PHYSICAL REVIEW D VOLUME 29, NUMBER 12 15 JUNE 1984

Quantum fluctuations of the relativistic scalar plasma in the Hartree-Vlasov approximation

J. Diaz Alonso and Rémi Hakim*
Groupe d'Astrophysique Relativiste, Observatoire de Paris-Meudon, 92190 Meudon, France
(Received 23 July 1981; revised manuscript received 18 August 1983)

The quantum fluctuations of the relativistic quantum scalar plasma (i.e., a system of spin-$\frac{1}{2}$ fermions interacting through the exchange of scalar particles via a Yukawa-type interaction) are considered within the context of the covariant Wigner-function approach studied elsewhere. The usual infinities occurring in the conventional many-body theory appear as a consequence of a vacuum Wigner function. They are removed in the Hartree-Vlasov approximation for thermal equilibrium. Results previously obtained by Chin are recovered. The effect of these quantum fluctuations on abnormal matter is briefly discussed. For the sake of illustration, numerical results are given and compared to those first obtained by Kalman.

I. INTRODUCTION

The relativistic quantum scalar plasma (an expression due to Kalman;[1,2] see also Ref. 3 for the classical case) is the many-body system characterized by the Lagrangian

$$\mathscr{L}=\overline{\psi}(i\gamma\cdot\partial-m)\psi+g\overline{\psi}\psi\phi+\frac{1}{8\pi}[(\partial\phi)^2-\mu_R{}^2\phi^2]-\mathscr{L}_c\ , \tag{1}$$

where ψ is a spin-$\frac{1}{2}$ field, ϕ is a scalar field, and $\mathscr{L}_c$ represents the most general counterterms that can be introduced within the context of perturbation theory (we come back to this question below), i.e.,

$$4\pi\mathscr{L}_c=\frac{\alpha}{1!}\phi+\frac{\delta\mu^2}{2!}\phi^2+\frac{\gamma}{3!}\phi^3+\frac{\lambda}{4!}\phi^4\ . \tag{2}$$

Although such a system is not supposed to represent any actual physical system, it is, however, extremely useful as a "laboratory" where methods and techniques can be used. Furthermore, if one insists on a possible physical role, one may think either of the SLAC bag model[4] or "abnormal nuclear matter,"[5] etc. However, particularly in this paper, it will be considered merely as a useful model.

In other papers[6,7] (and references quoted therein) we have developed new techniques—somewhat reminiscent of plasma physics techniques— for the study of relativistic dense matter, based essentially on the use of a covariant Wigner function which, for fermions, reads[6,7]

$$F(x,p)=\frac{1}{(2\pi)^4}\int d^4R\,\exp(-ip\cdot R)\langle\overline{\psi}(x+R/2)\otimes\psi(x-R/2)\rangle\ , \tag{3}$$

where the angular brackets denote a quantum-statistical-mechanical average (i.e., $\langle\cdots\rangle=\mathrm{Tr}[\rho\cdots]$, ρ being the density operator). From Eq. (3) one can derive, for instance, the fermion four-current

$$J^\mu(x)=\mathrm{Tr}\int d^4p\,\gamma^\mu F(x,p) \tag{4}$$

and the momentum-energy tensor

$$T^{\mu\nu}(x)=\mathrm{Tr}\int d^4p\,\gamma^\nu p^\mu F(x,p)\ . \tag{5}$$

As a result of this formalism, some nonperturbative approximation schemes suggested by plasma physics have been studied[6,7] and, in particular, the Hartree-Vlasov approximation of the relativistic quantum scalar plasma in thermodynamical equilibrium.[8] As expected, the usual infinities of quantum field theory show up and hence the model has to be renormalized (see Sec. III). However, the conventional renormalization procedure (i.e., absorption of infinities by appropriate counterterms) has been devised within the context of perturbation theory so that it remains to show that it may also be used (or possibly modified) within other approximation schemes which are not analytical in the coupling constant. For instance, if we deal with some cumulant expansion[6–8] then the Hartree-Vlasov approximation is the simplest approximation, dealing with collective effects only, while the next one also contains two-body correlations. Therefore, if one succeeds in getting rid of the infinities of the Hartree-Vlasov approximation—which is the main purpose of this paper—is this procedure consistent with the next approximation (two-body correlations) and the following ones (three-, four-, etc. body correlations)? While such a consistency can be proved in perturbation theory it is *a priori* not so for other approximation techniques.

In this paper, our aim is more modest since we only want to explore the source of the infinities occurring in the simplest approximation (Hartree-Vlasov) and the way they can be removed.

In Sec. II a brief summary of the results already obtained is given while in Sec. III the various infinities are removed within the usual procedure (regularization of divergent integrals and absorption of infinities in counterterms). Section IV deals with the determination of arbitrary constants occurring in the renormalization process while a brief discussion and a comparison with the semiclassical case[2,8] is provided in Sec. V.

II. THE HARTREE-VLASOV APPROXIMATION

In this section we briefly recall some previous[6,8] results on the relativistic quantum scalar plasma in the Hartree-Vlasov approximation and show how infinities come into play.

Using the equations of motion implicitly derived from the Lagrangian (1) and the definition (3) of the covariant Wigner function, one easily arrives at[6,7]

$$[i\gamma\cdot\partial+2(\gamma\cdot p-m)]F=-\frac{2^5 g}{(2\pi)^4}\int d^4x'd^4p'\exp[-2ip'\cdot(x-x')]\langle F_{\rm op}(x,p-p')\phi(x')\rangle\ , \tag{6}$$

$$(\Box+\mu_R{}^2)\langle\phi(x)\rangle=4\pi g\,{\rm Tr}\int d^4p\,F(x,p)\ , \tag{7}$$

and another equation similar to Eq. (6) but connecting F to $\langle\phi F_{\rm op}\rangle$ where $F_{\rm op}$ is defined as F in Eq. (3) *except* that the average value is not taken (note that $F\equiv\langle F_{\rm op}\rangle$).

The Hartree-Vlasov approximation is obtained by factorizing $\langle F_{\rm op}\phi\rangle$ as

$$\langle F_{\rm op}\phi\rangle\sim F\langle\phi\rangle \tag{8}$$

as in conventional plasma physics (without such a factorization we would have to solve an equation connecting $\langle F_{\rm op}\phi\rangle$ to $\langle F_{\rm op}\phi\phi\rangle$, etc.;[6] such a hierarchy has to be closed in some way and the Hartree-Vlasov ansatz is the simplest one).

The above ansatz, joined to the fact that in thermal equilibrium the system is invariant under spacetime translations, once introduced into Eqs. (6) and (7) leads to

$$\begin{aligned}&(\gamma\cdot p-M)F_{\rm eq}(p)=0\ ,\\ &F_{\rm eq}(p)(\gamma\cdot p-M)=0\ ,\end{aligned} \tag{9}$$

and

$$\mu_R{}^2(m-M)=4\pi g^2{\rm Tr}\int d^4p\,F_{\rm eq}(p)\ , \tag{10}$$

where M is an *effective* mass for the fermions:

$$M\equiv m-g\langle\phi\rangle\ . \tag{11}$$

Equations (9) represent the relativistic and quantum analog of the Liouville equation for free quasifermions[6] endowed with the effective mass M, while Eq. (10) is nothing but the Klein-Gordon equation (7). The equilibrium Wigner function $F_{\rm eq}(p)$ has the general form[6]

$$F_{\rm eq}(p)=\frac{\gamma\cdot p+M}{4M}f(p)\ , \tag{12}$$

where $f(p)$ is given by[9]

$$f(p)=\frac{4M}{(2\pi)^3}\left[\sum_{\pm}\frac{\theta(p^0)}{\exp[\beta(p^0\mp\epsilon_f)]+1}-\theta(-p^0)\right]\times\delta(p^2-M^2)\ , \tag{13}$$

