Structure of superspace in supersymmetry

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It is known that the scalar superspace in supersymmetry theory is the direct sum of chiral, antichiral, and isoscalar (or linear) fields, provided that the mass of the system is not zero. However, we show here that the situation changes drastically for the massless case. The whole superspace becomes reducible but not fully reducible. Moreover, it is indecomposable in a sense to be specified. The same reducible but indecomposable property is also shared by any chiral, antichiral, and isoscalar spaces. However, if we accept only unitary representations on positive-metric spaces, then only irreducible components of these spaces must be physically relevant. We demonstrate that these facts are intimately connected with the structure of the commutator algebra of the little Lie superalgebra as well as with a difference of Casimir invariants of the supersymmetry between the massive and massless cases.

I. INTRODUCTION

Supersymmetry[1,2] is a Lie superalgebra[3,4] with many remarkable properties. It combines both fermions and bosons with different Lorentz properties into a single multiplet. Moreover, the theory is renormalizable with fewer divergences. The simplest way to deal with supersymmetry is to start with a superspace formalism introduced by Salam and Strathdee.[5] For a review of these facts, see the article by Fayet and Ferrara.[6]

A theorem of Djoković and Hochschild[4,7] states that the only Lie superalgebra whose finite-dimensional representation is always fully reducible is the direct product of semisimple Lie algebras with finitely many simple Lie superalgebras of the type OSP(1,2n) ($n \geq 1$). In view of this theorem, we may wonder whether supersymmetry in particle physics may admit some physically nontrivial, reducible but not fully reducible, representations in superspace. The principal purpose of this paper is to answer this question. First we show in Sec. II that the so-called scalar superspace is a direct sum of three irreducible spaces consisting of chiral and antichiral spaces and one more space which we call isoscalar, provided that the mass of the system is nonzero. However, for the massless case, the situation drastically changes. We will show in Sec. IV that all of these chiral, antichiral, and isoscalar spaces are now not irreducible but indecomposable (and hence not fully reducible). Moreover, the superspace contains an additional subspace other than those mentioned above. The whole superspace is also reducible but still indecomposable. In the following sections, we show that these facts are intimately related to the existence of new Lie algebras in the superspace, as well as to different forms of Casimir invariants between the massive and massless cases.

In order to facilitate our discussion in the following sections, we will briefly recapitulate notations and definitions concerning Lie superalgebra and supersymmetry. Let L be a Lie superalgebra consisting of a bosonic sector L_B and a fermionic one L_F as

$$L = L_B \oplus L_F \tag{1.1}$$

so that we have

$$[L_B, L_B] \subset L_B, \quad [L_B, L_F] \subset L_F, \quad \{L_F, L_F\} \subset L_B \ . \tag{1.2}$$

It is convenient to introduce the signature function $\sigma(x)$ ($x \in L$) by

$$\sigma(x) = \begin{cases} 0, & \text{if } x \in L_B \\ 1, & \text{if } x \in L_F \ . \end{cases} \tag{1.3}$$

Setting

$$[x,y\} = xy - (-1)^{\sigma(x)\sigma(y)} yx \ , \tag{1.4}$$

we then have

$$[x,y\} = -(-1)^{\sigma(x)\sigma(y)}[y,x\} \tag{1.5a}$$

as well as the generalized Jacobi identity

$$(-1)^{\sigma(x)\sigma(z)}[[x,y\},z\} + (-1)^{\sigma(y)\sigma(x)}[[y,z\},x\} + (-1)^{\sigma(z)\sigma(y)}[[z,x\},y\} = 0 \ . \tag{1.5b}$$

If a subalgebra L_0 of L satisfies

$$[L_0, L\} \subset L_0 \ ,$$

then L_0 is called an ideal of L. If L contains no proper ideal, then L is simple. We define subalgebras $L^{(k)}$ and L_k ($k \geq 1$) inductively by

$$[L,L\} = L^{(1)} = L_1 \ ,$$

$$[L^{(k)}, L^{(k)}\} = L^{(k+1)}, \quad k \geq 1$$

$$[L_k, L\} = L_{k+1}, \quad k \geq 1 \ .$$

If we have $L_n = 0$ identically for some positive integer n, then L is called nilpotent, while L is said to be solvable when we have $L^{(n)} = 0$ for some n. A nilpotent algebra is automatically solvable. If L contains no solvable ideal, then L is defined to be semisimple. Schur's lemma is stated[3,4] as follows.

Schur's lemma. Let L be a Lie superalgebra in a finite-dimensional irreducible space V. If a linear operator J in the space V commutes with all elements of L, i.e., if

$[J,L\}=0$, then we have the following possibilities.

(i) If $\sigma(J)=0$, then $J=\lambda I$ where λ is a constant and I is the identity operator in V.

(ii) If $\sigma(J)=1$, then either J is identically zero in V or $\mathrm{Dim}\,V_B=\mathrm{Dim}\,V_F$, and $JV_B=V_F$, $JV_F=V_B$ with $J^2=\lambda I$. Here, Dim V_B and Dim V_F refer to the dimensions of the bosonic part V_B and the fermionic part V_F of V, respectively.

Next, for any finite-dimensional representation space V of L, we can define the supertrace[3,4] by

$$\mathrm{S\,Tr}x=\sum_b \langle b\,|\,x\,|\,b\rangle-\sum_f \langle f\,|\,x\,|\,f\rangle,\quad x\in L \tag{1.6}$$

where b and f refer to complete orthonormal sets in the bosonic part V_B and the fermionic part V_F of V, respectively. We then have[3,4]

$$\mathrm{S\,Tr}x=0 \quad \text{if } \sigma(x)=1 \tag{1.7a}$$

as well as

$$\mathrm{S\,Tr}[x,y\}=0,\quad x,y\in L\ . \tag{1.7b}$$

Supersymmetry in particle physics is a Lie superalgebra L such that its bosonic part L_B is the usual Poincaré Lie algebra consisting of P_a and J_{ab} $(=-J_{ba})$, satisfying the familiar commutation relations

$$\begin{aligned} &i[J_{ab},J_{cd}]=\eta_{bc}J_{ad}+\eta_{ad}J_{bc}-\eta_{ac}J_{bd}-\eta_{bd}J_{ac}\ ,\\ &i[J_{ab},P_c]=\eta_{bc}P_a-\eta_{ac}P_b\ ,\\ &[P_a,P_b]=0\ ,\end{aligned} \tag{1.8}$$

where latin indices a,b,c,d assume values 0,1,2,3 with the Minkowski metric η_{ab} satisfying

$$\eta_{11}=\eta_{22}=\eta_{33}=-\eta_{00}=1\ . \tag{1.9}$$

The fermionic part L_F of supersymmetry consists[6,8,9] of undotted spinor Q_α $(\alpha=1,2)$ and dotted spinor $\bar{Q}_{\dot\alpha}$ $(\dot\alpha=\dot 1,\dot 2)$ obeying the commutation relations

$$\begin{aligned} &[J_{ab},Q_\alpha]=-\frac{i}{4}(\sigma_a\bar\sigma_b-\sigma_b\bar\sigma_a)_\alpha{}^\beta Q_\beta\ ,\\ &[J_{ab},\bar Q_{\dot\alpha}]=\frac{i}{4}(\bar\sigma_a\sigma_b-\bar\sigma_b\sigma_a)^{\dot\beta}{}_{\dot\alpha}\bar Q_{\dot\beta}\ ,\\ &[P_m,Q_\alpha]=[P_m,\bar Q_{\dot\alpha}]=0\end{aligned} \tag{1.10}$$

as well as the anticommutation relations

$$\begin{aligned} &\{Q_\alpha,Q_\beta\}=\{\bar Q_{\dot\alpha},\bar Q_{\dot\beta}\}=0\ ,\\ &\{Q_\alpha,\bar Q_{\dot\beta}\}=2\sigma^m_{\alpha\dot\beta}P_m\ .\end{aligned} \tag{1.11}$$

In this paper, we are following the standard notation and summation conventions for Pauli matrices σ_a and $\bar\sigma_a$ and for spinor indices as those used by Wess[8] and by Ovrut.[9] Let

$$L_0=\langle P_m,Q_\alpha,\bar Q_{\dot\alpha}\rangle \tag{1.12}$$

be a subalgebra of L, spanned by P_m, Q_α, and $\bar Q_{\dot\alpha}$. We evidently have $[L_0,L\}\subset L_0$ and $[[L_0,L_0\},L_0\}=0$ so that L_0 is a nilpotent ideal of L. Actually, L_0 is the radical of L, i.e., the maximum solvable ideal of L. Thus, both L and L_0 are not semisimple in contrast to simple Lie superalgebras used in nuclear theory.[10]

Let θ_α $(\alpha=1,2)$ and $\bar\theta_{\dot\alpha}$ $(\dot\alpha=\dot 1,\dot 2)$ be mutually anticommuting Grassmann variables. Then, the superspace V is a vector space consisting of all smooth functions $\Phi(x,\theta,\bar\theta)$ of Minkowski coordinate x^m $(m=0,1,2,3)$, θ_α, and $\bar\theta_{\dot\alpha}$. If we set

$$\begin{aligned} &P_m=-i\frac{\partial}{\partial x^m}\equiv-i\partial_m\ ,\\ &Q_\alpha=\frac{\partial}{\partial\theta^\alpha}+\bar\theta^{\dot\alpha}\sigma^m_{\alpha\dot\alpha}P_m\ ,\\ &\bar Q_{\dot\alpha}=\frac{\partial}{\partial\bar\theta^{\dot\alpha}}+\theta^\alpha\sigma^m_{\alpha\dot\alpha}P_m\ ,\end{aligned} \tag{1.13}$$

these offer a realization of the Lie superalgebra L_0 in V. Note that these operators are derivations[3,4] in V, i.e., they satisfy the differential rule

$$D(\Phi_1\Phi_2)=(D\Phi_1)\Phi_2+(-1)^{\sigma_1\sigma_D}\Phi_1(D\Phi_2) \tag{1.14}$$

for two vectors Φ_1 and Φ_2 in V. Here, σ_1 and σ_D are the signatures of Φ_1 and D, respectively, and we assume commutativity of θ_α and $\bar\theta_{\dot\alpha}$ with any function $f(x)$ of the coordinate x^m alone which is contained in Φ_1. If we assume[8,9] contrarily anticommutativity of θ_α and $\bar\theta_{\dot\alpha}$ with any fermionic fields $\psi(x)$ contained in Φ_1, then we may set $\sigma_1=0$ in Eq. (1.14). Since the latter convention is more convenient, we will adopt it in discussions of Secs. II and III.

We may realize J_{ab} $(=-J_{ab})$, $a,b=0,1,2,3$, in V also as

$$\begin{aligned} J_{ab}=&-i(x_a\partial_b-x_b\partial_a)+\frac{i}{4}\theta(\sigma_a\bar\sigma_b-\sigma_b\bar\sigma_a)\frac{\partial}{\partial\theta}\\ &+\frac{i}{4}\bar\theta(\bar\sigma_a\sigma_b-\bar\sigma_b\sigma_a)\frac{\partial}{\partial\bar\theta}+S_{ab}\ .\end{aligned} \tag{1.15}$$

S_{ab} $(=-S_{ba})$ are the spin operators which commute with all other quantities, and which act only upon possible Lorentz indices contained in Φ. However, in this paper we consider only the case of the so-called scalar superspace where Φ contains no extra Lorentz indices and hence $S_{ab}=0$.

