)}\ ,\\ \hat{\varphi}_{\tau j}(\tau,\vec{k})&=-i(k^i/\omega_k)h_{ij}(\vec{k})e^{\omega_{\vec{k}}(\tau-t)}\ ,\\ \hat{\varphi}_{\tau\tau}(\tau,\vec{k})&=-h^i{}_i(\vec{k})e^{\omega_{\vec{k}}(\tau-t)}\ .\end{aligned} \tag{3.31}$$

It is at this point that the assumption of the constraint (3.29) enters the calculation in an essential way. Without it (3.31) would not be a solution of the field equations in the gauge chosen, the action would not be extremized, and a translation of the integration variables about (3.31) would lead to cross terms in the action between the integration variables and $\hat{\varphi}_{\alpha\beta}$. With the constraint, Eq. (3.31) is a solution and the discussion proceeds as before. The result is that, up to a normalization factor, the ground-state wave function on the extended configuration space of h_{ij} constrained by (3.29) is identical to the result in (3.28). That is, adjusting the normalization we can write

$$\Psi_0[h_{ij}(\vec{x}),t]=\psi_0[h_{ij}^{\text{TT}}(\vec{x}),t]\ . \tag{3.32}$$

From this explicit result we can see explicitly how the constraints of linearized gravity are satisfied. The Hamiltonian constraint (3.29) is satisfied because it is a constraint defining the extended configuration space. The constraints

$$\nabla_j \hat{\pi}^{ij}(\vec{x})\Psi_0 = -i\nabla_j \frac{\delta\Psi_0}{\delta h_{ij}(\vec{x})} = 0 , \tag{3.33}$$

which express the independence of the wave function on the choice of coordinates in the three-surface, are satisfied because from (3.32) Ψ_0 is a function only of the gauge-invariant combination $h_{ij}^{\mathrm{TT}}(\vec{x})$.

IV. SEMICLASSICAL APPROXIMATION FOR NEARLY FLAT THREE-SURFACES

In the preceding section we have used Euclidean functional-integral methods to evaluate the ground-state wave function for the linearized theory of gravity. The linearized theory was considered as a theory of a spin-two field on a flat background fully independent of general relativity. In this section we shall consider the semiclassical approximation to the ground-state wave function for the full, nonlinear Einstein theory evaluated on three-geometries which are flat asymptotically and nearly flat everywhere. One would expect these two wave functions to be identical and they are.

Equation (1.2) displays the functional integral which gives the ground-state wave function of the full theory with the action in Eq. (3.5). We seek to evaluate it semiclassically for three-surfaces which asymptotically approach a flat plane in Minkowski space labeled by time t on which the metric deviates by only a small amount h_{ij} from the flat metric,

$${}^3g_{ij}(\vec{x}) = \delta_{ij} + h_{ij}(\vec{x}) . \tag{4.1}$$

Consistent with our earlier discussion of the linearized theory we shall assume that the h_{ij} are restricted by the linearized Hamiltonian constraint of the full theory [Eq. (3.29)].

To evaluate the integral (1.2) semiclassically one needs first to find those Euclidean four-geometries which make the action stationary, are asymptotically flat, and match Eq. (4.1) on the boundary three-surface. We shall begin with the case $h_{ij}=0$ when the boundary three-surface is a flat plane. Euclidean four-geometries which make the action stationary satisfy the Euclidean field equations $R_{\alpha\beta}=0$. These field equations imply constraints on the intrinsic and extrinsic curvatures of the boundary surface exactly as do the Einstein equations in Lorentzian spacetimes. If K_{ij} is the extrinsic curvature of the three-surface, ${}^3R$ its three curvature scalar, and D_i the derivative in the surface, these constraints take the form

$$K_{ij}K^{ij} - K^2 + {}^3R = 0 , \tag{4.2a}$$

$$D_i(K^{ij} - \delta^{ij}K) = 0 . \tag{4.2b}$$

For a curved three-surface these constraints differ from the Lorentzian ones only by the sign of the ${}^3R$ term in (4.2a). For the flat boundary surface under consideration here the constraints therefore are identical to those in the Lorentzian theory.[12]

The only solution of these constraints for a flat surface which is asymptotically a plane in Minkowski space is a surface whose extrinsic curvature vanishes everywhere:

$$K_{ij} = 0 . \tag{4.3}$$

More precisely, in an asymptotically Cartesian coordinate system the metric of an asymptotically flat solution of the Euclidean field equations falls off as $g_{\alpha\beta} \sim O(1/r^4)$, where r is the asymptotic Euclidean distance. (See Ref. 13 for a discussion.) The induced extrinsic curvature on a surface which asymptotically coincides with one of the coordinate planes must therefore also fall off as $K_{ij} \sim O(1/r^4)$. This is enough to ensure that the only regular solution of Eqs. (4.2) for a flat three-geometry is (4.3). (See the Appendix for demonstration and references.)