θ being the Heaviside step function and ϵ_f the chemical potential of the fermions while $\beta\equiv(kT)^{-1}$. Equation (13) contains two terms, a term representing the thermal equilibrium of matter, which vanishes in the limit $T\to 0$ and $\epsilon_f\to 0$, and a vacuum term which remains in this limit [the second one on the right-hand side of Eq. (13)]. Discarding provisionally this vacuum term and inserting Eq. (13) into Eq. (12) one gets the "gap equation"

$$M=m-\Gamma\frac{M^3}{m^2}\sum_{\pm}\int_0^\infty\frac{\xi^2d\xi}{(1+\xi^2)^{1/2}}\frac{1}{\exp\{\beta[M(1+\xi^2)^{1/2}\pm\epsilon_f]\}+1}\ , \tag{14}$$

i.e., a self-consistent equation for the effective mass M. In Eq. (14) one has set

$$\Gamma\equiv\frac{4}{\pi}g^2\left[\frac{m}{\mu_R}\right]^2 . \tag{15}$$

Once M is known then the pressure and the energy density of the fermions is obtained via Eq. (5) while the same quantities for the scalar field are given through[6,8]

$$\begin{aligned}T^{\mu\nu}_{\rm scal}&=\frac{\mu_R{}^2}{8\pi}\langle\phi\rangle^2g^{\mu\nu}\\ &=\frac{\mu_R{}^2}{8\pi g^2}(M-m)^2g^{\mu\nu}\ .\end{aligned} \tag{16}$$

This is essentially a mean-field approximation and we would like to examine the effects of quantum fluctuation on the thermal properties of the scalar plasma. Indeed, while in this approximation (Hartree-Vlasov—mean-field) the scalar field appears to be classical (i.e., $\langle\phi^n\rangle\sim\langle\phi\rangle^n$; this means that the quantum fluctuations of the scalar field can be neglected since they would imply considering terms like $\langle\phi\phi\rangle\neq\langle\phi\rangle\langle\phi\rangle$), yet there exist the quantum fluctuations of the fermion field itself. They are implemented in the vacuum Wigner function corresponding to the last term of Eq. (13), i.e, in

$$F_{\rm vac}(p)=-\frac{1}{(2\pi)^3}(\gamma\cdot p+M)\theta(-p^0)\delta(p^2-M^2)\ , \tag{17}$$

which expresses the fact that the negative-energy states of the quasifermions of effective mass M are uniformly occupied.

III. REMOVAL OF THE INFINITIES

The introduction of the vacuum Wigner function (17) into Eq. (10) or (14), besides the matter part already included in this last equation, gives rise to a divergent integral. Furthermore, introduced in Eq. (5) for the momentum-energy tensor of the fermions it also leads to another infinity. Consequently, we have to remove these infinities and we shall do it in the spirit of current renormalization procedures: regularization of divergent integrals and absorption of infinities in suitable counterterms. Although this can be done successfully (see below) this was not *a priori* obvious owing to the fact that we use an approximation scheme quite different from perturbation theory.

Adding now the counterterms $\mathscr{L}_c$, deriving the field equation for the scalar field ϕ, taking the average value of this latter, and using the Hartree-Vlasov approximation, one gets the following new equation for the effective mass of the fermions:

$$(\mu_R{}^2+\delta\mu^2)(m-M)+\alpha g+\frac{\gamma}{2g}(m-M)^2+\frac{\lambda}{6g^2}(m-M)^3 =4\pi g^2\mathrm{Tr}\int d^4p[F_{\mathrm{mat}}(p)+F_{\mathrm{vac}}(p)]\ , \quad (18)$$

where $F_{\mathrm{mat}}(p)$ is nothing but the "matter part" of Eqs. (12) and (13). Next, the integral involving $F_{\mathrm{vac}}(p)$, i.e.,

$$-\frac{16\pi g^2}{(2\pi)^3}M\int d^4p\,\theta(-p^0)\delta(p^2-M^2)\ , \quad (19)$$

is evaluated in $4-\epsilon$ dimensions as[10] (see Appendix A)

$$\frac{2m^3}{\pi\epsilon}g^2\left[\frac{M}{m}\right]^3\left[1-\epsilon\ln\left[\frac{M}{m}\Lambda\right]\right]+O(\epsilon)\ , \quad (20)$$

where use has been made of[10]

$$\int\frac{d^nk}{(k^2+b^2)^p}=\pi^{n/2}\frac{\Gamma(p-n/2)}{\Gamma(p)}\frac{1}{(b^2)^{p-n/2}}\ . \quad (21)$$

In Eq. (20) Λ is an arbitrary constant, as usual. The pole term of Eq. (20) can clearly be absorbed into the counterterms in the left-hand side of Eq. (18) so that, finally, the gap equation (18) can be rewritten as

$$gA_F+(B_F+\mu_R{}^2)(m-M)+\frac{C_F}{2g}(m-M)^2+\frac{D_F}{6g^2}(m-M)^3=4\pi g^2\mathrm{Tr}\int d^4p\,F_{\mathrm{mat}}(p)-\frac{2m^3g^2}{\pi}\left[\frac{M}{m}\right]^3\ln\left[\frac{M}{m}\Lambda\right]\ , \quad (22)$$

where A_F, B_F, C_F, and D_F are arbitrary finite constants occurring in the regularization/renormalization process (see Appendix A) and have to be related to the experimental renormalized constants $\mu_R{}^2$, α_R, and γ_R, and λ_R (see below). Note that Λ is a redundant constant which could be absorbed into A_F, B_F, C_F, and D_F; it is, however, more convenient to keep it at this stage. Nevertheless if we demand that our vacuum be normal, i.e., $\langle\phi\rangle=0$, in the absence of matter, and hence that $m=M$, then one must have

$$A_F=-\frac{2m^3g}{\pi}\ln\Lambda\ . \quad (23)$$

So far we have dealt with only one infinity, namely, the one occurring in the gap equation (18). However, there still exists another infinity that occurs in the momentum-energy tensor of the fermions,

$$T^{\mu\nu}_{\mathrm{fermions}}=\mathrm{Tr}\int d^4p[F_{\mathrm{mat}}(p)+F_{\mathrm{vac}}(p)]p^\mu\gamma^\nu\ , \quad (24)$$

also due to the vacuum term $F_{\mathrm{vac}}(p)$, and it is *a priori* not obvious that this new infinity can also be absorbed in the *same counterterms as above.* In fact, the same kind of calculation as above—dimensional regularization of the divergent integral and absorption of the pole term into $T^{\mu\nu}_c$ (the contribution of $\mathscr{L}_c$ to the momentum-energy tensor)—shows immediately that the infinite part of $T^{\mu\nu}_{\mathrm{fermions}}$ not only can actually be absorbed into $T^{\mu\nu}_c$ but also, what is more important, with the *same* counterterms (see Appendix B).

Finally, the renormalized momentum-energy tensor of the system reads

$$T^{\mu\nu}_R=T^{\mu\nu}_{\mathrm{mat}}+\frac{g^{\mu\nu}}{4\pi}\left[\frac{A_F}{1!}\langle\phi\rangle+\frac{B_F}{2!}\langle\phi\rangle^2+\frac{C_F}{3!}\langle\phi\rangle^3+\frac{D_F}{4!}\langle\phi\rangle^4+\frac{\mu_R{}^2}{2!}\langle\phi\rangle^2-\frac{M^4}{2\pi}\ln\left[\frac{M}{m}\Lambda\exp(-\tfrac{1}{4})\right]\right] \quad (25)$$

$$=T^{\mu\nu}_{\mathrm{mat}}+\frac{g^{\mu\nu}}{4\pi}\left[\frac{A_F}{g}(m-M)+\frac{B_F}{2g^2}(m-M)^2+\frac{\mu_R{}^2}{2g^2}(m-M)^2+\frac{C_F}{3!g^3}(m-M)^3+\frac{D_F}{4!g^4}(m-M)^4 -\frac{M^4}{2\pi}\ln\left[\frac{M}{m}\Lambda\exp(-\tfrac{1}{4})\right]\right]\ . \quad (26)$$

IV. DETERMINATION OF THE ARBITRARY CONSTANTS

We now have to determine the arbitrary constants A_F, B_F, C_F, D_F, and Λ (we emphasize once more that Λ is redundant and can be absorbed in A_F, B_F, C_F, and D_F) and, to this end, suppose that our scalar plasma does represent an actual physical system. Then through various physical experiments (diffusions, measures of binding energies, etc.), we could, at least in principle, determine the numerical values of the renormalized constants α_R, $\mu_R{}^2$, γ_R, and λ_R (of which some of them could possibly vanish).