Let A be a polynomial algebra generated by derivatives $\partial/\partial\theta^\alpha$, $\partial/\partial\bar\theta^{\dot\alpha}$, and $\partial/\partial x^m$ as well as by multiplication operations of θ_α and $\bar\theta_{\dot\alpha}$. We denote A_0 as a subalgebra of A, generated by polynomials of Q_α, $\bar Q_{\dot\alpha}$, and P_m given by Eq. (1.13). Clearly, A_0 is a realization of the universal enveloping algebra[3,4] of L_0 in the superspace V. The commutator algebra A_0' of A_0 is defined then by

$$A_0'=\{Z\,|\,[Z,A_0\}=0,\ Z\in A\}\ . \tag{1.16}$$

It is not difficult to prove that A_0' is generated by three elements D_α, $\bar D_{\dot\alpha}$, and P_m where

$$\begin{aligned} &P_m=-i\frac{\partial}{\partial x^m}\ ,\\ &D_\alpha=\frac{\partial}{\partial\theta^\alpha}-\bar\theta^{\dot\alpha}\sigma^m_{\alpha\dot\alpha}P_m\ ,\\ &\bar D_{\dot\alpha}=-\frac{\partial}{\partial\bar\theta^{\dot\alpha}}+\theta^\alpha\sigma^m_{\alpha\dot\alpha}P_m\ .\end{aligned} \tag{1.17}$$

Note that our notations here follow those of Wess and Ovrut.[11] They satisfy

$$\{D_\alpha,D_\beta\}=\{\bar{D}_{\dot{\alpha}},\bar{D}_{\dot{\beta}}\}=0\ ,$$
$$\{D_\alpha,\bar{D}_{\dot{\beta}}\}=2\sigma^m_{\alpha\dot{\beta}}P_m\ , \tag{1.18}$$
$$[P_m,D_\alpha]=[P_m,\bar{D}_{\dot{\alpha}}]=0\ ,$$

which are exactly the same as Eq. (1.11) with replacement $Q_\alpha \leftrightarrow D_\alpha$ and $\bar{Q}_{\dot{\alpha}} \leftrightarrow \bar{D}_{\dot{\alpha}}$. This proves that A'_0 is isomorphic to A_0.

In passing, we briefly remark the following. The realization Eq. (1.13) is by no means unique. Instead of simple spinors θ_α and $\bar{\theta}_{\dot{\alpha}}$, we could have utilized more complicated Grassmann variables such as θ^m_α and $\bar{\theta}^m_{\dot{\alpha}}$ where $m=0,1,2,3$ is the Lorentz vector index, and α, $\dot{\alpha}$ refer to spinor indices. When we set

$$Q_\alpha=\sigma^m_{\alpha\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}^m_{\dot{\alpha}}}+\theta^m_\alpha P_m\ ,$$
$$\bar{Q}_{\dot{\alpha}}=\sigma^m_{\alpha\dot{\alpha}}\frac{\partial}{\partial\theta^m_\alpha}+\bar{\theta}^m_{\dot{\alpha}}P_m\ , \tag{1.19}$$

as well as

$$D_\alpha=\sigma^m_{\alpha\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}^m_{\dot{\alpha}}}-\theta^m_\alpha P_m\ ,$$
$$\bar{D}_{\dot{\alpha}}=-\sigma^m_{\alpha\dot{\alpha}}\frac{\partial}{\partial\theta^m_\alpha}+\bar{\theta}^m_{\dot{\alpha}}P_m\ , \tag{1.20}$$

then we can easily verify the validity of Eqs. (1.11) and (1.18) as well as anticommutativity between pairs $(Q_\alpha,\bar{Q}_{\dot{\alpha}})$ and $(D_\alpha,\bar{D}_{\dot{\alpha}})$, although the isomorphism of A_0 and A'_0 is no longer true for this case. Since the use of θ^m_α and $\bar{\theta}^m_{\dot{\alpha}}$ will introduce many higher-spin fields in theory, we will not consider this possibility in this paper. However, many of our results in this paper are based only upon the use of commutation relations, Eq. (1.11) and (1.18), but not upon their explicit realizations, such as Eqs. (1.13) and (1.17) or (1.19) and (1.20), so that many results given in this paper are equally applicable to both realizations.

Since $P^2=P^mP_m$ commutes with all elements of L, it is a Casimir invariant of L. If M is the mass, then we have

$$M^2=-P^2=\Box\ . \tag{1.21}$$

We will consider only physically relevant cases of $M^2\geq 0$. As we shall observe in Sec. V, L possesses another Casimir invariant whose form changes drastically for the two cases $M\neq 0$ and $M=0$.

In this paper, we are mainly concerned with finite-dimensional representations of the subalgebra L_0 defined by Eq. (1.12), rather than infinite-dimensional unitary representations of larger superalgebra L itself. Since the four-momenta P_m commute with all elements of L_0, we regard P_m to be constants p_m. Then, we are dealing effectively with subvector space $V(p)$ of V where P_m assume eigenvalues p_m. Therefore, any vector $\Phi(x,\theta,\bar{\theta})$ appearing in $V(p)$ must have the form

$$\Phi(x,\theta,\bar{\theta})=\exp(ipx)\Phi(\theta,\bar{\theta})\ . \tag{1.22}$$

However, we will use the same notation V instead of $V(p)$ for simplicity throughout this paper. If we use the realization Eq. (1.13) based upon θ_α and $\bar{\theta}_{\dot{\alpha}}$, then the dimension of V is 16. Contrarily, V will have dimension of 65 536 for another realization Eq. (1.19) based upon θ^m_α and $\bar{\theta}^m_{\dot{\alpha}}$. Also, we will not consider a physically trivial case of $p_m=0$ for all m in this paper, which could occur for the case $M=0$.

Let $V^{(0)}$ be any finite-dimensional representation space of L_0. We designate its bosonic and fermionic parts by $V_B^{(0)}$ and $V_F^{(0)}$, respectively. Then, in view of Eqs. (1.7b) and (1.11), we calculate

$$0=\mathrm{STr}[Q_\alpha,\bar{Q}_{\dot{\beta}}\}=\mathrm{STr}\{Q_\alpha,\bar{Q}_{\dot{\beta}}\}=2\sigma^m_{\alpha\dot{\beta}}p_m\mathrm{STr}1\ ,$$

where 1 is the identity. However, since we have

$$\mathrm{STr}\ 1=\mathrm{Dim}\ V_B^{(0)}-\mathrm{Dim}\ V_F^{(0)}\ ,$$

and since p_m are not identically zero, this requires

$$\mathrm{Dim}\ V_B^{(0)}=\mathrm{Dim}\ V_F^{(0)}\ . \tag{1.23}$$

In other words, the dimensions of bosonic and fermionic parts are always the same in any representation $V^{(0)}$ of L_0. Therefore, the dimension of any representation space of L_0 is always even. Especially, L_0 cannot have any one-dimensional representation space. Then, any two-dimensional representation of L_0 is automatically irreducible, since otherwise it must contain one-dimensional invariant subspace. These facts will be useful in Sec. IV, since these statements are valid for both massive and massless cases as well as when $V^{(0)}$ is not necessarily fully reducible. The validity of Eq. (1.23) also implies that representations of L_0 are always typical in terminology[3] by Kac, although L_0 is not simple.

II. DECOMPOSITION OF MASSIVE SUPERSPACE

First, we note the following proposition for later purposes.

Proposition. Let V_0 be a subspace of the superspace V, such that the Lie superalgebra L_0 [see Eq. (1.12)] is irreducible in V_0. If a linear operator J in V satisfies conditions $[J,L_0\}=0$ and $J^2=0$, then we have either $JV_0=0$ or JV_0 is disjoint from V_0. Especially, if $JV_0\subset V_0$, then $JV_0=0$ and we find [$(\bar{D}\bar{D})\equiv\bar{D}\bar{D}$ hereafter]

(i) $D_\alpha V_0=0$ if $D_\alpha V_0\subset V_0$,

(ii) $\bar{D}_{\dot{\alpha}}V_0=0$ if $\bar{D}_{\dot{\alpha}}V_0\subset V_0$,

(iii) $(DD)V_0=0$ if $(DD)V_0\subset V_0$,

(iv) $(\bar{D}\bar{D})V_0=0$ if $(\bar{D}\bar{D})V_0\subset V_0$.

The proof is simple. We first observe that $L_0V_0\subset V_0$, and $L_0(JV_0)=J(L_0V_0)\subset JV_0$ since $[J,L_0\}=0$. Therefore, when we set $V_1=V_0\cap JV_0$, we find $L_0V_1\subset V_1$. In other words, V_1 is an invariant subspace of V_0. Hence, we must have either $V_1=0$ or $V_1=V_0$ because of the irreducibility of V_0. However, $V_1=V_0$ is impossible because of the following reason. Suppose $V_1=V_0$ and hence $JV_0\supset V_0$. Then, we find $0=J^2V_0\supset JV_0\supset V_0$ since $J^2=0$. This proves $V_0\cap JV_0=0$. Thus, unless $JV_0=0$ identically, JV_0 is disjoint from V_0. Especially, if $JV_0\subset V_0$, then we must have $JV_0=0$. We remark that we could have used Schur's lemma for a proof of the second half of our proposition.

We note that we have used only basic commutation relations, Eq. (1.18), but not explicit realization, such as Eqs. (1.17) or (1.20), in our proof of this proposition. Hence, its validity is independent of any particular realization of V discussed in Sec. I. The same remark also applies to most results of this section, unless it is stated otherwise.

As we may observe from the result of this proposition, the subspaces V_0 such as $\bar{D}_{\dot\alpha}V_0=0$ are important for our discussion. We set

$$\begin{aligned} V_C&=\{\Phi \mid \bar{D}_{\dot\alpha}\Phi=0,\ \ \Phi\in V,\ \ \dot\alpha=\dot 1 \text{ and } \dot 2\}\ ,\\ V_A&=\{\Phi \mid D_\alpha\Phi=0,\ \ \Phi\in V,\ \alpha=1 \text{ and } 2\}\ ,\\ V_T&=\{\Phi \mid (DD)\Phi=(\bar{D}\bar{D})\Phi=0,\ \ \Phi\in V\} \end{aligned} \tag{2.1}$$

and call V_C, V_A, and V_T chiral, antichiral, and isoscalar (or isotopic) spaces, respectively. In view of anticommutativity between pairs $(Q_\alpha,\bar{Q}_{\dot\alpha})$ and $(D_\alpha,\bar{D}_{\dot\alpha})$, each of V_C, V_A, and V_T defines a representation of L_0. The first question we have to ask is whether they are irreducible or not. The answer depends upon two cases of $M\neq0$ and $M=0$. We shall prove that for the massive case $M\neq0$ with realization of Eqs. (1.13) and (1.17), they are indeed irreducible. However, for the massless case $M=0$, we will show in Sec. IV that they are not irreducible but nevertheless indecomposable (hence not fully reducible).

Hereafter in this section, we assume $M\neq0$. Although many of the results in this section are more or less known in the literature, we will discuss the decomposition of V in some detail here, since it is relevant for comparison with the case of $M=0$, and since some results are certainly new with some generalizations.

We first define I_+, I_-, and I_3 by

$$\begin{aligned} I_+&=-\frac{1}{4M}(\bar{D}\bar{D})=-\frac{1}{4M}\epsilon^{\dot\alpha\dot\beta}\bar{D}_{\dot\alpha}\bar{D}_{\dot\beta}\ ,\\ I_-&=-\frac{1}{4M}(DD)=\frac{1}{4M}\epsilon^{\alpha\beta}D_\alpha D_\beta\ ,\\ I_3&=-\frac{1}{8M^2}\sigma^m_{\alpha\dot\alpha}[D^\alpha,\bar{D}^{\dot\alpha}]P_m\\ &=\frac{1}{4M^2}\bar{D}\bar\sigma^m DP_m-\tfrac{1}{2}=-\frac{1}{4M^2}D\sigma^m\bar{D}P_m+\tfrac{1}{2} \end{aligned} \tag{2.2}$$

in the notation of Refs. 8 and 9. Using the commutation relation Eq. (1.18), it is not difficult to verify that they satisfy quadratic identities:

$$\begin{aligned} I_3I_+&=-I_+I_3=\tfrac{1}{2}I_+\ ,\\ I_3I_-&=-I_-I_3=-\tfrac{1}{2}I_-\ ,\\ (I_+)^2&=(I_-)^2=0\ ,\\ I_+I_-&=2(I_3)^2+I_3\ ,\\ I_-I_+&=2(I_3)^2-I_3\ . \end{aligned} \tag{2.3}$$

In particular, I_+, I_-, and I_3 satisfy an SU(2) commutation relation:

$$\begin{aligned} [I_3,I_+]&=I_+\ ,\\ [I_3,I_-]&=-I_-\ ,\\ [I_+,I_-]&=2I_3\ . \end{aligned} \tag{2.4}$$

For simplicity, we call the SU(2) a pseudo-isotropic spin group or simply isotropic spin group. The Casimir invariant $(\vec{\mathrm{I}})^2$ of SU(2) is given by

$$(\vec{\mathrm{I}})^2=(I_3)^2+\tfrac{1}{2}(I_+I_-+I_-I_+)\ , \tag{2.5}$$

which is rewritten as

$$(\vec{\mathrm{I}})^2=3(I_3)^2 \tag{2.6}$$

because of Eq. (2.3). Moreover, Eq. (2.3) imposes constraints

$$(I_3)^3=\tfrac{1}{4}I_3\ , \tag{2.7a}$$

$$(\vec{\mathrm{I}})^2[(\vec{\mathrm{I}})^2-\tfrac{3}{4}]=0\ . \tag{2.7b}$$

Equations (2.7) imply that the total isospin I can assume only values 0 and $\frac{1}{2}$, while I_3 can have values 0, $\frac{1}{2}$, and $-\frac{1}{2}$ only. From Eqs. (2.1) and (2.2), we see that chiral and antichiral fields are eigenstates of I_3 with eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Also, since $(DD)\Phi_T=(\bar{D}\bar{D})\Phi_T=0$, the isoscalar field Φ_T belongs to $I=0$ and hence it has a zero eigenvalue for I_3. As we shall see shortly, we can actually prove the following stronger statements:

$$\begin{aligned} V_C&=\{\Phi \mid I_3\Phi=\tfrac{1}{2}\Phi\}\ ,\\ V_A&=\{\Phi \mid I_3\Phi=-\tfrac{1}{2}\Phi\}\ ,\\ V_T&=\{\Phi \mid I_3\Phi=0\}\ . \end{aligned} \tag{2.8}$$

Then, both V_C and V_A belong to $I=\frac{1}{2}$, while V_T has $I=0$. This is the reason why we call V_T an isoscalar here instead of the terminologies[6,12-14] of linear or transverse fields used in the literature.