Suppose there were a nonflat solution of the Euclidean field equations with a flat asymptotically planar boundary surface and which was asymptotically flat to the past of this surface. By the above argument this surface would have vanishing extrinsic curvature. The "time-reversed" solution would have a flat asymptotically planar boundary and be asymptotically flat to its future. The two solutions could be joined together at the boundary surface to produce a new solution of the Euclidean field equations because the junction conditions[12]—the matching of intrinsic geometry and extrinsic curvature—would be clearly satisfied at the boundary surface. Such a solution would be asymptotically flat in all Euclidean directions. But flat space is the unique asymptotically flat solution of the Euclidean field equations.[14,13] Thus flat space is the unique stationary point of the Euclidean action which is asymptotically flat with a flat boundary three-surface.

The Euclidean four-geometries which make the action stationary when the boundary is slightly curved [Eq. (4.3)] will be small perturbations about flat space. Were this not the case one would not recover flat space as the unique stationary geometry when the boundary curvature is taken to zero. The perturbations in the metric satisfy the linearized Euclidean field equations, vanish at infinity, and match the prescribed $h_{ij}(\vec{x})$ on the boundary. We have already shown that there is a unique solution of the linearized Euclidean field equations which matches a given constrained perturbation in the three-metric on the boundary surface and which vanishes asymptotically to the past of this surface. We conclude that the unique stationary points of the Euclidean action for slightly curved three-geometries are the unique perturbations of flat space already discussed in the context of the linear theory.

The rest of the calculation of the semiclassical approximation is straightforward. We have already evaluated the action to lowest order in the h_{ij} at the stationary solution; it is the action of the linear theory [Eq. (3.7)]. There remains the calculation of the functional integral (1.2) over the fluctuations about the stationary configuration. To obtain the semiclassical approximation we should expand the action to quadratic order in the deviations from the stationary configuration and evaluate the resulting Gaussian function integral. Denoting this integral by $P[h_{ij}]$ we can write for the semiclassical approximation

$$\psi_0[h_{ij}^{\rm TT},t] \approx P[h_{ij}]\exp(-I_2[h_{ij}]) \ . \tag{4.4}$$

The prefactor P can be expanded in the deviations h_{ij}. In lowest order it is the functional integral of the fluctuations about a semi-infinite half of flat space and therefore contributes only a normalization factor to (4.4). Thus, in the approximation where the prefactor and the exponent of the semiclassical approximation are expanded to their lowest nonvanishing orders in h_{ij}, the semiclassical ground-state wave function for the full theory of gravitation coincides with the exact ground-state wave function of the linearized theory.

ACKNOWLEDGMENTS

Special thanks are due Stephen Hawking for many conversations on this subject. This paper grew out of joint work with him. The author is grateful for useful conversations with D. Eardley, D. Gross, K. Kuchar, G. Horowitz, and J. York. He would like to thank D. Eardley and G. Horowitz for permission to publish their demonstration here. This work was supported in part by the National Science Foundation under Grant No. PHY 81-07384.

APPENDIX: A FLAT SLICE IS A PLANE

Consider a slice of spacetime whose intrinsic geometry is flat and which is asymptotically planar in the sense that its extrinsic curvature vanishes at infinity faster than $r^{-3/2}$,

$$K_{ij} \sim O(1/r^{3/2+\epsilon}) \tag{A1}$$

(in a rectangular system of coordinates). On such a slice the unique solution of the initial-value equations of general relativity has vanishing extrinsic curvature everywhere:

$$K_{ij}=0 \ , \tag{A2}$$

i.e., it is a plane. This result appears widely known among workers on the initial-value problem[15] and a proof of it has been given by Arnowitt and Deser.[16] As the author has learned from D. M. Eardley and G. T. Horowitz, a simple proof can be given starting from the positive-energy theorem. For completeness, the proof kindly supplied to the author by Dr. D. M. Eardley and Dr. G. T. Horowitz is sketched here.