How are they related to, say, A_F, B_F, C_F, and D_F? First, we fix Λ in such a way that in a normal vacuum where $\langle\phi\rangle=0$, or $m=M$, $T^{\mu\nu}_R=0$. This is achieved by choosing

$$\Lambda=\exp(\tfrac{1}{4})\ . \quad (27)$$

Next, we remark that the "experimental" constants α_R, $\mu_R{}^2$, γ_R, and λ_R are, by definition, the coefficients of $\phi/1!$, $\phi^2/2!$, $\phi^3/3!$, and $\phi^4/4!$, respectively, in a renormalized Lagrangian and that the coefficient of $g^{\mu\nu}$ [in Eq. (25) or (26)] is nothing but an effective Lagrangian that takes account of quantum fluctuations in the Hartree-

Vlasov approximation (rather it is an effective potential since there is no kinetic term in this coefficient, owing to the spacetime translational invariance of our equilibrium state). In this effective Lagrangian the coefficients of $\langle\phi\rangle$, $\langle\phi\rangle^2$, $\langle\phi\rangle^3$, and $\langle\phi\rangle^4$ *are not* A_F, B_F, C_F, and D_F since the term $\sim M^4 \ln M$ also contains such terms when expanded in a power series of $\langle\phi\rangle$. The Taylor formula up to fourth order in $\langle\phi\rangle$ for the function

$$\eta(\langle\phi\rangle) = -\frac{M^4}{8\pi^2}\ln\left[\frac{M}{m}\right] \tag{28}$$

is

$$\begin{aligned}\eta(\langle\phi\rangle) = {} & \langle\phi\rangle\frac{gm^3}{8\pi^2} - \langle\phi\rangle^2\frac{7}{16\pi^2}g^2m^2 \\ & + \langle\phi\rangle^3\frac{13}{24\pi^2}mg^3 - \langle\phi\rangle^4\frac{25g^4}{96\pi^2} \\ & + \frac{3g^5\langle\phi\rangle^5}{\pi^2 m[1-(g/m)\theta\langle\phi\rangle]}\frac{1}{5!} ,\end{aligned} \tag{29}$$

where θ is an unknown constant such that $0<\theta<1$. Identifying now the various renormalized constants with the corresponding coefficients of $\langle\phi\rangle, \ldots, \langle\phi\rangle^4$, one obtains

$$\frac{\alpha_R}{1!} = \frac{A_F}{1!} + \frac{gm^3}{2\pi} , \tag{30}$$

$$0 = \frac{B_F}{2!} - \frac{7g^2m^2}{4\pi} , \tag{31}$$

$$\frac{\gamma_R}{3!} = \frac{C_F}{3!} + \frac{13g^3m}{6\pi} , \tag{32}$$

$$\frac{\lambda_R}{4!} = \frac{D_F}{4!} - \frac{25g^4}{24\pi} . \tag{33}$$

Finally, the renormalized gap equation and the renormalized momentum-energy tensor can, respectively, be written as

$$\begin{aligned} & g\left[\alpha_R - \frac{gm^3}{2\pi}\right] + \left[\mu_R{}^2 + \frac{7g^2m^2}{2\pi}\right](m-M) + \left[\gamma_R - \frac{13g^3m}{\pi}\right]\frac{(m-M)^2}{2g} + \left[\lambda_R + \frac{25g^4}{\pi}\right]\frac{(m-M)^3}{6g^2} \\ & \qquad = 4\pi g^2 \mathrm{Tr}\int d^4p\, F_{\mathrm{mat}}(p) - \frac{2m^3g^2}{\pi}\left[\frac{M}{m}\right]^3 \ln\left[\frac{M}{m}e^{1/4}\right] \end{aligned} \tag{34}$$

and

$$\begin{aligned} T_R^{\mu\nu} = T_{\mathrm{mat}}^{\mu\nu} + g^{\mu\nu}\frac{1}{4\pi}\Bigg[& \left[\alpha_R - \frac{gm^3}{2\pi}\right]\langle\phi\rangle + \left[\frac{\mu_R{}^2}{2!} + \frac{7g^2m^2}{4\pi}\right]\langle\phi\rangle^2 + \left[\frac{\gamma_R}{3!} - \frac{13g^3m}{6\pi}\right]\langle\phi\rangle^3 \\ & + \left[\frac{\lambda_R}{4!} + \frac{25g^4}{24\pi}\right]\langle\phi\rangle^4 - \left[\frac{(m-g\langle\phi\rangle)^4}{2\pi}\right]\ln\left[1 - \frac{g\langle\phi\rangle}{m}\right]\Bigg] . \end{aligned} \tag{35}$$

Note that, owing to Eqs. (23) and (30), $\alpha_R \equiv 0$.

Finally, a further determination of Eqs. (34) and (35) demands the "experimental" determination of $\mu_R{}^2$, γ_R, and λ_R. However, since we want to compare these "renormalized" results with the semiclassical ones[2,8] we have to take the same finite constants in both cases, i.e., $\mu_R{}^2$ given and

$$\gamma_R = \lambda_R = 0 . \tag{36}$$

Doing so, results first obtained by Chin[11,12] in an interesting article are then recovered exactly. Numerical results with the values (36) are given in the next section and compared to those obtained in Refs. 2 and 8.

Let us look at the effects of the quantum fluctuations in the Hartree-Vlasov approximation on the state of "nuclear" matter, i.e., whether in a normal or abnormal[5] state. To this end we consider the energy density T_R^{00} of the system,

$$\begin{aligned} T_R^{00} = T_{\mathrm{mat}}^{00} + \frac{1}{4\pi}\Bigg[& \frac{\mu_R{}^2}{2!}\langle\phi\rangle^2 + \frac{\gamma_R}{3!}\langle\phi\rangle^3 + \frac{\lambda_R}{4!}\langle\phi\rangle^4 \\ & + \frac{3g^5\langle\phi\rangle^5}{\pi^2 m[1-(g/m)\langle\phi\rangle\theta]5!}\Bigg] , \end{aligned} \tag{37}$$

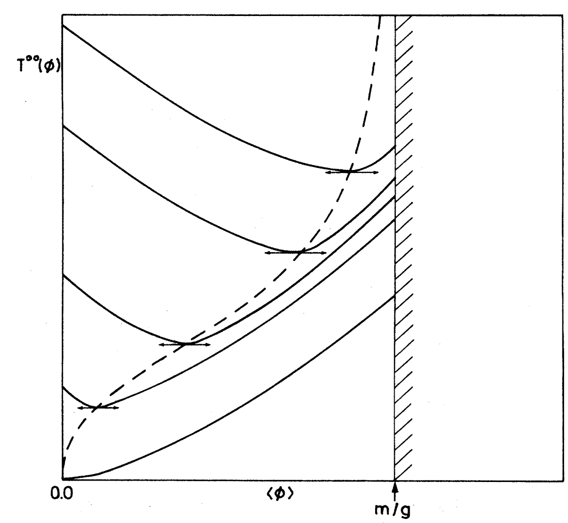

FIG. 1. The energy density is plotted as a function of $\langle\phi\rangle$. The various continuous curves correspond to various Fermi energies. On the dashed line lie the minima of the continuous lines. At $T=0$ K the equation providing these minima is the gap equation: $(\partial/\partial\langle\phi\rangle)T^{00}(\langle\phi\rangle)=0$. At $T\neq0$ K we should plot the free energy.

where use has been made of Eqs. (29)—(33) and (35). The effect of quantum fluctuations is twofold: (i) it leads to the observable values of the various constants $\mu_R{}^2$, γ_R, and λ_R, and (ii) it adds to the total energy density T_R^{00} a vacuum contribution, i.e., the last term on the right-hand side of Eq. (37). It is not difficult to realize that this contribution is always positive since $m-g\langle\phi\rangle>0$ and $0<\theta<1$.