The projection operators P_1, P_2, and P_T for $I_3=\frac{1}{2}$, $-\frac{1}{2}$, and 0 states, respectively, are evidently given by

$$\begin{aligned} P_1&=2(I_3)^2+I_3=I_+I_-=\frac{1}{16M^2}(\bar{D}\bar{D})(DD)\ ,\\ P_2&=2(I_3)^2-I_3=I_-I_+=\frac{1}{16M^2}(DD)(\bar{D}\bar{D})\ ,\\ P_T&=1-4(I_3)^2=-\frac{1}{8M^2}D^\alpha(\bar{D}\bar{D})D_\alpha\\ &=-\frac{1}{8M^2}\bar{D}_{\dot\alpha}(DD)\bar{D}^{\dot\alpha}\ , \end{aligned} \tag{2.9}$$

which satisfy the orthogonality condition

$$P_jP_k=\delta_{jk}P_j\quad (j,k=1,2,T)\ , \tag{2.10}$$

as well as the completeness condition

$$P_1+P_2+P_T=1\ . \tag{2.11}$$

These properties of P_1, P_2, and P_T together with P_+ $(=I_+)$ and P_- $(=I_-)$ are already well known in the literature (e.g., see Ref. 11). The new fact is that P_1, P_2, and P_T are projection operators for three eigenvalues $I_3=\frac{1}{2}$, $-\frac{1}{2}$, and 0 of the new isospin SU(2) group.

We now prove Eq. (2.8). In view of Eqs. (2.10) and (2.11), the superspace V is split as a direct sum,

$$V=V_1\oplus V_2\oplus V_3 , \tag{2.12}$$

when we set

$$V_1=P_1V,\quad V_2=P_2V,\quad V_3=P_TV . \tag{2.13}$$

Our task is to prove

$$V_1=V_C,\quad V_2=V_A,\quad V_3=V_T \tag{2.14}$$

so that Eq. (2.12) can be written as

$$V=V_C\oplus V_A\oplus V_T . \tag{2.15}$$

First of all, it is almost evident that we have

$$V_C\subset V_1,\quad V_A\subset V_2,\quad V_T\subset V_3 , \tag{2.16}$$

since V_C, V_A, and V_T are eigenspaces for $I_3=\frac{1}{2}$, $-\frac{1}{2}$, and 0, respectively. To prove Eq. (2.14), let us first note $\bar{D}_{\dot\alpha}P_1=0=D_\alpha P_2$ from Eq. (2.9) so that we find $\bar{D}_{\dot\alpha}V_1=0=D_\alpha V_2$. This proves $V_1\subset V_C$ and $V_2\subset V_A$, and hence $V_1=V_C$ and $V_2=V_A$ in view of Eq. (2.16). The last identity $V_T=V_3$ is almost self-evident. This shows the validity of Eq. (2.14) and hence of Eq. (2.8). The decomposition, Eq. (2.15), for V is known in the literature when V is constructed by Grassmann variables θ_α and $\bar\theta_{\dot\alpha}$. However, since we did not utilize any particular realization for D_α and $\bar{D}_{\dot\alpha}$, our conclusion is also valid even when we use θ_α^m and $\bar\theta_{\dot\alpha}^m$ as in Sec. I.

Before proceeding further, we can also prove Eq. (2.14) as follows. With this aim, we introduce F_α and $\bar{F}_{\dot\alpha}$:

$$\begin{aligned} F_\alpha&=\frac{1}{M}\bar{D}^{\dot\alpha}\sigma^m_{\alpha\dot\alpha}P_m ,\\ \bar{F}_{\dot\alpha}&=-\frac{1}{M}D^\alpha\sigma^m_{\alpha\dot\alpha}P_m . \end{aligned} \tag{2.17}$$

Then, we can readily show that the two pairs (F_α,D_α) and $(\bar{D}_{\dot\alpha},\bar{F}_{\dot\alpha})$ behave as isospinors with respect to the SU(2). Indeed, we find

$$\begin{aligned} &[I_3,F_\alpha]=\tfrac{1}{2}F_\alpha ,\\ &[I_3,D_\alpha]=-\tfrac{1}{2}D_\alpha ,\\ &[I_+,D_\alpha]=F_\alpha ,\\ &[I_-,F_\alpha]=D_\alpha ,\\ &[I_+,F_\alpha]=[I_-,D_\alpha]=0 \end{aligned} \tag{2.18a}$$

as well as

$$\begin{aligned} &[I_3,\bar{D}_{\dot\alpha}]=\tfrac{1}{2}\bar{D}_{\dot\alpha} ,\\ &[I_3,\bar{F}_{\dot\alpha}]=-\tfrac{1}{2}\bar{F}_{\dot\alpha} ,\\ &[I_+,\bar{F}_{\dot\alpha}]=\bar{D}_{\dot\alpha} ,\\ &[I_-,\bar{D}_{\dot\alpha}]=\bar{F}_{\dot\alpha} ,\\ &[I_+,\bar{D}_{\dot\alpha}]=[I_-,\bar{F}_{\dot\alpha}]=0 . \end{aligned} \tag{2.18b}$$

Suppose now that $\Phi\in V_1$, i.e., $I_3\Phi=\frac{1}{2}\Phi$, then, in view of Eq. (2.18b), we calculate

$$I_3(\bar{D}_{\dot\alpha}\Phi)=\bar{D}_{\dot\alpha}\Phi$$

so that $\bar{D}_{\dot\alpha}\Phi$ is an eigenstate of I_3 with eigenvalue one. But, we have already noted that this is not possible so that $\bar{D}_{\dot\alpha}\Phi=0$. This proves $V_1\subset V_C$ as before, and hence $V_1=V_C$. Similarly, we find $V_2=V_A$. It is interesting to see that we have

$$\begin{aligned} &\{F_\alpha,F_\beta\}=\{\bar{F}_{\alpha\beta},\bar{F}_{\dot\beta}\}=0 ,\\ &\{F_\alpha,\bar{F}_{\dot\alpha}\}=-2\sigma^m_{\alpha\dot\alpha}P_m . \end{aligned} \tag{2.19}$$

Therefore, P_m, F_α, and $-\bar{F}_{\dot\alpha}$ satisfy exactly the same commutation relations as P_m, D_α, and $\bar{D}_{\dot\alpha}$ with identification $D_\alpha\leftrightarrow F_\alpha$ and $\bar{D}_{\dot\alpha}\leftrightarrow-\bar{F}_{\dot\alpha}$. This implies that we could have used F_α and $-\bar{F}_{\dot\alpha}$ instead of D_α and $\bar{D}_{\dot\alpha}$ from the beginning. Also, P_m, D_α, $\bar{D}_{\dot\alpha}$, F_β, and $\bar{F}_{\dot\beta}$ define a new type of Lie superalgebra. For example, we have

$$\begin{aligned} &\{F_\alpha,D_\beta\}=-2M\epsilon_{\alpha\beta} ,\\ &\{\bar{F}_{\dot\alpha},\bar{D}_{\dot\beta}\}=+2M\epsilon_{\dot\alpha\dot\beta} . \end{aligned} \tag{2.20}$$

However, we will not discuss this here, since a more general case will be presented in Sec. V. We simply remark that two isodoublets of SU(2) are related to each other by

$$\begin{bmatrix}\bar{D}_{\dot\alpha}\\ \bar{F}_{\dot\alpha}\end{bmatrix}=\frac{i}{M}\sigma^m_{\alpha\dot\alpha}\partial_m\begin{bmatrix}F^\alpha\\ D^\alpha\end{bmatrix} . \tag{2.21}$$

In view of mutually anticommuting properties between the two pairs $(Q_\alpha,\bar{Q}_{\dot\alpha})$ and $(D_\alpha,\bar{D}_{\dot\alpha})$, we find $[I_3,L_0]=0$ so that any of V_C, V_A, and V_T is a representation space of L_0. The next question is whether these are irreducible or not. The answer now depends upon a choice of superspace. We consider here the case that V is defined in terms of Grassmann variables θ_α and $\bar\theta_{\dot\alpha}$ as in Eq. (1.13) but not in terms of more complicated variables θ_α^m and $\bar\theta_{\dot\alpha}^m$ as in Eq. (1.19). In that instance we can show the irreducibility of V_1 and V_2, while V_T is also irreducible in a sense to be specified shortly. For this purpose, we first observe that any representation of L_0 is fully reducible as long as $M\neq0$. This is due to the fact that L_0 is isomorphic to algebra $\tilde{L}_0$ with the same structure as L_0 but with its four-momenta P_m evaluated at the rest frame, i.e., $P_m=(M,0,0,0)$ because of Lorentz covariance of the theory. However, $\tilde{L}_0$ forms a basis of a Dirac-Clifford algebra, where the full reducibility of its representations is well established[15,16] with only one four-dimensional irreducible representation for the present case of $\tilde{L}_0$. Therefore, the same remark also applies to L_0.

The criteria of the irreducibility of spaces V_C, V_A, and V_T is equivalent to the primitivity[17] of the corresponding projection operators P_1, P_2, and P_T in the commutator algebra A_0' discussed in Sec. I. In other words, we have to study whether any of these can be written as a sum of two nontrivial mutually orthogonal idempotents (i.e., projection operators) or not. It is not difficult to verify the primitivity of P_1 and P_2 so that both chiral space V_C and antichiral space V_A are irreducible with respect to L_0. Moreover, their dimensions must be four. However, P_T is not primitive in A'_0. Indeed, it can be split into the sum of two nontrivial mutually orthogonal idempotents e_1 and e_2:

$$P_T = e_1 + e_2 \ , \quad e_j e_k = \delta_{jk} e_j \quad (j,k=1,2) \ . \tag{2.22}$$

The most general forms of e_1 and e_2 are given by

$$e_1 = \tfrac{1}{4} D(\sigma^m l_m)\bar{D} - P_2 \ , \quad e_2 = \tfrac{1}{4}\bar{D}(\bar{\sigma}^m l_m) D - P_1 \ . \tag{2.23}$$

Here, l^m is an arbitrary but fixed four-dimensional constant vector satisfying the conditions

$$l^2 = l^m l_m = 0, \quad lp = l^m p_m = -1 \ . \tag{2.24}$$

Such a vector l^m always exists. However, l^m can never be proportional to p^m. Because of Eq. (2.22), V_T splits as

$$V_T = V'_T \oplus V''_T \ , \tag{2.25a}$$

where we have set

$$V'_T = e_1 V_T = D(\sigma l)\bar{D} V_T \ , \quad V''_T = e_2 V_T = \bar{D}(\bar{\sigma} l) D V_T \tag{2.25b}$$

since $P_1 V_T = P_2 V_T = 0$. Moreover, e_1 and e_2 are now primitive, since any idempotent e satisfying $e P_T = P_T e = e$ must be expressed in the forms specified by Eq. (2.23) for some l^m. This proves that V_T is a direct sum of two four-dimensional irreducible representation spaces with respect to L_0. However, the decomposition, Eq. (2.25), for V_T is *not* Lorentz invariant, since e_1 and e_2 are not Lorentz invariant because of the presence of the constant four-vector l^m. Therefore, if we insist upon Lorentz-convariant decomposition, then V_T is irreducible. More precisely, we have to adjoin some linear combinations of J_{ab} to L_0, which leave the four-momenta p_m invariant. The resulting algebra $L_0(p)$ is the little Lie superalgebra of L. With respect to $L_0(p)$, V_T is now irreducible with dimension eight. The irreducibility of V_T under $L_0(p)$ will be seen also from the explicit construction of V_T which is given in Sec. III. In Sec. IV, we show that a similar situation also exists for the massless case. In conclusion, we find that the massive superspace V is a direct sum of three irreducible spaces V_C, V_A, and V_T with dimensions four, four, and eight, respectively. We should remark that this fact is more or less known in literature, although its connection with the SU(2) algebra in A'_0 is perhaps new.