A flat three-geometry has zero energy. From the positive-energy theorem[17,13] it follows that a flat three-geometry and any K_{ij} satisfying (A1) must be initial data for flat spacetime. Since the flat surface can be embedded in flat spacetime the Gauss-Codazzi equations imply

$$K_{ij}K_{lk}-K_{ik}K_{lj}=0 \ , \tag{A3}$$

$$\nabla_{[i}K_{j]k}=0 \ . \tag{A4}$$

Equation (A3) implies that K_{ij} is of rank one, that is,

$$K_{ij}=\pm V_i V_j \tag{A5}$$

for some V_i. Inserting this in (A4), one finds

$$V_k \nabla_{[i}V_{j]}+V_{[j}\nabla_{i]}V_k=0 \ . \tag{A6}$$

Contract Eq. (A6) with V^k to find

$$\nabla_{[i}VV_{j]}=0 \ , \tag{A7}$$

where $V=(V^iV_i)^{1/2}$. In flat R^3 this implies that VV_j is the gradient of some function f:

$$VV_i=\nabla_i f \ . \tag{A8}$$

The vector V^i is therefore orthogonal to a family of nonintersecting two-surfaces which are the level surfaces of f. The extrinsic curvature of these two-surfaces may be computed by projecting (A6) perpendicular to V^i on the indices i and k. One finds that $\nabla_i V_j$ projected in the surface vanishes, but, since V_i lies along the unit normal, this implies the extrinsic curvature vanishes. The intrinsic curvature of these two-surfaces may be computed by using the Gauss-Codazzi equations. Since the embedding space is flat and the extrinsic curvature vanishes the intrinsic curvature is zero. Thus V^i is orthogonal to a family of nonintersecting, flat, two-planes. It must, therefore, have the form

$$V^i=WZ^i \ , \tag{A9}$$

where W is some function and Z^i a constant vector, $\nabla_i Z_j=0$. Substitute (A9) into (A6) to find

$$Z_{[i}\nabla_{j]}W=0 \tag{A10}$$

so that $\nabla_i W$ lies along Z_i and W is a function only of the Cartesian coordinate z in this direction, $W=W(z)$.

The argument from Eqs. (A3) and (A4) to this point has been essentially local. To proceed further the condition (A1) that the slice be asymptotically planar must be invoked because

$$K_{ij}=W^2(z)Z_iZ_j \tag{A11}$$

is a nontrivial solution of (A3) and (A4) corresponding to "cylindrical" deformations of a flat three-surface in a flat four-dimensional space. However, Eq. (A11) vanishes at infinity in the directions perpendicular to Z^i only if $W=0$. Thus, the only solution of (A3) and (A4) which vanishes at infinity is $K_{ij}=0$ and the result is proved.

[1] J. B. Hartle and S. W. Hawking, Phys. Rev. D 28, 2960 (1983).

[2] S. W. Hawking (unpublished).

[3] S. W. Hawking, in Proceedings of the 1983 Les Houches Summer School on Quantum Gravity (unpublished).

[4] K. Kuchar, J. Math. Phys. 11, 3322 (1970).

[5] A general reference for the detailed issues involved in constructing amplitudes in gauge theories using functional integrals is L. D. Faddeev, Teor. Mat. Fiz. 1, 3 (1969) [Theor. Math. Phys. 1, 1 (1970)].

[6] See, e.g., R. Arnowitt and S. Deser, Phys. Rev. 113, 745 (1959); or R. Arnowitt, S. Deser, and C. Misner, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962).

[7] We thus suspend the convention of Refs. 1—3 of using h_{ij} to denote the full three-metric and use it here in the familiar convention for the deviations of the three-metric from the flat metric.

[8] For a review of the Euclidean functional-integral approach to

quantum gravity, see S. W. Hawking, in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).
[9]G. W. Gibbons and M. Perry, Nucl. Phys. B146, 90 (1978).
[10]G. W. Gibbons, S. W. Hawking, and M. Perry, Nucl. Phys. B138, 141 (1978).
[11]J. B. Hartle (unpublished).
[12]See, e.g., C. Misner, K. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, California, 1973), Chap. 21.
[13]E. Witten, Commun. Math. Phys. 80, 381 (1981).
[14]R. Schoen and S.-T. Yau, Phys. Rev. Lett. 42, 547 (1979).
[15]In particular, Professor J. W. York has kindly informed the author that he was familiar with the result and has provided Ref. 16.
[16]R. Arnowitt and S. Deser, Ann. Phys. (N.Y.) 23, 318 (1963).
[17]R. Schoen and S.-T. Yau, Commun. Math. Phys. 65, 45 (1979); Phys. Rev. Lett. 43, 1457 (1979).