First, we note that the minimum (or minima) of the energy density T_R^{00} with respect to $\langle\phi\rangle$, i.e.,

$$\frac{\partial}{\partial\langle\phi\rangle}T_R^{00}(\langle\phi\rangle)=0\ , \tag{38}$$

leads to the renormalized gap equation (34) (see Appendix C) as it should be. The qualitative features of $T_R^{00}(\langle\phi\rangle)$ are shown on Fig. 1. Next let us discuss the effect of the vacuum contribution to the energy density [see Eq. (37)]. When the various constants $\mu_R{}^2$, γ_R, and λ_R are such that matter is in a normal state (only one minimum in T_R^{00}), then the vacuum energy term does not change the general shape of the energy density curve: it only displaces the position of the minimum and also the numerical value of the corresponding energy density, which is increased since we add a positive quantity.

The situation is, however, not so simple in the case of abnormal matter. In such a case, depending on the values of the constants $\mu_R{}^2$, γ_R, and λ_R (a detailed discussion of the various cases has been given by Lee and Margulies[5]), there are two minima (separated by a maximum) corresponding to two possibilities: either the lowest minimum

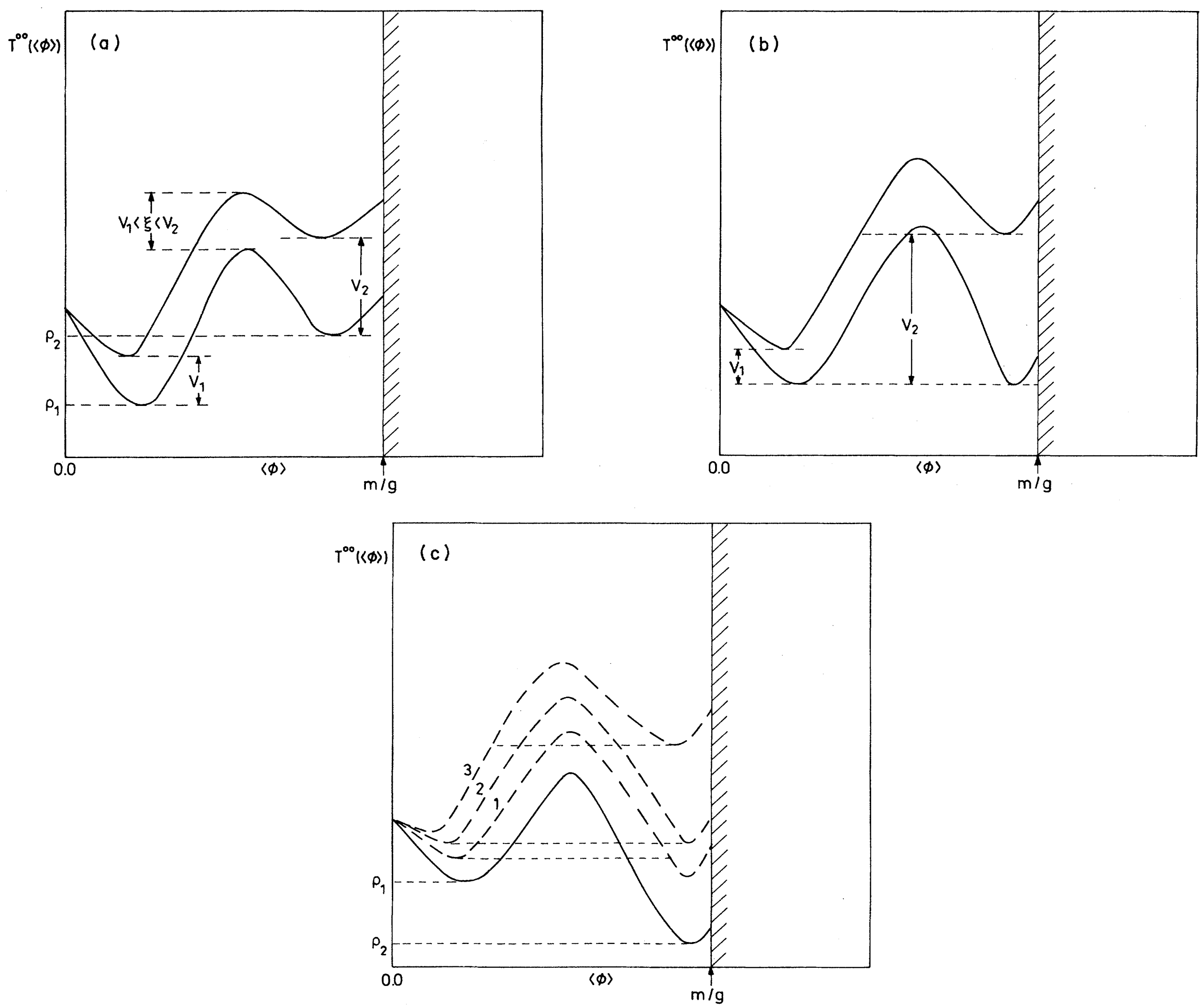

FIG. 2. The effect of quantum fluctuations on the energy density $T^{00}(\langle\phi\rangle)$ at $T=0$ K and for a given value of p_f is represented. The quantum term in $T^{00}(\langle\phi\rangle)$ has two effects: (i) it increases the energy density (the larger $\langle\phi\rangle$ and the larger the increase) and (ii) it shifts the minima in the direction of decreasing $\langle\phi\rangle$. Various cases are shown in diagrams (a), (b), and (c). (a) Normal matter remains normal. (b) The degenerate ground state becomes nondegenerate and the first minimum is the lowest: matter becomes normal. (c) In the semiclassical case (continuous curve) matter is in an abnormal state; the effect of quantum fluctuations may *a priori* be represented by one of the dashed lines labeled (1), (2), or (3): (1) Matter remains abnormal, (2) the ground state becomes degenerate, and (3) matter becomes normal. The existence of these various cases depend on the possible value of m/g.

is obtained for the smallest values of $\langle \phi \rangle$ (in such a case it corresponds to a stable normal state while the other minima is metastable and the maximum unstable) or it is obtained at a larger value $\langle \phi \rangle$ [in this case, the first minimum (see Fig. 2) is metastable while the second is stable and represents abnormal matter]. Moreover, whether these two cases (shown on Fig. 2) exist or not does depend on the possible limiting value of $\langle \phi \rangle$, i.e., of m/g (see Ref. 5). What is now the effect of quantum fluctuations on this situation? In fact, it is easy to realize that the term brought by the vacuum fluctuations is not only positive but also monotonically increasing with $\langle \phi \rangle$. This means that the value of the energy density corresponding to the first minimum to the right of $\langle \phi \rangle = 0$ is *less* increased than for the second one, corresponding to a larger value of $\langle \phi \rangle$. It follows that several cases have to be considered. For the sake of the discussion we label ρ_1 and ρ_2 the minima corresponding to $\langle \phi \rangle_1$ and $\langle \phi \rangle_2$, respectively, and take $0 < \langle \phi \rangle_1 \leq \langle \phi \rangle_2$; let us also call V_1 and V_2 the corresponding vacuum contribution (ρ_1 refers to the quasiclassical case). Therefore, we examine the following different cases:

(i) $\rho_1 < \rho_2$ (metastable abnormal state: ρ_2; stable normal state: ρ_1). In this case $\rho_1 + V_1 < \rho_2 + V_2$ and hence the quantum fluctuations enhance the stability of the normal state; also they can possibly suppress the second (metastable) minimum depending on its depth relative to the maximum (the energy density of the maximum is *less* increased than is the one of the metastable minimum).

(ii) $\rho_1 = \rho_2$ (and $\langle \phi \rangle_1 < \langle \phi \rangle_2$). In this case, the quasiclassical degenerate normal state is split into a normal state (state 1) and possibly a metastable state 2 (or no state 2 at all).