The following peculiar property of our isospin group SU(2) should be mentioned. First, let Φ_1 and Φ_2 be two chiral fields. Then, the product $\Phi_1\Phi_2$ is also a chiral field as is well known. This implies that a product of two fields with $I_3 = \frac{1}{2}$ gives another field with $I_3 = \frac{1}{2}$. Similarly, let Φ_1 and Φ_2 be isoscalar fields. Then, $\Phi_1\Phi_2$ is no longer isoscalar as we may see from

$$(DD)(\Phi_1\Phi_2) = (DD\Phi_1)\Phi_2 + \Phi_1(DD\Phi_2) + 2(D^\alpha\Phi_1)(D_\alpha\Phi_2) \ . \tag{2.26}$$

In other words, a product of two $I=0$ fields does not give a pure $I=0$ field but contains a mixture of $I_3 = \frac{1}{2}$ and $I_3 = -\frac{1}{2}$ components. These apparently peculiar results are due to the fact that I_+, I_-, and I_3 are second-order (instead of the first-order) differential operators. Especially, they do not satisfy the differential rule, Eq. (1.14), as we may see from Eq. (2.26). In other words, they are not derivations in V, although they are elements of A'_0.

In ending this section, we may impose the Hermiticity constraint

$$(\theta_\alpha)^\dagger = \bar{\theta}_{\dot{\alpha}}, \quad (Q_\alpha)^\dagger = \bar{Q}_{\dot{\alpha}}$$

in some sense. We can then impose the reality condition

$$(\Phi_T)^\dagger = \Phi_T \tag{2.27}$$

for isoscalar field Φ_T, if we wish. However, this notion of the real isoscalar field should not be confused with the so-called vector field[6,8,9] used in literature, which satisfies the same condition, Eq. (2.27). The latter is not irreducible, but a real combination of all real isoscalar, chiral, and antichiral fields. The use of nonirreducible representation for the vector field appears to be necessary for maintaining local gauge symmetry. As we shall see in Sec. IV, it corresponds to a indecomposable representation when it is massless.

III. ISOSCALAR SPACE

Although uses of the isocalar field (often called linear or transverse) have been occasionally considered[6,12–14] in the literature, its systematic study does not appear to have been seriously undertaken. We will investigate the structure of V_T in some detail, since its knowledge is important for the proof in Sec. IV of indecomposability of the superspace V for the massless case.

When we note

$$[DD, \bar{D}\bar{D}] = -16M^2 + 8D\sigma^m\bar{D}P_m \tag{3.1}$$

then the isoscalar space V_T is defined by

$$(DD)\Phi_T = (\bar{D}\bar{D})\Phi_T = (2M^2 - D\sigma^m\bar{D}P_m)\Phi_T = 0 \tag{3.2}$$

for $\Phi_T \in V_T$, irrespective of $M \neq 0$ or $M=0$.

In this section, we restrict ourselves to the superspace constructed in terms of θ_α and $\bar{\theta}_{\dot{\alpha}}$. Then, using Eq. (1.17) for explicit expressions for D_α and $\bar{D}_{\dot{\alpha}}$, the general solution of Eq. (3.2) is found to have the form $[(\overline{\theta\theta}) \equiv \bar{\theta}\bar{\theta}$ hereafter$]$

$$\Phi_T = A(x) + (\theta\sigma^m\bar{\theta})B_m(x) - \tfrac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box A(x) + \sqrt{2}\theta\psi(\bar{y}) + \sqrt{2}\bar{\theta}\bar{\psi}(y) \ , \tag{3.3}$$

where y^m and $\bar{y}^m$ are defined by

$$y^m = x^m + i\theta\sigma^m\bar{\theta} \ , \quad \bar{y}^m = x^m - i\theta\sigma^m\bar{\theta} \tag{3.4}$$

and where $B_m(x)$ must satisfy the constraint

$$\partial^m B_m(x) = 0 \ . \tag{3.5}$$

In view of Eq. (3.3) with this constraint, Eq. (3.5), the dimension of V_T is clearly eight in agreement with the result of Sec. II for the case of $M\neq 0$.

Now, we operate Q_α and $\bar{Q}_{\dot\alpha}$ given by Eq. (1.13) to the expression, Eq. (3.3), for Φ_T. This induces linear transformations among these eight functions $A(x)$, $B_m(x)$, $\psi_\alpha(x)$, and $\bar\psi_{\dot\alpha}(x)$. Following the usual convention,[8] we use the same notations Q_α and $\bar{Q}_{\dot\alpha}$ for these induced linear transformations in V_T. Also, the same symbol V will be used for similarly induced representation spaces hereafter. Then, we can easily find

$$\begin{aligned} &Q_\alpha A(x)=\sqrt{2}\psi_\alpha(x)\ ,\\ &Q_\alpha\bar\psi_{\dot\alpha}(x)=-\frac{1}{\sqrt{2}}\sigma^m_{\alpha\dot\alpha}[B_m(x)-i\partial_m A(x)]\ ,\\ &Q_\alpha\psi_\beta(x)=0\ ,\\ &Q_\alpha\bar\psi_{\dot\alpha}(x)=-\frac{1}{\sqrt{2}}\sigma^m_{\alpha\dot\alpha}[B_m(x)-i\partial_m A(x)] \end{aligned} \tag{3.6}$$

for the operation of Q_α and

$$\begin{aligned} &\bar{Q}_{\dot\alpha}A(x)=-\sqrt{2}\bar\psi_{\dot\alpha}(x)\ ,\\ &\bar{Q}_{\dot\alpha}B_m(x)=\frac{i}{\sqrt{2}}(\bar\sigma_m\sigma_l-\bar\sigma_l\sigma_m)_{\dot\alpha\dot\beta}\partial^l\bar\psi^{\dot\beta}(x)\ ,\\ &\bar{Q}_{\dot\alpha}\psi_\alpha(x)=-\frac{1}{\sqrt{2}}\sigma^m_{\alpha\dot\alpha}[B_m(x)+i\partial_m A(x)]\ ,\\ &\bar{Q}_{\dot\alpha}\bar\psi_{\dot\beta}(x)=0 \end{aligned} \tag{3.7}$$

for $\bar{Q}_{\dot\alpha}$, from which we can again verify the validity of Eq. (1.11).

We see that $B_m(x)$ is invariant under the action of both Q_α and $\bar{Q}_{\dot\alpha}$, apart from the total divergence. Therefore,

$$\int d^4x\, B_m(x) \text{ and } \int d^3x\, B_0(x)$$

are invariant against the actions of the supersymmetry L_0. Note that the latter is also Lorentz invariant because of Eq. (3.5). The constraint, Eq. (3.5), i.e., $\partial^m B_m(x)=0$, is a direct consequence of $I_3\Phi_T=0$. We can rewrite the latter in the form

$$\partial^m V_m(x)=0\ , \tag{3.8}$$

where we have set

$$V_m(x)=-\tfrac{1}{4}(\sigma_m)_{\alpha\dot\alpha}[D^\alpha,\bar{D}^{\dot\alpha}]\Phi_T\ . \tag{3.9}$$

Inserting the explicit solution Eq. (3.3) into Eq. (3.9), we calculate

$$\begin{aligned} V^m(x)=&B^m(x)-e^{mlab}(\theta\sigma_l\bar\theta)\partial_a B_b(x)\\ &+(\theta\sigma_l\bar\theta)(\eta^{ml}\Box-\partial^m\partial^l)A(x)\\ &+\frac{i}{\sqrt{2}}(\sigma^m\bar\sigma^l-\sigma^l\bar\sigma^m)_\beta{}^\alpha\theta_\alpha\partial_l\psi^\beta(\bar{y})\\ &-\frac{i}{\sqrt{2}}(\bar\sigma^m\sigma^l-\bar\sigma^l\sigma^m)^{\dot\alpha}{}_{\dot\beta}\bar\theta^{\dot\beta}\partial_l\bar\psi_{\dot\alpha}(y) \end{aligned} \tag{3.10}$$

so that Eq. (3.8) is equivalent to Eq. (3.5).

As we noted in Sec. II, the space V_T is not strictly irreducible even for the case $M\neq 0$, since we can split V_T as in Eq. (2.25). However, it is clear that the decomposition is not Lorentz covariant.

Let Φ_T and Φ'_T be two isoscalar fields. Then, only the bilinear supersymmetric invariant is proportional to[8,9]

$$\begin{aligned} L_0(x)=&\int\int d^2\theta\, d^2\bar\theta\,\Phi_T(x,\theta,\bar\theta)\Phi'_T(x,\theta,\bar\theta)\\ =&\tfrac{1}{2}[\partial^m A(x)\partial_m A'(x)-B^m(x)B'_m(x)]\\ &-i\psi(x)\sigma^m\partial_m\bar\psi'(x)-i\bar\psi(x)\bar\sigma^m\partial_m\psi'(x)\ , \end{aligned} \tag{3.11}$$

apart from some terms proportional to total divergence. If we choose $\Phi'_T=\Phi_T$ or $\Phi'_T=\Phi^\dagger_T$, then we see that this represents free massless spin-0 and spin-$\frac{1}{2}$ fields. Note that $B_m(x)$ has no kinetic term. However, once we introduce self-interactions, the theory will in general become unrenormalizable. Although this fact is known,[14] we show that the reason for it has a simple dimensional basis. We consider here the case of a real isoscalar field Φ, dropping the subscript T. Then, we have already noted that both Φ,Φ and $\Phi\Phi\Phi$ are no longer pure isoscalar fields, and the only Lorentz-invariant Lagrangian must be[8,9] of the form

$$L(x)=\int\int d^2\theta\, d^2\bar\theta(\tfrac{1}{2}\Phi^2+\tfrac{1}{6}g\Phi^3)\ , \tag{3.12}$$

where g is a coupling parameter. We note that $d\theta_\alpha$ and $d\bar\theta_{\dot\alpha}$ must have[8] the length dimension of $l^{-1/2}$ where l is a length. Since the Lagrangian L must possess the canonical dimension l^{-4}, the field Φ must have the dimension of l^{-1}. Therefore, Eq. (3.12) requires the coupling parameter g to possess the dimension l. In other words, the cubic interaction is unrenormalizable[14] on the basis of dimensional counting, since it corresponds to the second-kind interaction.[18] We should contrast this fact with the cubic interaction containing only chiral field Φ_C, since the Lagrangian is now replaced by[8,9]

$$\begin{aligned} L(x)=&\int\int d^2\theta\, d^2\bar\theta\,\Phi_C^\dagger\Phi_C+g_0\int d^2\theta(\Phi_C)^3\\ &+g_0\int d^2\bar\theta(\Phi_C^\dagger)^3\ . \end{aligned}$$

The coupling parameter g_0 is dimensionless, so that L is renormalizable by dimensional counting.

In spite of the unrenormalizability, the theory described by Eq. (3.12) is of some theoretical interest. To be definite, we assume that we are dealing with a massive field. The equation of motion may be derived from the action principle

$$\delta\int d^4x\, L(x)=0\ . \tag{3.13}$$

However, we have to impose a constraint

$$P_T\Phi=\Phi \tag{3.14}$$

for Φ, since Φ is supposed to be isoscalar. Here, P_T is given by

$$P_T=-\frac{1}{8\Box}D^\alpha\bar{D}^2D_\alpha\ . \tag{3.15}$$

Therefore, the variation $\delta\Phi$ must obey the same constraint, so that it can be written as

$$\delta\Phi=P_T\delta X$$

for arbitrary superfield X. Then, the variational problem, Eq. (3.13), for arbitrary variation $\delta\mathcal{X}$ leads to the equation of motion

$$\Phi=-\tfrac{1}{2}gP_T(\Phi^2) \tag{3.16}$$

when we note Eq. (3.14). Multiplying $\Box$ on both sides of Eq. (3.16) and noting

$$\Box\Phi=-\tfrac{1}{8}D^{\alpha}(\bar{D}^2)D_{\alpha}\Phi$$

by Eqs. (3.14) and (3.15), we can rewrite Eq. (3.16) as

$$(1+g\Phi)\Box\Phi=-\tfrac{1}{4}g(D^{\alpha}\bar{D}^{\dot\beta}\Phi)(\bar{D}_{\dot\beta}D_{\alpha}\Phi)+\frac{i}{2}g\sigma^m_{\alpha\dot\alpha}[(D^{\alpha}\partial_m\Phi)(\bar{D}^{\dot\alpha}\Phi)-(\bar{D}^{\dot\alpha}\partial_m\Phi)(D^{\alpha}\Phi)]\ . \tag{3.17}$$

This is somewhat reminiscent of nonlinear realization[19] of the chiral $SU_L(2)\otimes SU_R(2)$ theory. We should mention the fact that the special case $g=0$ leads to $\Phi=0$ identically in view of Eq. (3.16). For such a case, we cannot use the present formulation. However, Ferrara and Zumino[13] constructed a constraint-free Lagrangian for the free massive isoscalar field.