(iii) $\rho_1 > \rho_2$ (normal metastable state and abnormal stable state). This is the most complicated case since there are several possibilities: (a) $\rho_1 + V_1 > \rho_2 + V_2$, (b) $\rho_1 + V_1 = \rho_2 + V_2$, and (c) $\rho_1 + V_1 < \rho_2 + V_2$; of course these various cases depend on the values of the constants at hand.

Finally, it should be mentioned that all these cases have to be reconsidered according to the value of m/g, since, e.g., when $\langle \phi \rangle_2 > m/g$ there is no physical second minimum.

V. DISCUSSION AND CONCLUSION

(1) In Sec. III we have shown that the Hartree-Vlasov approximation could be renormalized with the *same* kind of counterterms as in perturbation theory, i.e., with a polynomial of fourth degree in the scalar field. This is hardly a surprise since this approximation can also be obtained by summing the so-called tadpole diagrams of the many-body theory,[11] at least formally. Nevertheless, this possibility *does not prove* that the *next* approximation (i.e., the one that retains two-body correlations and, of course, three-, four-, etc. body correlations) can also be rendered finite with such counterterms.

(2) In order to compare our renormalized equations to the previous semiclassical ones,[6,8] the various constants α_R, γ_R, and λ_R have been chosen to be zero. As was pointed out above, they should be determined by high-energy-physics experiments; however, since this scalar plasma is not intended to represent a real physical situation but is only a "laboratory" useful to test techniques, these constants can be chosen freely as far as comparisons have to be made and also for the sake of numerical calculations.

The effects of quantum fluctuations can be inferred[13] from the renormalized gap equation (34) and from the expression of the renormalized momentum-energy tensor (35). Apart from providing finite coupling constants (γ_R, λ_R) and mass $\mu_R{}^2$, the renormalization process modifies these two basic equations by terms of order $\langle \phi \rangle^4$ and $\langle \phi \rangle^5$, respectively. It follows that, at low densities where $\langle \phi \rangle \sim 0$ (we consider the case where $\gamma_R = \lambda_R = 0$) the modifications brought by renormalization are certainly weak.

On the other hand, in the high-density case (where $p_f \gg m > M$) where M tends to zero (or, equivalently, where $\langle \phi \rangle$ tends to m/g), the gap equation (34) can be approximated by

$$-\frac{g^2 m^3}{2\pi} + \left[\mu_R{}^2 + \frac{7g^2 m^2}{2\pi}\right](m-M) - \frac{13 g^2 m}{2\pi}(m^2 - 2Mm) + \frac{25 g^2}{6\pi}(m^3 - 3m^2 M) \simeq \frac{2g^2}{\pi}\left[M\epsilon_f p_f - M^3 \ln\left|\frac{\epsilon_f + p_f}{M}\right|\right] + O(M^2) , \quad (39)$$

where the $M^3 \ln M$ term has been dropped and where only terms linear in M have been kept. Similarly, the last term of this equation should be dropped. Finally, Eq. (39) reads

$$M \sim m \frac{1+\frac{1}{6}\Gamma}{1+\frac{3}{4}\Gamma + \Gamma p_f{}^2/2m^2} \sim 2m^3 \frac{1+\frac{1}{6}\Gamma}{\Gamma p_f{}^2} , \quad (40)$$

while the corresponding equation in the semiclassical case is

$$M_{\mathrm{cl}} \sim \frac{2m^3}{\Gamma p_f{}^2} . \quad (41)$$

This shows (i) that M and M_{cl} both vanish at high p_f, (ii) that—still at high p_f—the quantum mass M can be obtained from M_{cl} simply by scaling the coupling constant Γ [i.e., $M(\Gamma) = M_{\mathrm{cl}}(\Gamma')$ with $\Gamma' = \Gamma/(1+\frac{1}{6}\Gamma)$], and (iii) that the ratio M/M_{cl} tends to a definite limit at large p_f:

$$\lim_{p_f \to \infty} \frac{M}{M_{\mathrm{cl}}} = 1 + \frac{1}{6}\Gamma .$$

A similar scaling can be obtained for the energy density and for the pressure, although with a different Γ'. In fact, one has ρ_{quant} or $P_{\mathrm{quant}}(\Gamma) = \rho_{\mathrm{cl}}$ or $P_{\mathrm{cl}}(\Gamma'')$ with

$$\Gamma'' = \frac{\Gamma}{1+\Gamma/16} .$$

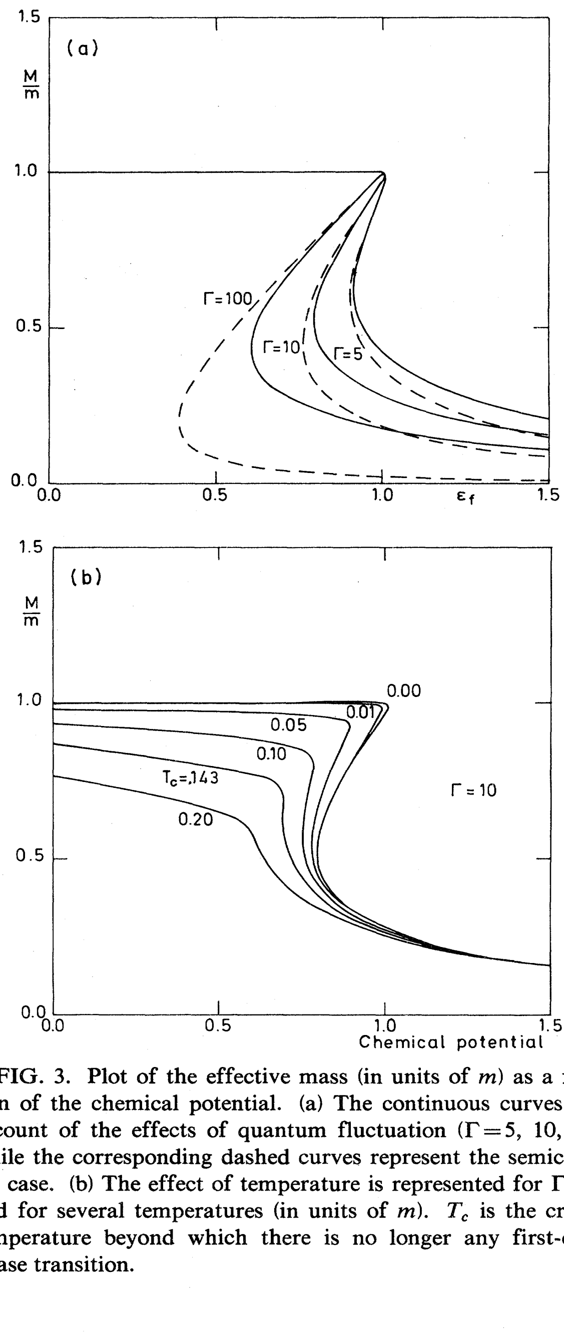

FIG. 3. Plot of the effective mass (in units of m) as a function of the chemical potential. (a) The continuous curves take account of the effects of quantum fluctuation ($\Gamma=5$, 10, 100) while the corresponding dashed curves represent the semiclassical case. (b) The effect of temperature is represented for $\Gamma=10$ and for several temperatures (in units of m). T_c is the critical temperature beyond which there is no longer any first-order phase transition.

Note also that, in the above remarks, the zero-temperature case was dealt with: at $T\neq 0$ K, the asymptotic regime $p_f \gg m$ or kT amounts to considering $T=0$ K.

All these remarks are confirmed by a numerical calculation of the effective mass (see Fig. 3) performed for several values of the constant Γ; in any case only a limited range of densities is concerned with the modification brought by the quantum term. It appears that the effective mass is higher when quantum fluctuations are taken into account than without. Therefore the fermion's pressure should be lower when quantum fluctuations are taken into account, owing to the mass dependence of the free Fermi gas pressure. However, the situation is not so simple and we must also take account of the other terms of

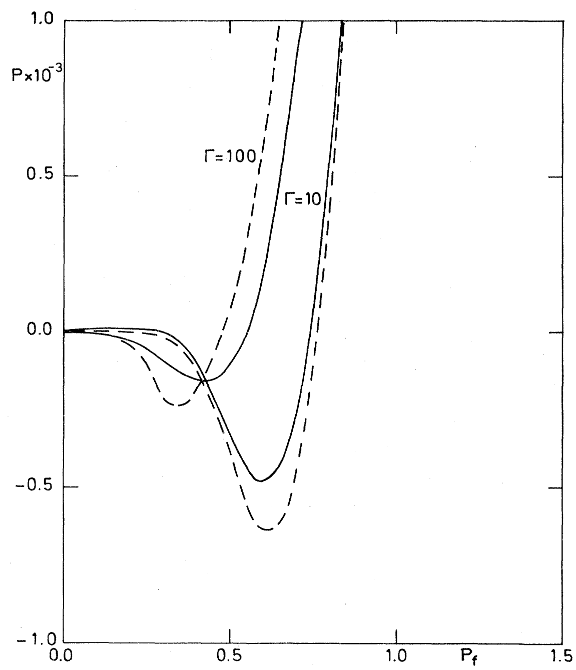

FIG. 4. The pressure (in units of m^4) is plotted versus the Fermi momentum at $T=0$ K and for $\Gamma=10$ and 100. Continuous lines: with quantum fluctuations; dashed lines: without.

Eq. (35) besides the one for fermions in $T^{\mu\nu}_{\rm mat}$ [let us recall that the pressure and the energy density are related to the momentum-energy tensor through

$$\text{pressure}=-\tfrac{1}{3}T^{ii}, \quad \text{energy density}=T^{00}\,,$$

while the fermion (minus antifermion) density is related to the chemical potential as in Refs. 6 and 8]. In fact, there is a density region where the pressure is lower with quantum fluctuations than without (at smaller densities) and

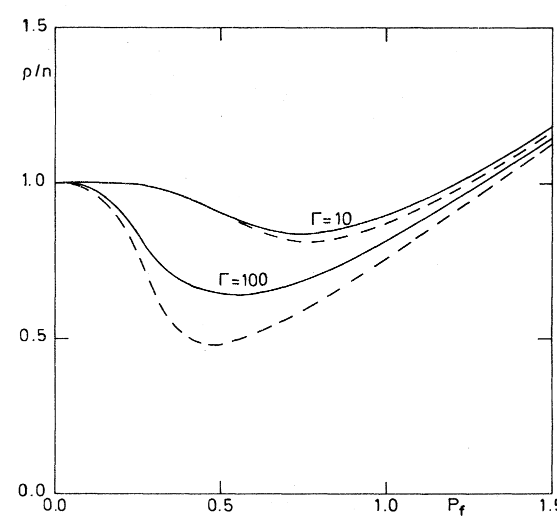

FIG. 5. The energy per particle (in units of m) is plotted versus the Fermi momentum at $T=0$ K and for $\Gamma=10$ and 100. Continuous lines: with quantum fluctuations; dashed lines: without.

another one (at higher densities) where it is higher. This is shown in the various figures. The pressure versus the Fermi momentum is plotted in Fig. 4 at $T=0$ K for two values of the coupling constant $\Gamma=4g^2m^2/\pi\mu_R{}^2$, i.e., for $\Gamma=100$ and 10. The continuous line (quantum fluctuations included) and the dashed line (semiclassical case) cross at a given value of p_f for $\Gamma=100$ while this is not apparent for $\Gamma=10$. In fact, there is also such a crossing but outside the figure. In Fig. 5 a plot of the energy per particle versus the Fermi momentum, still at $T=0$ K and for $\Gamma=10$ and 100, has been drawn: in these two cases it is lower than 1 (in units of m) and hence this corresponds to a collective bound state. In conclusion, there is not much difference (other than quantitative) between the semiclassical case and the quantum case. This is reflected in Fig. 6 where we have drawn a phase diagram for the case $T=0$ K, where the various possible regimes (depending on Γ and p_f) have been indicated, in the semiclassical (dashed lines) case first given by Kalman[2] and in the quantum case. We refer to Refs. 2 and 8 for a more detailed discussion of this diagram.

(3) A direct comparison with the interesting work by Chin (who dealt with a similar model) cannot be made for the following reason: besides the scalar field, Chin also considers the effect of a massive vector field (supposed to take account in a phenomenological way of the short-distance repulsive forces between nucleons); technically, the effect of this field is to shift the chemical potential ϵ_f to

$$\epsilon_f-g_vA^0=\epsilon_f-4\pi g_V{}^2n/\mu_V{}^2 ,$$

where A^0 is the zeroth component of the vector field whose mass is $\mu_V{}^2$, the coupling constant with the fermions is g_V, and n is the fermionic density; it follows that

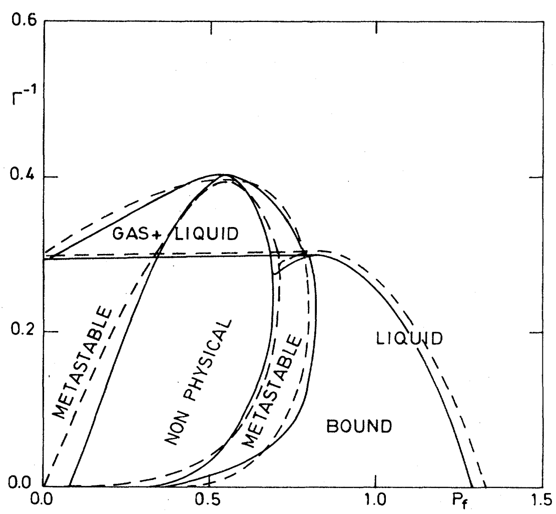

FIG. 6. The phase diagram Γ^{-1}/p_f of the scalar plasma with quantum fluctuations (continuous lines) and without [dashed lines; see Kalman's diagram (Ref. 2)]. The "metastable regions" correspond to the metastable states of the first-order phase transition corresponding to a given Γ. In the "nonphysical" zone the Maxwell's construction is not feasible at $T=0$ K (see Ref. 8).

a direct comparison is not feasible. However, as far as the scalar field and the fermions are concerned, our gap equation (34) is identical with Chin's similar equation once identical Lagrangians[12] are dealt with. Thus, we think that both approaches, which are very different (i.e., Wigner functions/Green's functions), are mutually illuminating.

(4) We would like to emphasize that our approach to the relativistic quantum scalar plasma is valid as well at $T\neq0$ K as at $T=0$ K. There is no particular difficulty in dealing with the effects of temperature: it is sufficient to replace the Fermi distribution function at $T=0$ K by the corresponding expression at $T\neq0$ K. For the sake of illustration we have calculated the effective mass of the fermions [see Fig. 3(b)] for $\Gamma=10$ and for various temperatures ranging from $0.01m$ to $0.2m$, still with $\lambda_R=\gamma_R=0$. The other thermodynamical quantities—all of them do depend on M—can be easily obtained and, in the absence of a particular physical problem do not present much interest: they are quite similar to those obtained in the semiclassical case.

(5) We have not considered in this paper the interesting case of abnormal nuclear matter; this would require a particular study which is in the course of active investigation. Also the important question of the restoration of a broken gauge theory through a Higgs mechanism has been left aside and necessitates a particular consideration within the context of the standard big-bang cosmology.

(6) Finally it is worth mentioning that our procedure for extracting finite results from the infinite vacuum terms cannot be considered as a full renormalization. Though it is a renormalization of the Hartree approximation, yet it would remain to show its consistency with a similar procedure for higher-order approximations that would take account of N-body correlations. However, this is far from trivial the more so since our approximations (cluster expansions) are typically nonperturbative.[6,7]

ACKNOWLEDGMENT

We are indebted to Dr. P. Bakshi who pointed out an error in a previous version and whose long and detailed comments have been quite helpful in the improvement of this work.