Concluding this section, we simply mention general forms of chiral field Φ_C and antichiral field Φ_A here for the sake of comparison. The general solutions of $\bar{D}_{\dot\alpha}\Phi_C=0$ and $D_{\alpha}\Phi_A=0$ are well known to be

$$\begin{aligned}\Phi_A&=A(y)+\sqrt{2}\theta\psi(y)+(\theta\theta)F(y)\ ,\\ \Phi_C&=A'(\bar{y})+\sqrt{2}\bar\theta\bar\psi(\bar{y})+(\bar\theta\bar\theta)F'(\bar{y})\end{aligned} \tag{3.18}$$

for some scalar functions $A(x)$, $F(x)$, $A'(x)$, $F'(x)$, and spinors $\psi_{\alpha}(x)$, $\bar\psi_{\dot\alpha}(x)$.

IV. INDECOMPOSABILITY OF MASSLESS SUPERSPACE

Here, we consider the massless case $M=0$. However, we exclude the trivial case where all four-momenta P^m are identically zero. In Sec. II, it has been shown that the superspace is a direct sum of chiral V_C, antichiral V_A, and isoscalar V_T spaces. However, the situation changes drastically for the massless case. First, I_+, I_-, and I_3 of Sec. II have no meaning at all. The simplest way to obtain a new algebra for $M=0$ is to use the method of contraction. We multiply suitable powers of M to Eqs. (2.3), (2.4), and (2.7) and take the limit $M\to 0$. Then, the SU(2) algebra will contract to

$$\begin{aligned}&[DD,J]=[\bar{D}\bar{D},J]=0\ ,\\ &[\bar{D}\bar{D},DD]=16J\ ,\end{aligned} \tag{4.1}$$

where we have set

$$J=-\tfrac{1}{2}(D\sigma^m\bar{D})P_m=\tfrac{1}{2}(\bar{D}\bar\sigma^mD)P_m\ . \tag{4.2}$$

Similarly, we find

$$J^2=J(DD)=(DD)J=J(\bar{D}\bar{D})=(\bar{D}\bar{D})J=0\ . \tag{4.3}$$

Let $\hat{L}_0$ be the Lie algebra defined by Eq. (4.1). Then, $\hat{L}_0$ is nilpotent and hence solvable, since we find $[[\hat{L}_0,\hat{L}_0],\hat{L}_0]=0$ identically. In view of Lie's theorem[20] on solvable Lie algebras, any irreducible representation is one-dimensional, and moreover, representations are not in general fully reducible. In particular, there cannot exist analogs of projection operators P_1, P_2, and P_T of Sec. II, which are based upon the simple Lie algebra SU(2). Therefore, we have to study the problem in a different way. We shall prove in this section that each of V, V_C, V_A, and V_T is not irreducible but not fully reducible. Moreover, V_C and V_A are indecomposable, while V_T and V are also indecomposable in a sense to be specified shortly. We will prove these facts in several steps below.

A. V not fully reducible

First of all, J defined by Eq. (4.2) can be written also as

$$J=\tfrac{1}{2}Q\sigma^m\bar{Q}P_m=-\tfrac{1}{2}\bar{Q}\bar\sigma^mQP_m \tag{4.4}$$

when we note Eqs. (1.13) and (1.17) together with $(\sigma P)(\bar\sigma P)=(\bar\sigma P)(\sigma P)=-P^2=0$. Since pairs $(Q_{\alpha},\bar{Q}_{\dot\alpha})$ and $(D_{\alpha},\bar{D}_{\dot\alpha})$ anticommute with each other, Eq. (4.2) implies

$$[J,Q_{\alpha}]=[J,\bar{Q}_{\dot\alpha}]=[J,P_m]=0 \tag{4.5}$$

so that J is a Casimir invariant of L_0. Actually, J is also a Casimir invariant of the larger algebra L since $[J,J_{ab}]=0$. We will return to this point in Sec. V.

Now, we suppose that the whole superspace V is fully reducible with respect to L_0 as a direct sum,

$$V=\sum_j\oplus V_j\ , \tag{4.6}$$

of some irreducible representation spaces V_j of L_0. We then prove that it will lead to a contradiction. Because of Eqs. (4.4) and (4.5), we see $JV_j\subset V_j$ and $[J,L_0]=0$. Therefore, the proposition of Sec. II or Schur's lemma requires

$$JV_j=0$$

for each index j when we note $J^2=0$ from Eq. (4.3). As a consequence, we must have

$$JV=0$$

if V is fully reducible as in Eq. (4.6). However, this is impossible, since for example

$$\Phi=(\theta\theta)(\bar\theta\bar\theta)G(x)\in V$$

obeys $J\Phi\neq 0$. This proves that V cannot be fully reducible.

In the proof given above, we could have used J_{α} and $\bar{J}_{\dot\alpha}$ instead of J, where J_{α} and $\bar{J}_{\dot\alpha}$ are fermionic Casimir invariants of L_0 (but not L) for $M=0$, given by

$$\begin{aligned}J_{\alpha}&=\tfrac{1}{2}P_m\sigma^m_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha}=-\tfrac{1}{2}P_m\sigma^m_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\ ,\\ \bar{J}_{\dot\alpha}&=\tfrac{1}{2}Q^{\alpha}\sigma^m_{\alpha\dot\alpha}P_m=\tfrac{1}{2}D^{\alpha}\sigma^m_{\alpha\dot\alpha}P_m\ .\end{aligned} \tag{4.7}$$

We can readily verify $[J_{\alpha},L_0\}=[\bar{J}_{\dot\alpha},L_0\}=0$ as well as $(J_{\alpha})^2=(\bar{J}_{\dot\alpha})^2=0$, when we note $(\sigma P)(\bar\sigma P)=(\bar\sigma P)(\sigma P)=-P^2=0$. We show in Sec. IV E that V is indecomposable if we allow only Lorentz-covariant decompositions for V_j in Eq. (4.6).

We remark that any irreducible as well as any fully reducible subrepresentation space $V^{(0)}$ of V must satisfy

$$JV^{(0)}=J_\alpha V^{(0)}=\bar{J}_{\dot\alpha}V^{(0)}=0 \tag{4.8}$$

by the same reasoning used in the proof of V being not fully reducible. Examples of $V^{(0)}$ satisfying Eq. (4.8) are spaces V_{CT} and V_{AT} to be given in Sec. IV B. In this connection, we note the validity of

$$\{J_\alpha,\bar{J}_{\dot\alpha}\}=0=J^2$$

for the massless case. Therefore, if we restrict ourselves to consideration of only the so-called unitary representations which satisfy the self-adjointness conditions $\bar{Q}_{\dot\alpha}=(Q_\alpha)^\dagger$ and hence $J^\dagger=J$ as well as $(J_\alpha)^\dagger=\bar{J}_{\dot\alpha}$, then any physical subspace $\tilde{V}$ must obey the same condition,

$$J\tilde{V}=J_\alpha\tilde{V}=\bar{J}_{\dot\alpha}\tilde{V}=0\ , \tag{4.8$'$}$$

as Eq. (4.8), whether $\tilde{V}$ is fully reducible or not. We still see at the end of this section that $\tilde{V}$ coincides, however, with the direct sum of all irreducible subspaces contained in V. Since $JV\neq0$ and $J_\alpha V\neq0$, we cannot impose the positive-metric condition to all of the superspace. Presumably, we have to introduce an indefinite-metric structure in V. Note that V contains massless particles possibly with local gauge symmetry. Then, $\tilde{V}$ will be identified with the maximum positive-metric subspace of V.

B. Spaces V_0, V_{CT}, and V_{AT}

In contrast to the massive case, V_C, V_A, and V_T can now have common intersections, although we still have $V_C\cap V_A=0$, i.e., V_C and V_A have no common intersection. We now define V_{CT} and V_{AT} by

$$\begin{aligned}V_{CT}&=V_C\cap V_T\equiv\{\Phi\mid \bar{D}_{\dot\alpha}\Phi=(DD)\Phi=0\quad \Phi\in V\}\ ,\\ V_{AT}&=V_A\cap V_T\equiv\{\Phi\mid D_\alpha\Phi=(\bar{D}\bar{D})\Phi=0,\quad \Phi\in V\}\ .\end{aligned} \tag{4.9}$$

All linear transformations in V spanned by D_α, J_β, and $\bar{D}\bar{D}$ are nilpotent, and form a Lie superalgebra. Therefore, there exists a nonzero vector Φ in V, such that it satisfies $D_\alpha\Phi=(\bar{D}\bar{D})\Phi=J_\alpha\Phi=0$ by the generalized Engel's theorem.[3,4] This proves the nonemptiness of V_{AT}. We can prove similarly that V_{CT} is not null. The generic element $\Phi_{CT}\in V_{CT}$ and $\Phi_{AT}\in V_{AT}$ can be easily calculated to be

$$\begin{aligned}\Phi_{CT}&=A_0(y)+\sqrt{2}\theta\phi(y)\ ,\\ \Phi_{AT}&=A_0'(\bar{y})+\sqrt{2}\bar\theta\bar\phi(\bar{y})\ ,\end{aligned} \tag{4.10}$$

where $A_0(x)$, $A_0'(x)$, $\phi_\alpha(x)$, and $\bar\phi_{\dot\alpha}(x)$ are some functions of the coordinate x^m alone, subject to constraints

$$\begin{aligned}&\Box A_0(x)=\Box A_0'(x)=0\ ,\\ &\sigma^m_{\alpha\dot\alpha}\partial_m\phi^\alpha(x)=\sigma^m_{\alpha\dot\alpha}\partial_m\bar\phi^{\dot\alpha}(x)=0\ .\end{aligned} \tag{4.11}$$

In view of Eq. (4.11), $\phi_\alpha(x)$ ($\alpha=1,2$) and $\bar\phi_{\dot\alpha}(x)$ ($\dot\alpha=\dot1,\dot2$) contain only one linearly independent component, respectively, so that the dimensions of V_{CT} and V_{AT} are two for both. By the result of Sec. I, both V_{CT} and V_{AT} are irreducible then, since $L_0V_{CT}\subset V_{CT}$ and $L_0V_{AT}\subset V_{AT}$ are evident. The explicit actions of Q_α and $\bar{Q}_{\dot\alpha}$ in V_{CT} are easily found to be

$$\begin{aligned}&Q_\alpha\phi_\beta(x)=\bar{Q}_{\dot\alpha}A_0(x)=0\ ,\\ &Q_\alpha A_0(x)=\sqrt{2}\phi_\alpha(x)\ ,\\ &\bar{Q}_{\dot\alpha}\phi_\alpha(x)=-\sqrt{2}i\sigma^m_{\alpha\dot\alpha}\partial_mA_0(x)\ .\end{aligned} \tag{4.12}$$

A similar result also holds for V_{AT}.

Next, we define a subspace V_0 of V by

$$V_0=\{\Phi\mid J\Phi=0\}\ . \tag{4.13}$$

Then, it is not difficult to show that V_0 is a space generated by V_C, V_A, and V_T, i.e.,

$$V_0=V_C\cup V_A\cup V_T\ . \tag{4.14}$$

Since $\mathrm{Dim}V_{CT}=\mathrm{Dim}V_{AT}=2$, and $\mathrm{Dim}V_T=8$, we have $\mathrm{Dim}V_0=12$. It is simple to show that any state Φ satisfying $J_\alpha\Phi=0$ or $\bar{J}_{\dot\alpha}\Phi=0$ automatically belongs to V_0, but the converse is not necessarily correct. Using this, it is obvious that V_0 cannot be fully reducible, since we will otherwise have $J_\alpha V_0=\bar{J}_{\dot\alpha}V_0=0$ by the same reasoning as in Sec. IV A. Actually, V_0 is moreover indecomposable in a sense specified shortly. This fact will be shown in Sec. IV E.

The quotient space V/V_0 is four-dimensional, whose basis modulo V_0 consists of four representatives $(\theta\sigma^m\bar\theta)K_m(x)$, $(\bar\theta\bar\theta)\theta\xi(x)$, $(\theta\theta)\bar\theta\bar\xi(x)$, and $(\theta\theta)(\bar\theta\bar\theta)G(x)$, subject to constraints

$$\partial^mK_m(x)\neq0,\quad \bar\sigma^m\partial_m\xi(x)\neq0,\quad \sigma^m\partial_m\bar\xi(x)\neq0\ .$$

Since $JV\subset V_0$ in view of $J^2=0$, the quotient space V/V_0 is a zero eigenspace of J. We can prove that V/V_0 as a representation space of L_0 is a direct sum of two two-dimensional irreducible representations and is isomorphic to $V_{CT}\oplus V_{AT}$. More explicitly, their basis (modulo V_0) is given by

$$\begin{aligned}A_0(x)&=G(x)-\frac{i}{2}\partial^mK_m(x)\ ,\\ \phi_\alpha(x)&=\frac{i}{\sqrt{2}}\sigma^m_{\alpha\dot\alpha}\partial_m\bar\xi^{\dot\alpha}(x)\end{aligned} \tag{4.15a}$$

just as for V_{CT} and

$$\begin{aligned}A_0'(x)&=G(x)+\frac{i}{2}\partial^mK_m(x)\ ,\\ \bar\phi_{\dot\alpha}(x)&=\frac{i}{\sqrt{2}}\sigma^m_{\alpha\dot\alpha}\partial_m\xi^\alpha(x)\end{aligned} \tag{4.15b}$$

for V_{AT}.

C. Indecomposability of V_C and V_A

Both V_C and V_A are not irreducible but indecomposable by the following reason. Consider V_C. If it is irreducible, then we must have $J_\alpha V_C=\bar{J}_{\dot\alpha}V_C=0$ by Eq. (4.8). However, this cannot be correct, since we calculate

$$\bar{J}_{\dot\alpha}V_C=-\frac{i}{\sqrt{2}}\sigma^m_{\alpha\dot\alpha}\partial_m[\psi^\alpha(y)+\sqrt{2}\theta^\alpha F(y)]\ .$$

Next, suppose that V_C is decomposable as a direct sum

$$V_C = V_1 \oplus V_2 \ . \quad (4.16)$$

Since L_0 cannot admit any one-dimensional invariant space, both V_1 and V_2 must be two-dimensional, and hence irreducible by the result of Sec. I. Then, we have $J_\alpha V_C = \bar{J}_{\dot\alpha} V_C = 0$ by Eq. (4.8), since V_C would then be fully reducible. However, this has been shown to be incorrect. The same proof equally applies to V_A, and we conclude that both V_C and V_A are not irreducible but indecomposable. Note that we need not assume the Lorentz covariance of the decomposition for this proof.

Now, we study in some detail the structure of V_C. The general solution of the chiral field Φ_C is given by Eq. (3.18), i.e.,

$$\Phi_C = A(y) + \sqrt{2}\theta\psi(y) + (\theta\theta)F(y) \ . \quad (3.18)$$

The actions of Q_α and $\bar{Q}_{\dot\alpha}$ on these components are well known to be

$$\begin{aligned} &Q_\alpha A(x) = \sqrt{2}\psi_\alpha(x) \ , \\ &Q_\alpha \psi_\beta(x) = \sqrt{2}\epsilon_{\alpha\beta}F(x) \ , \\ &Q_\alpha F(x) = \bar{Q}_{\dot\alpha} A(x) = 0 \ , \\ &\bar{Q}_{\dot\alpha}\psi_\alpha(x) = -\sqrt{2}i\sigma^m_{\alpha\dot\alpha}\partial_m A(x) \ , \\ &\bar{Q}_{\dot\alpha} F(x) = \sqrt{2}i\sigma^m_{\alpha\dot\alpha}\partial_m \psi^\alpha(x) \ . \end{aligned} \quad (4.17)$$

Now, V_C contains the following unique two-dimensional invariant subspace $\tilde{V}_0$, spanned by $F(x)$ and $\bar\phi_{\dot\alpha}(x)$, where

$$\bar\phi_{\dot\alpha}(x) = i\sigma^m_{\alpha\dot\alpha}\partial_m\psi^\alpha(x) \quad (4.18)$$

obviously satisfies the constraint

$$\sigma^m_{\alpha\dot\alpha}\partial_m\bar\phi^{\dot\alpha}(x) = 0 \ . \quad (4.19)$$

It may be emphasized that $\bar\phi_{\dot\alpha}(x)$ defined here is not the same quantity given in Eqs. (4.10) and (4.15), although we use the same symbol for both cases because of the shortage of suitable symbols. The same remark also applies in what follows without mentioning it explicitly.

Now, we calculate

$$\begin{aligned} &Q_\alpha F(x) = \bar{Q}_{\dot\alpha}\bar\phi_{\dot\beta}(x) = 0 \ , \\ &\bar{Q}_{\dot\alpha} F(x) = \sqrt{2}\bar\phi_{\dot\alpha}(x) \ , \\ &Q_\alpha \bar\phi_{\dot\alpha}(x) = -\sqrt{2}i\sigma^m_{\alpha\dot\alpha}\partial_m F(x) \ . \end{aligned} \quad (4.20)$$

Comparing this with the result of Sec. IV B, we see that this subspace V_{1C} spanned by $F(x)$ and $\bar\phi_{\dot\alpha}(x)$ is isomorphic to V_{AT}. Also, it satisfies $\bar{J}_{\dot\alpha} V_{1C} = 0$, and is a subspace of V_C obeying the positive-metric condition $J\tilde{V} = J_\alpha\tilde{V} = \bar{J}_{\dot\alpha}\tilde{V} = 0$ discussed in Sec. IV A.

From the result obtained above, we can also prove the indecomposability of V_C as follows. Suppose that V_C is decomposed as in Eq. (4.16). Then, both V_1 and V_2 there must be two-dimensional. But the only two-dimensional invariant subspace $\tilde{V}_C$ of V_C is the V_{1C} spanned by $F(x)$ and $\bar\phi_{\dot\alpha}(x)$. Therefore, Eq. (4.15) cannot be correct and hence V_C is indecomposable.

The quotient space V_C/V_{1C} is spanned by two elements $A(x)$ and $\psi_\alpha(x)$ satisfying $\bar\sigma^m\partial_m\psi(x) = 0$ modulo V_{1C}. It is isomorphic to V_{CT} discussed in Sec. IV B.

D. Indecomposability of V_T

The isoscalar space V_T cannot be fully reducible, since otherwise we will have $J_\alpha V_T = \bar{J}_{\dot\alpha} V_T = 0$ by Eq. (4.8) which is not correct as we will see shortly. However, the question of the decomposability of V_T is more subtle and requires studies of the structure of V_T in some detail. As we shall see shortly, V_T is indecomposable only if we insist on the use of only Lorentz-covariant decompositions as in Sec. II for the massive case.

The generic element Φ_T of V_T is given by Eq. (3.3), on which Q_α and $\bar{Q}_{\dot\alpha}$ act as in Eqs. (3.6) and (3.7). First, we can show that V_T contains two independent two-dimensional irreducible subspaces which are designated here by V_{1T} and V_{2T}, respectively. V_{1T} is spanned by $\xi_\alpha(x)$ and $S_{\alpha\beta}(x)$ defined by

$$\begin{aligned} &\xi_\alpha(x) = \frac{i}{\sqrt{2}}\sigma^m_{\alpha\dot\alpha}\partial_m\bar\psi^{\dot\alpha}(x) \ , \\ &S_{\alpha\beta}(x) = \frac{i}{4}(\sigma^l\bar\sigma^m - \sigma^m\bar\sigma^l)_{\alpha\beta}\partial_m B_l(x) \ , \end{aligned} \quad (4.21)$$

on which Q_α and $\bar{Q}_{\dot\alpha}$ operate as

$$\begin{aligned} &Q_\alpha\xi_\beta(x) = S_{\alpha\beta}(x) \ , \\ &Q_\alpha S_{\mu\nu}(x) = \bar{Q}_{\dot\alpha}\xi_\alpha(x) = 0 \ , \\ &\bar{Q}_{\dot\alpha} S_{\mu\nu}(x) = i\sigma^m_{\mu\dot\alpha}\partial_m\xi_\nu(x) + i\sigma^m_{\nu\dot\alpha}\partial_m\xi_\mu(x) \ . \end{aligned} \quad (4.22)$$

Since $\xi_\alpha(x)$ obeys

$$\sigma^m_{\alpha\dot\alpha}\partial_m\xi^\alpha(x) = 0 \ ,$$

it contains only one linearly independent component. The same statement applies also to $S_{\alpha\beta}(x)$ when we note constraints

$$\begin{aligned} &S_{\alpha\beta}(x) = S_{\beta\alpha}(x) \ , \\ &\sigma^m_{\alpha\dot\mu}\partial_m S^{\alpha\nu}(x) = 0 \ . \end{aligned} \quad (4.23)$$

To understand this fact more clearly, we consider a special Lorentz frame in which the four-vector p_m assumes a form $p_m = (p_0, 0, 0, p_3)$ with $p_3 = p_0 \neq 0$. Using the standard matrix representations[8] for Pauli matrices σ^m, we see that only S_{11} is nonzero. Moreover, the helicity (i.e., eigenvalue of J_{12}) is easily computed to be 1 for $S_{\alpha\beta}$ and $\frac{1}{2}$ for ξ_α when we use Eq. (1.15) with $x_a = i\partial/\partial p_a$.

Next, let V_{2T} be a subspace of V_T spanned by $\bar\xi_{\dot\alpha}(x)$ and $\bar{S}_{\dot\alpha\dot\beta}(x)$ defined by

$$\begin{aligned} &\bar\xi_{\dot\alpha}(x) = \frac{i}{\sqrt{2}}\sigma^m_{\alpha\dot\alpha}\partial_m\psi^\alpha(x) \ , \\ &\bar{S}_{\dot\alpha\dot\beta}(x) = \frac{i}{4}(\bar\sigma^m\sigma^l - \bar\sigma^l\sigma^m)_{\dot\alpha\dot\beta}\partial_m B_l(x) \ . \end{aligned} \quad (4.24)$$

We calculate

$$Q_\alpha \bar{\xi}_{\dot\alpha}(x) = \bar{Q}_{\dot\alpha}\bar{S}_{\dot\mu\dot\nu}(x) = 0 ,$$
$$\bar{Q}_{\dot\alpha}\bar{\xi}_{\dot\beta}(x) = \bar{S}_{\dot\alpha\dot\beta}(x) , \tag{4.25}$$
$$Q_\alpha \bar{S}_{\dot\mu\dot\nu}(x) = i\sigma^m_{\alpha\dot\mu}\partial_m\bar{\xi}_{\dot\nu}(x) + i\sigma^m_{\alpha\dot\nu}\partial_m\bar{\xi}_{\dot\mu}(x) .$$

Moreover, the constraints are

$$(\sigma^m\partial_m)_{\alpha\dot\alpha}\bar{\xi}^{\dot\alpha}(x) = 0 ,$$
$$\bar{S}_{\dot\alpha\dot\beta}(x) = \bar{S}_{\dot\beta\dot\alpha}(x) , \tag{4.26}$$
$$(\sigma^m\partial_m)_{\alpha\dot\mu}\bar{S}^{\dot\mu\dot\nu}(x) = 0 .$$

Again V_{2T} is a two-dimensional irreducible subspace of V_T, whose basis consists of two states with helicities -1 and $-\frac{1}{2}$, respectively. We may also note

$$J_\alpha\Phi_T = \xi_\alpha(y) - \theta^\beta S_{\alpha\beta}(x) \simeq V_{1T} ,$$
$$\bar{J}_{\dot\alpha}\Phi_T = -\bar{\xi}_{\dot\alpha}(\bar{y}) + \bar{\theta}^{\dot\beta}\bar{S}_{\dot\beta\dot\alpha}(x) \simeq V_{2T} . \tag{4.27}$$