APPENDIX A: REGULARIZATION OF THE GAP EQUATION

It is necessary to indicate the origin of Eq. (19): it comes from Eq. (18) with

$$F_{\rm vac}(p)=\frac{\gamma\cdot p+M}{4M}f_{\rm vac}(p) , \tag{A1}$$

where

$$f_{\rm vac}(p)=-\frac{4M}{(2\pi)^3}\theta(-p^0)\delta(p^2-M^2) . \tag{A2}$$

Indeed, only those primary expressions that enter into various equations have to be written in n dimensions rather than derived quantities such as Eq. (18).

Owing to the fact that in n dimensions one has[14]

$$\mathrm{Tr}\gamma^{\mu}\gamma^{\nu}=\varphi(n)g^{\mu\nu},\quad \mathrm{Tr}I_n=\varphi(n)\;, \tag{A3}$$

where $\varphi(n)$ is an arbitrary function with a continuous derivative, such that $\varphi(4)=4$, $F_{\rm vac}(p)$ has to be generalized as

$$F_{\rm vac}(p)=\frac{\gamma\cdot p+M}{\varphi(n)M}f_{\rm vac}(p)\;, \tag{A4a}$$

$$f_{\rm vac}(p)=-\chi(n)\frac{4M}{(2\pi)^3}\theta(-p^0)\delta(p^2-M^2)\;, \tag{A4b}$$

where $\chi(n)$ is still a continuous function with a continuous derivative and such that $\chi(4)=1$. In fact, $\varphi(n)$ does not play any role since by taking the trace of $F_{\rm vac}(p)$, it is eliminated. Furthermore $\chi(n)$ can always be taken to be

$$\chi(n)=(\mathrm{const})^{n-4}\;, \tag{A5}$$

as is made clear below.

Let us now look at the integral (19) in n dimensions which, omitting multiplicative factors, we write as

$$I\equiv\tfrac{1}{2}\int\frac{dp^{n-1}}{p_0}=\tfrac{1}{2}\pi^{n/2-1}\Gamma\left[1-\frac{n}{2}\right]M^{2(n/2-1)}\;, \tag{A6}$$

where use has been made of Eq. (21).[10] With $n-4=-\epsilon$, it reads

$$I=\tfrac{1}{2}\pi M^2\exp(-\tfrac{1}{2}\epsilon\ln\pi)\exp(-\epsilon\ln M)\Gamma(-1+\epsilon/2)\;. \tag{A7}$$

Using the well-known functional relation $\Gamma(x+1)=x\Gamma(x)$ for defining $\Gamma(-1+\epsilon/2)$ and expanding the various terms of the right-hand side of Eq. (A7) in powers of ϵ, one gets

$$I=-\frac{\pi M^2}{\epsilon}+\tfrac{1}{2}\pi M^2(\ln(\pi M^2)-\ln\{\exp[1+\Gamma'(1)]\})+O(\epsilon)\;. \tag{A8}$$

Similarly, for $\chi(4-\epsilon)I$ one obtains

$$\chi(4-\epsilon)I=-\frac{\pi M^2}{\epsilon}+\pi M^2\ln\left[\frac{M}{m}\Lambda\right]+O(\epsilon) \tag{A9}$$

with

$$\Lambda/m\equiv\sqrt{\pi}/\exp[\tfrac{1}{2}+\tfrac{1}{2}\Gamma'(1)-\chi'(4)] \tag{A10}$$

[χ being arbitrary, so is χ' and hence Λ is an arbitary constant; moreover, since only the first derivative of χ is involved in this calculation, then χ can always be chosen as in Eq. (A5)].

Let us now introduce Eq. (A9) into the gap equation *with* those counterterms that come from $\mathscr{L}_c$. One obtains

$$(\mu_R{}^2+\delta\mu^2)\langle\phi\rangle+\alpha+\frac{\gamma}{2}\langle\phi\rangle^2+\frac{\lambda}{6}\langle\phi\rangle^3 =4\pi g\left[\int d^4p\,f_{\rm mat}(p)+\frac{M^3}{2\pi^2}\frac{1}{\epsilon}-\frac{M^3}{2\pi^2}\ln\left[\frac{M}{m}\Lambda\right]\right]. \tag{A11}$$

We now suppress the pole term $M^3/2\pi^2\epsilon$ in the right-hand side of this last equation with a redefinition of the various constants appearing in the left-hand side. This is of course possible since the pole term is a polynomial of degree 3 in $\langle\phi\rangle$, $(M=m-g\langle\phi\rangle)$, as is the left-hand side of Eq. (A11). Doing this, one is led to

$$\delta\mu^2=-\frac{6m^2g^2}{\pi\epsilon}+B_F\;, \tag{A12}$$

$$\alpha=\frac{2gm^3}{\pi\epsilon}+A_F\;, \tag{A13}$$

$$\gamma=\frac{12g^3m}{\pi\epsilon}+C_F\;, \tag{A14}$$

$$\lambda=-\frac{12g^4}{\pi\epsilon}+D_F\;, \tag{A15}$$

where A_F, B_F, C_F, and D_F are arbitrary *finite* constants to be connected with the renormalized parameters $\mu_R{}^2$, α_R, γ_R, and λ_R in Sec. IV. From Eqs. (A11)—(A15) one easily obtains the "renormalized gap equation" (22).

APPENDIX B: REGULARIZATION OF THE MOMENTUM-ENERGY TENSOR

The total momentum-energy tensor of the scalar plasma is given by

$$T^{\mu\nu}=T^{\mu\nu}_{\rm mat}+T^{\mu\nu}_{\rm vac}+T^{\mu\nu}_{\rm scal}\;, \tag{B1}$$

where $T^{\mu\nu}_{\rm mat}$ is the finite-temperature- and density-dependent part of the momentum-energy tensor of the fermions;

$$T^{\mu\nu}_{\rm vac}=\mathrm{Tr}\int d^4p\,p^{\mu}\gamma^{\nu}F_{\rm vac}(p) \tag{B2}$$

and $T^{\mu\nu}_{\rm scal}$ is the momentum-energy tensor of the spin-0 particles

$$T^{\mu\nu}_{\rm scal}=\frac{g^{\mu\nu}}{4\pi}\left[\alpha\phi+\tfrac{1}{2}(\mu_R{}^2+\delta\mu^2)\phi^2+\frac{\gamma}{3!}\phi^3+\frac{\lambda}{4!}\phi^4\right]\;, \tag{B3}$$

which includes the various counterterms.

Only $T^{\mu\nu}_{\rm vac}$ is infinite and owing to Lorentz invariance it is necessarily proportional to $g^{\mu\nu}$,

$$T^{\mu\nu}_{\rm vac}=Xg^{\mu\nu}\;, \tag{B4}$$

so that X is given by

$$X=\frac{1}{n}\mathrm{Tr}\int d^4p\,p\cdot\gamma F_{\rm vac}(p)\;, \tag{B5}$$

where use has been made of[14]

$$g^{\mu\nu}g_{\mu\nu}=n\;. \tag{B6}$$

Finally, using Eqs. (A1)—(A4) one is led to

$$X=-\frac{4M^2}{(2\pi)^3n}I\chi(n)\;, \tag{B7}$$

where I is the same integral as the one occurring in Appendix A [i.e., Eqs. (A6)—(A9)], so that

$$X=-\frac{M^2}{(2\pi)^3}\left[-\frac{\pi M^2}{\epsilon}+\pi M^2\ln\left[\frac{M}{m}\Lambda\right]-\frac{\pi M^2}{4}\right]. \tag{B8}$$

[Note that the last term in this equation stems from the factor $n^{-1}=(4-\epsilon)^{-1}\sim\frac{1}{4}(1+\epsilon/4)$ occurring in Eq. (B7).] Since the pole term $M^4/8\pi^2\epsilon$ is a fourth-degree polynomial in ϕ, it can be absorbed in the counterterms, themselves a polynomial with the same degree.