From this, we can show, for example, that any state Φ_T satisfying the condition $\bar{J}_{\dot\alpha}\Phi_T = 0$ forms a vector space spanned by $\psi_\alpha(x)$, $\bar{\xi}_{\dot\alpha}(x)$, $S_{\alpha\beta}(x)$, and $\bar{S}_{\dot\alpha\dot\beta}(x)$. If the physical subspace $\tilde{V}_T$ must obey the positive-metric condition $J_\alpha\tilde{V}_T = \bar{J}_{\dot\alpha}\tilde{V}_T = J\tilde{V}_T = 0$ as in Eq. (4.8′), then $\tilde{V}_T$ is spanned by $\xi_\alpha(x)$, $\bar{\xi}_{\dot\alpha}(x)$, $S_{\alpha\beta}(x)$, and $\bar{S}_{\dot\alpha\dot\beta}(x)$ so that

$$\tilde{V}_T = V_{1T} \oplus V_{2T} .$$

Then, the unphysical quotient space $V_T/\tilde{V}_T$ is a four-dimensional space isomorphic to $V_{CT}\oplus V_{AT}$. The basis of $V_T/\tilde{V}_T$ is given by $A(x)$, $B(x)$, $\psi_\alpha(x)$, and $\bar{\psi}_{\dot\alpha}(x)$ modulo $\tilde{V}_T$, where $B(x)$ is the longitudinal component of $B_m(x)$ with identification $B_m(x) = -i\partial_m B(x)$ in $V_T/\tilde{V}_T$. Then, $\frac{1}{2}(A-B)(x)$ and $\psi_\alpha(x)$ are the analogs of $A_0(x)$ and $\phi_\alpha(x)$ of V_{CT}, while $\frac{1}{2}(A+B)(x)$ and $\bar{\psi}_{\dot\alpha}(x)$ correspond to $A_0'(x)$ and $\bar{\phi}_{\dot\alpha}(x)$ in V_{CT}.

The decomposition of $B_m(x)$ into the transverse parts $S_{\alpha\beta}(x)$ and $\bar{S}_{\dot\alpha\dot\beta}(x)$ as well as into its longitudinal part $B(x)$ cannot be done Lorentz covariantly. We have to introduce an arbitrary but fixed constant four-vector l_m satisfying

$$lp = l^m p_m = 1 . \tag{4.28}$$

The usual choice of l_m is to choose $l_m = (\underline{p}/|\underline{p}|^2, 0)$ when $p_m = (\underline{p}, p_0)$. However, without assuming any particular realization for l_m it is not difficult to prove the decomposition

$$B_m(x) = \tfrac{1}{2}[(\sigma l)\bar{\sigma}_m - \sigma_m(\bar{\sigma}l)]_{\alpha\beta}S^{\alpha\beta}(x) + \tfrac{1}{2}[\bar{\sigma}_m(\sigma l) - (\bar{\sigma}l)\sigma_m]_{\dot\alpha\dot\beta}\bar{S}^{\dot\alpha\dot\beta}(x) - i\partial_m B(x) , \tag{4.29}$$

where we have set

$$B(x) = l^m B_m(x) . \tag{4.30}$$

Now, let V_1 and V_2 be

$$V_1 = \langle \tfrac{1}{2}(A+B)(x), [(\sigma l)(\bar{\sigma}p)]_{\alpha\beta}\psi^\beta(x), [(\bar{\sigma}p)(\sigma l)]_{\dot\alpha\dot\beta}\bar{\psi}^{\dot\beta}(x), (\sigma l)_{\mu\dot\lambda}\bar{S}^{\dot\lambda\dot\nu}(x)\rangle , \tag{4.31a}$$
$$V_2 = \langle \tfrac{1}{2}(A-B)(x), [(\sigma p)(\bar{\sigma}l)]_{\alpha\beta}\psi^\beta(x), [(\bar{\sigma}l)(\sigma p)]_{\dot\alpha\dot\beta}\bar{\psi}^{\dot\beta}(x), (\sigma l)_{\lambda\dot\mu}S^{\lambda\nu}(x)\rangle , \tag{4.31b}$$

which are subspaces of V_T, spanned by elements inside the angular brackets. It can be readily verified that they are four-dimensional invariant subspaces of L_0 and that V_T is their direct sum, i.e.,

$$V_T = V_1 \oplus V_2 . \tag{4.32}$$

Therefore, we conclude that the isoscalar space V_T is decomposable and is a direct sum of two four-dimensional representations which will be soon shown to be indecomposable. However, the decompositions, Eqs. (4.29) and (4.31), are evidently not Lorentz covariant just as for the massive case discussed in Sec. II. Therefore, if we insist on the Lorentz-covariant decomposition, or if we extend L_0 into the little superalgebra $L_0(p)$ as in Sec. II, then V_T can be regarded as being indecomposable, as will be shown shortly.

The decomposition Eq. (4.32) can be also derived as follows. Let us set

$$K = -\tfrac{1}{4}D(\sigma l)\bar{D} + \tfrac{1}{8}\lambda(DD + \bar{D}\bar{D}) \tag{4.33}$$

for any four-vector l_m satisfying Eq. (4.28), i.e., $l^m p_m = 1$. Here, λ is the square root of l_m, so that

$$\lambda = \pm(l^2)^{1/2}, \quad l^2 = l^m l_m . \tag{4.34}$$

Then, for the present massless case $M=0$, it is easy to prove

$$K^2 = K \tag{4.35}$$

so that K is a projection operator in V for each choice of two possible signs in Eq. (4.34). Moreover, if we set

$$Y = DD + 2\lambda J, \quad \bar{Y} = \bar{D}\bar{D} + 2\lambda J , \tag{4.36}$$

we find

$$[J,K] = [J,Y] = [J,\bar{Y}] = 0 ,$$
$$[K,Y] = Y, \quad [K,\bar{Y}] = -\bar{Y}, \quad [\bar{Y},Y] = 16J , \tag{4.37}$$

which define another solvable (but not nilpotent) Lie algebra in V. In passing, we remark that this algebra is isomorphic to the example of a quasiclassical Lie algebra given in Ref. 21, where it possesses an invariant invertible metric tensor g_{jk} $(j,k = J,K,Y,\bar{Y})$ in spite of its solvability with the second-order Casimir invariant $I_2 = \frac{1}{16}(Y\bar{Y} + \bar{Y}Y) - 2KJ$. For the present case, we find I_2 to be

$$I_2 = J + \tfrac{1}{8}[(DD)(\bar{D}\bar{D}) + (\bar{D}\bar{D})(DD)] .$$

Returning to the original problem, we note

$YV_T=\bar{Y}V_T=JV_T=0$, so that $KV_T\subset V_T$ from Eq. (4.37). Therefore, when we denote the restriction of K in V_T by K_T, it also satisfies

$$(K_T)^2=K_T \ . \tag{4.38}$$

Hence, we can decompose V_T as a direct sum $V_1\oplus V_2$ where V_1 and V_2 correspond to eigenspaces of K_T with eigenvalues 1 and 0, respectively.

Since V_T is not fully reducible by Eq. (4.8), at least one of V_1 or V_2 must be not fully reducible. We can show that both are actually not irreducible but indecomposable. To prove this, we first observed $\mathrm{Dim}\,V_1=\mathrm{Dim}\,V_2=4$. If V_1 is either irreducible or decomposable, then it must satisfy $J_\alpha V_1=\bar{J}_{\dot\alpha}V_1=0$ by the same reasoning as in the proof of the indecomposability of V_C in Sec. IV C. Since the condition can be seen to be unfulfilled, we conclude that V_1 is not irreducible but indecomposable. The same conclusion is equally applicable to V_2. We can prove further that the decomposition, Eq. (4.32), is the only possible way V_T can be written as a direct sum of two nontrivial representation spaces. Thus, it is the unique decomposition, apart from isomorphism. However, since its demonstration requires a somewhat lengthy calculation, we simply sketch its proof here. Since the only bosonic elements of V_T are $A(x)$ and $B_m(x)$, we may assume $A(x)+l^mB_m(x)$ for a constant vector l^m to belong to V_1. Then, applying Q_α and $\bar{Q}_{\dot\alpha}$ to this, we can prove that it will generate the whole space V_T unless l^m satisfies $(l^mP_m)^2=1$. Assuming $l^mP_m=1$, we conclude that $A+l^mB_m$ and $A-l^mB_m$ must belong to V_1 and V_2, respectively. Applying Q_α and $\bar{Q}_{\dot\alpha}$ to these, we recover Eq. (4.31). If we consider only Lorentz-invariant decomposition, then V_T may be regarded as indecomposable. The situation is very much analogous to the massive case discussed in Sec. II, although we need not assume here the extra condition $l^2=0$ in contrast to Eq. (2.24).

E. Indecomposability of V_0 and V

First, we note that any of V_T, V_0, and V (but not V_C and V_A) may be regarded as a representation space of the solvable Lie algebra defined by Eq. (4.37). Then, both V_0 and V are actually decomposable. For instance, V can be decomposed as a direct sum,

$$V=V_1\oplus V_2 \ , \tag{4.39}$$

when we set $V_1=KV$, and $V_2=(1-K)V$ for the operator K defined by Eq. (4.33). Clearly, both V_1 and V_2 are representation spaces of V since $[K,L_0]=0$. However, this example is again not a Lorentz-invariant decomposition. We will prove here that there is no Lorentz-invariant decomposition for V_0, and V. First, we assume that this statement is correct for V_0, but that V can be written as a Lorentz-invariant direct sum of some V_1 and V_2 as in Eq. (4.39). Let e_1 and e_2 be projection operators into subspaces V_1 and V_2, respectively. Then, since V_1 and V_2 are representations of L_0, we must have $[e_1,L_0]=[e_2,L_0]=0$, and hence

$$[e_j,J]=[e_j,J_\alpha]=[e_j,\bar{J}_{\dot\alpha}]=0, \ \ j=1,2 \ . \tag{4.40}$$

This proves $JV_1\subset V_1$ and $JV_2\subset V_2$. Therefore, if we define $V_{0j}=\{\Phi\,|\,J\Phi=0,\Phi\in V_j\}$, then $V_0=V_{01}\oplus V_{02}$. But, since V_0 is assumed indecomposable in a Lorentz-invariant way, this implies that at least one of V_{01} and V_{02} is zero. Suppose $V_{02}=0$. In this case, V_0 is wholly contained in V_1. In particular, we must have $V_0\cap V_2=0$ and $\mathrm{Dim}\,V_2\le 4$. Since the dimension of V_2 must be even, this requires $\mathrm{Dim}\,V_2=2$ or 4. Moreover, V_2 cannot be irreducible or fully reducible since we will then have $JV_2=0$ by Eq. (4.8) and hence $V_2\subset V_0\subset V_1$. This proves that V_2 must be four-dimensional and reducible. But, then V_2 must contain a two-dimensional invariant subspace $\tilde{V}_2$ which must be necessarily irreducible. As a consequence, $J\tilde{V}_2=0$ again by Eq. (4.8). However, this implies $\tilde{V}_2\subset V_0$ which contradicts $V_2\cap V_0=0$. This proves the impossibility of the Lorentz-invariant decomposition, Eq. (4.39), for V.

We have yet to prove the analogous indecomposability property of V_0. It is more difficult and we simply sketch its outline here. Let $V_0=V_1\oplus V_2$. Just as in the proof of Eq. (4.40), we see $J_\alpha V_j\subset V_j$ and $\bar{J}_{\dot\alpha}V_j\subset V_j$ for $j=1,2$. Also, both V_1 and V_2 contain only Lorentz-covariant combinations of bosonic elements such as $A(x)$ and $B_m(x)$. From these, we can show $V_1=V_0$ or $V_2=V_0$ after some calculations, proving the desired indecomposability of V_0.