An elementary calculation leads exactly to Eqs. (A12) and (A15) once again. In the above pole term there is also the *constant* $m^4/8\pi^2\epsilon$ which cannot be eliminated with a corresponding counterterm. However, it can be eliminated on the ground that the energy is defined up to a constant. The final form of $T^{\mu\nu}$ is thus given in Eq. (25).

APPENDIX C: THE GAP EQUATION AS A MINIMUM CONDITION

In this appendix it is briefly shown that the gap equation expresses the fact that the free energy density is minimum in thermal equilibrium (at $T=0$ K the free energy density reduces to the energy density). We start from the usual thermodynamical relations.

$$a=\rho-Ts\ , \tag{C1}$$

$$\epsilon_f=(\rho-Ts+P)/n_{\rm eq}\ , \tag{C2}$$

where a is the free energy density, s is the entropy density, $n_{\rm eq}$ is the fermion density, and ϵ_f is the chemical potential. Explicitly, one has

$$\rho=\frac{M^4}{\pi^2}\int_0^\infty \epsilon^2(\epsilon^2+1)^{1/2}(n^++n^-)d\epsilon+\frac{\mu_R{}^2}{8\pi}\langle\phi\rangle^2+\frac{\gamma_R}{24\pi}\langle\phi\rangle^3+\frac{\lambda_R}{96\pi}\langle\phi\rangle^4-\frac{M^4}{8\pi^2}\ln\left[\frac{M}{m}\right]$$
$$-\frac{gm^3}{8\pi^2}\langle\phi\rangle+\frac{7m^2g^2}{16\pi^2}\langle\phi\rangle^2-\frac{13mg^3}{24\pi^2}\langle\phi\rangle^3+\frac{25g^4}{96\pi^2}\langle\phi\rangle^4\ , \tag{C3}$$

$$P=\frac{M^4}{3\pi^2}\int_0^\infty d\epsilon\,\epsilon^4(\epsilon^2+1)^{-1/2}(n^++n^-)-\frac{\mu_R{}^2}{8\pi}\langle\phi\rangle^2-\frac{\gamma_R}{24\pi}\langle\phi\rangle^3-\frac{\lambda_R}{96\pi}\langle\phi\rangle^4+\frac{M^4}{8\pi^2}\ln\left[\frac{M}{m}\right]$$
$$+\frac{gm^3}{8\pi^2}\langle\phi\rangle-\frac{7m^2g^2}{16\pi^2}\langle\phi\rangle^2+\frac{13mg^3}{24\pi^2}\langle\phi\rangle^3-\frac{25g^4}{96\pi^2}\langle\phi\rangle^4\ , \tag{C4}$$

$$n_{\rm eq}=\frac{M^3}{\pi^2}\int_0^\infty d\epsilon\,\epsilon^2(n^+-n^-)\ . \tag{C5}$$

In Eqs. (C3)—(C5) $n^\pm$ are the usual Fermi factors: $n^\pm=\{\exp[\beta(E-\epsilon_f)]+1\}^{-1}$. From Eqs. (C1) and (C2) one gets

$$a=n_{\rm eq}\epsilon_f-P \tag{C6}$$

and we write that, in thermal equilibrium, one has

$$\left.\frac{\partial a}{\partial\langle\phi\rangle}\right|_{T,n_{\rm eq}}=0=\left.\frac{\partial a}{\partial M}\right|_{T,n_{\rm eq}}\ . \tag{C7}$$

On the other hand it is not difficult to check that

$$\frac{\partial P_{\rm mat}}{\partial M}=-\frac{M^3}{\pi^2}\int_0^\infty d\epsilon\,\epsilon^2(\epsilon^2+1)^{-1/2}(n^++n^-)$$
$$+\frac{M^3}{\pi^2}\frac{\partial\epsilon_f}{\partial M}\int_0^\infty d\epsilon\,\epsilon^2(n^+-n^-)$$
$$=-\frac{M^3}{\pi^2}\int_0^\infty d\epsilon\,\epsilon^2(\epsilon^2+1)^{-1/2}(n^++n^-)+n_{\rm eq}\frac{\partial\epsilon_f}{\partial M}\ , \tag{C8}$$

which has been obtained with various integrations by parts and also using the properties of the Fermi factors $n^\pm$. In Eq. (C8) $P_{\rm mat}$ is the pressure of the fermions, i.e., the first term in Eq. (C4). Finally, Eq. (C7) yields

$$0=\frac{\partial a}{\partial M}\bigg|_{T,n_{\rm eq}}=n_{\rm eq}\frac{\partial\epsilon_f}{\partial\langle\phi\rangle}-\frac{\partial P}{\partial\langle\phi\rangle} \tag{C9}$$

$$\begin{aligned}&=n_{\rm eq}\frac{\partial\epsilon_f}{\partial\langle\phi\rangle}-g\frac{M^3}{\pi^2}\int_0^\infty d\epsilon\,\epsilon^2(\epsilon^2+1)^{-1/2}(n^++n^-)-n_{\rm eq}\frac{\partial\epsilon_f}{\partial\langle\phi\rangle}\\&\quad+\frac{1}{4\pi}\left[\mu_R{}^2\langle\phi\rangle+\frac{\gamma_R}{2}\langle\phi\rangle^2+\frac{\lambda_R}{6}\langle\phi\rangle^3+\frac{2gM^3}{\pi}\ln\left[\frac{M}{m}\right]+\frac{gM^3}{2\pi}-\frac{gm^3}{2\pi}+\frac{7m^2g^2}{2\pi}\langle\phi\rangle\right.\\&\quad\left.-\frac{13mg^3}{2\pi}\langle\phi\rangle^2+\frac{25g^4}{6\pi}\langle\phi\rangle^3\right],\end{aligned} \tag{C10}$$

which is nothing but the renormalized gap equation.

---

*Send reprint requests to R. Hakim.

[1]G. Kalman, Phys. Rev. 161, 156 (1967).

[2]G. Kalman, Phys. Rev. D 9, 1656 (1974).

[3]R. Hakim, Phys. Rev. 166, 75 (1968).

[4]W. A. Bardeen, M. Chanowitz, S. D. Drell, R. Giles, M. Weinstein, and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974); J. Rafelski, *ibid.* 9, 2358 (1974).

[5]T. D. Lee and G. C. Wick, Phys. Rev. D 9, 2291 (1974); T. D. Lee and M. Margulies, *ibid.* 11, 1591 (1975); T. D. Lee, Rev. Mod. Phys. 47, 267 (1975); C.-G. Källman, Phys. Lett. 55B, 178 (1975); 57B, 183 (1975); S. A. Moszkowski and C.-G. Källman, Nucl. Phys. A287, 495 (1977); E. M. Nyman and M. Rho, Nucl. Phys. A268, 408 (1976); M. Wakamatsu and A. Hayashi, Prog. Theor. Phys. 63, 1688 (1980).

[6]R. Hakim, Riv. Nuovo Cimento 1, No. 6 (1978).

[7]R. Hakim, in *Statistical Mechanics of Quarks and Hadrons,* proceedings of the International Symposium on Statistical Mechanics of Hadrons and Quarks, Bielefeld, 1980, edited by H. Satz (North-Holland, Amsterdam, 1981).

[8]J. Diaz Alonso and R. Hakim, Phys. Lett. 66A, 476 (1978).

[9]R. Hakim and J. Heyvaerts, Phys. Rev. A 18, 1250 (1978).

[10]See, e.g., C. Nash, *Relativistic Quantum Fields* (Academic, New York, 1978), p. 71.

[11]S. A. Chin, Ann. Phys. (N.Y.) 108, 301 (1977).

[12]Chin's Lagrangian and ours differ by factors 4π, g, etc. Chin's gap equation is recovered from ours with the following changes: $\phi_{\rm chin}{}^2\to\phi^2/4\pi$ and $g_{\rm chin}{}^2\to4\pi g^2$; with these changes $\mu_R{}^2$, α_R, γ_R, λ_R have the same values in both cases.

[13]We are quite indebted to Dr. P. Bakshi for the following remarks.

[14]C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1981).