So far, we have used the same symbols Q_α and $\bar{Q}_{\dot\alpha}$, etc., both for the differential operators, Eq. (1.13), and for their linear representation matrices as we indicated in Sec. III. In general, this fact will not cause any confusion, since we can readily infer their respective use for any particular case from the context under discussion. For example, J in Eq. (4.13) stands for the differential operator, while the operations Q_α and $\bar{Q}_{\dot\alpha}$ in Eq. (4.12) are evidently meant to be their representation matrices. However, some cautions may be necessary for the precise definition of the physical space $\tilde{V}$. In order to avoid possible confusion, we reserve hereafter in this section the symbols Q_α and $\bar{Q}_{\dot\alpha}$ (and also J, J_α, and $\bar{J}_{\dot\alpha}$) for the differential operators as in Eq. (1.13), while we designate their linear realizations in a representation ρ as $\rho(Q_\alpha)$ and $\rho(\bar{Q}_{\dot\alpha})$, etc. Then, the definition, Eq. (4.8′) for the physical space $\tilde{V}$ is properly understood to be

$$\rho(J)\tilde{V}=\rho(J_\alpha)\tilde{V}=\rho(\bar{J}_{\dot\alpha})\tilde{V}=0 \ . \tag{4.41}$$

If a state $\Phi\in V$ satisfies

$$J\Phi=J_\alpha\Phi=\bar{J}_{\dot\alpha}\Phi=0 \ , \tag{4.42}$$

then we have clearly $\Phi\in\tilde{V}$. However, the converse may not necessarily be true. Consider, for example, the case of the chiral space V_C, where $\Phi\in V_C$ can be expressed as

$$\Phi=A(y)+\sqrt{2}\theta\psi(y)+(\theta\theta)F(y) \ .$$

By definition, we have then, for example,

$$Q_\alpha\Phi=\rho(Q_\alpha)A(y)+\sqrt{2}\theta[\rho(Q_\alpha)\psi(y)]+(\theta\theta)[\rho(Q_\alpha)F(y)] \ .$$

From this, we find Eq. (4.17) where Q_α and $\bar{Q}_{\dot\alpha}$ there are now replaced by $\rho(Q_\alpha)$ and $\rho(\bar{Q}_{\dot\alpha})$, so that we calculate

$$\rho(J)F=\rho(J_\alpha)F=\rho(\bar{J}_{\dot\alpha})F=0 ,$$
$$\rho(J)\bar\phi_{\dot\beta}=\rho(J_\alpha)\bar\phi_{\dot\beta}=\rho(\bar J_{\dot\alpha})\bar\phi_{\dot\beta}=0 .$$

Here, $\bar\phi_{\dot\beta}(x)$ is defined by Eq. (4.18). However, we have

$$\bar J_{\dot\alpha}\{\sqrt{2}[\theta\psi(y)]+(\theta\theta)F(y)\}\neq 0$$

as we see from the result of Sec. IV C or by direct computation.

The physical condition, Eq. (4.41), is satisfied by any irreducible representation discussed so far. If we define[6,8] W_α and $\bar W_{\dot\alpha}$ by

$$W_\alpha=-\tfrac{1}{4}(\bar D\bar D)D_\alpha\Phi_0 , \quad \bar W_{\dot\alpha}=-\tfrac14(DD)\bar D_{\dot\alpha}\Phi_0 \tag{4.43}$$

for some $\Phi_0\in V$, it satisfies

$$JW_\alpha=J_\beta W_\alpha=J\bar W_{\dot\alpha}=\bar J_{\dot\beta}\bar W_{\dot\alpha}=0 .$$

However, we find

$$J_\alpha\bar W_{\dot\alpha}=-\tfrac{1}{16}(\sigma P)_{\alpha\dot\alpha}(DD)(\bar D\bar D)\Phi_0 ,$$
$$\bar J_{\dot\alpha}W_\alpha=-\tfrac{1}{16}(\sigma P)_{\alpha\dot\alpha}(\bar D\bar D)(DD)\Phi_0 .$$

We can prove now that a necessary and sufficient condition for the validity of Eq. (4.42) for $\Phi=W_\alpha$ and $\bar W_{\dot\alpha}$ is to have $J\Phi_0=0$, i.e., $\Phi_0\in V_0$. Indeed, if $(DD)(\bar D\bar D)\Phi_0=(\bar D\bar D)(DD)\Phi_0=0$, then we have $J\Phi_0=0$ since $[\bar D\bar D,DD]=16J$. Conversely, assume the validity of $J\Phi_0=0$. Then $\Phi_0\in V_C\cup V_A\cup V_T$ by Eq. (4.14), from which we can prove $(DD)(\bar D\bar D)\Phi_0=(\bar D\bar D)(DD)\Phi_0=0$. However, possible physical relevance of this fact is yet to be studied.

In concluding this section, we remark the following. Since the Casimir invariant J assumes only a zero eigenvalue for any irreducible representation by Eq. (4.8), it is useless to label the representation. Similarly, two fermionic Casimir invariants J_α and $\bar J_{\dot\alpha}$ of L_0 (but not of L) assume again zero eigenvalues for all irreducible representations. Even if another fermionic Casimir invariant of L_0 exists, Schur's lemma tells us that either it has a zero eigenvalue or it simply interchanges the bosonic subspace and the fermionic subspace. As the result, any fermionic Casimir invariant of L_0 including both J_α and $\bar J_{\dot\alpha}$ will not help us in specifying irreducible representations. A further discussion of Casimir invariants is found in Sec. V.

V. GENERALIZED SUPERSYMMETRY AND CASIMIR INVARIANTS

We now consider the following slight generalization of the Lie superalgebra of the previous sections. Let $Q_{\alpha A}$ and $\bar Q^A_{\dot\alpha}$ $(A=1,2,\dots,n)$ satisfy

$$\{Q_{\alpha A},Q_{\beta B}\}=\{\bar Q^A_{\dot\alpha},\bar Q^B_{\dot\beta}\}=0 , \quad \{Q_{\alpha A},\bar Q^B_{\dot\beta}\}=2\sigma^m_{\alpha\dot\beta}P_m\delta^B_A , \tag{5.1}$$

which replaces Eq. (1.11). However, all other relations in Eq. (1.10) are assumed to remain intact when we replace Q_α and $\bar Q_{\dot\alpha}$ by $Q_{\alpha A}$ and $\bar Q^A_{\dot\alpha}$, respectively, there. Although Eq. (5.1) is not the most general extended Lie superalgebra allowed by the theorem of Haag *et al.*,[22] a study of Eq. (5.1) will still be of some interest especially in connection with decomposition of its superspace. We introduce Grassmann variables θ^A_α and $\bar\theta_{\dot\alpha A}$ $(A=1,2,\dots,n)$. Then, the generalizations of Eqs. (1.13) and (1.17) are

$$Q_{\alpha A}=\frac{\partial}{\partial\theta^{\alpha A}}+\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}_A P_m , \quad \bar Q^A_{\dot\alpha}=\frac{\partial}{\partial\bar\theta^{\dot\alpha}_A}+\theta^{\alpha A}\sigma^m_{\alpha\dot\alpha}P_m \tag{5.2}$$

as well as

$$D_{\alpha A}=\frac{\partial}{\partial\theta^{\alpha A}}-\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}_A P_m , \quad \bar D^A_{\dot\alpha}=-\frac{\partial}{\partial\bar\theta^{\dot\alpha}_A}+\theta^{\alpha A}\sigma^m_{\alpha\dot\alpha}P_m . \tag{5.3}$$

Note that pairs $(Q_{\alpha A},\bar Q^A_{\dot\alpha})$ and $(D_{\alpha A},\bar D^A_{\dot\alpha})$ anticommute with each other. When we want to suppress the spinor indices, we simply write these as Q_A, $\bar Q^A$, D_A, and $\bar D^A$.

First, we consider the massive case $M\neq 0$ and set

$$R^B_A=\frac{1}{4M^2}(\bar D^B\bar\sigma^m D_A-D_A\sigma^m\bar D^B)P_m ,$$
$$S_{AB}=S_{BA}=\frac{1}{2M}(D_AD_B)=\frac{1}{2M}\epsilon^{\beta\alpha}D_{\alpha A}D_{\beta B} , \tag{5.4}$$
$$S^{AB}=S^{BA}=\frac{1}{2M}(\bar D^A\bar D^B)=\frac{1}{2M}\epsilon^{\dot\alpha\dot\beta}\bar D^A_{\dot\alpha}\bar D^B_{\dot\beta} .$$

These satisfy the commutation relations of the Lie algebra C_n corresponding to the Sp($2n$) group:

$$[R^B_A,S^{CD}]=\delta^C_AS^{BD}+\delta^D_AS^{CB} ,$$
$$[R^B_A,S_{CD}]=-\delta^B_CS_{AD}-\delta^B_DS_{CA} ,$$
$$[S^{CD},S_{AB}]=\delta^C_BR^D_A+\delta^C_AR^D_B+\delta^D_BR^C_A+\delta^D_AR^C_B , \tag{5.5}$$
$$[R^B_A,R^D_C]=\delta^D_AR^B_C-\delta^B_CR^D_A ,$$
$$[S_{AB},S_{CD}]=[S^{AB},S^{CD}]=0$$

for $A,B,C,D=1,2,\dots,n$.

The special case $n=1$ reproduces the case discussed in Sec. II since Sp(2) is isomorphic to SU(2) with correspondence $2I_+=S^{11}$, $2I_-=S_{11}$, and $I_3=\frac12R^1_1$. Clearly, R^B_A, S_{AB}, and S^{AB} satisfy many complicated algebraic identities which are generalizations of Eq. (2.3) For, example, we have $(S_{AA})^2=0$, $S_{AB}S_{AC}S_{AD}=0$, etc. These constraints restrict possible irreducible representations of the symplectic group, just as only $I=0$ and $I=\frac12$ are allowed for the SU(2) of Sec. II. To find all allowed representations of the Sp($2n$), we note first that it contains a SU(2) subalgebra consisting of R^1_1, S_{11}, and S^{11}. Moreover, the latter obeys the same algebraic identities as in Eq. (2.3), permitting only $I=0$ and $I=\frac12$. Therefore, when we decompose any irreducible space of the Sp($2n$) in our superspace V into subrepresentations of this SU(2) subalgebra, it contains $I=0$ and $I=\frac12$ but nothing else. Hence, by a theorem proved elsewhere,[23] only irreducible representations of the

$Sp(2n)$ to be allowed are restricted to n fundamental representations which correspond to n completely antisymmetric-tensor representations in the terminology of the Young tableau.

Note that $R^1_1, R^2_2, \dots, R^n_n$ form a Cartan subalgebra of C_n, whose simultaneous eigenvalues define a weight system. Since any representation of a simple Lie algebra is fully reducible and since any state in a given irreducible representation is specified by its weight, we can decompose our superspace V now by weights belonging to n fundamental irreducible representations of C_n. This generalizes the result of Sec. II. The chiral state satisfying $\bar{D}_{\dot{\alpha}A}\Phi=0$ for all $A=1,2,..,n$ and $\dot{\alpha}=\dot{1},\dot{2}$ corresponds to the highest weight Λ_n with $R^A_A\Phi=\Phi$ for all $A=1,2,\dots,n$, while the antichiral state $D_{\alpha A}\Phi=0$ belongs to the lowest-weight state with $R^A_A\Phi=-\Phi$ for all $A=1,2,\dots,n$ in the irreducible representation with the highest weight Λ_n.

The symplectic algebra C_n can be extended to a superalgebra. To demonstrate this, we set

$$\begin{aligned}
&\phi_{\alpha A}=D_{\alpha A}\ ,\\
&\bar{\psi}^A_{\dot{\alpha}}=\bar{D}^A_{\dot{\alpha}}\ ,\\
&\phi^A_\alpha=\frac{1}{M}(\sigma^m)_\alpha{}^{\dot{\beta}}\bar{D}^A_{\dot{\beta}}P_m\ ,\\
&\bar{\psi}_{\dot{\alpha}A}=-\frac{1}{M}(\bar{\sigma}^m)_{\dot{\alpha}}{}^{\beta}D_{\beta A}P_m\ .
\end{aligned} \tag{5.6}$$

Then, we see that these satisfy

$$\begin{aligned}
&[R^B_A,\phi_{\alpha C}]=-\delta^B_C\phi_{\alpha A}\ ,\\
&[R^B_A,\phi^C_\alpha]=\delta^C_A\phi^B_\alpha\ ,\\
&[S_{AB},\phi_{\alpha C}]=[S^{AB},\phi^C_\alpha]=0\ ,\\
&[S^{AB},\phi_{\alpha C}]=\delta^A_C\phi^B_\alpha+\delta^B_C\phi^A_\alpha\ ,\\
&[S_{AB},\phi^C_\alpha]=\delta^C_A\phi_{\alpha B}+\delta^C_B\phi_{\alpha A}\ ,\\
&\{\phi_{\alpha A},\phi_{\beta B}\}=\{\phi^A_\alpha,\phi^B_\beta\}=0\ ,\\
&\{\phi_{\alpha A},\phi^B_\beta\}=2M\epsilon_{\alpha\beta}\delta^B_A\ ,
\end{aligned} \tag{5.7}$$

as well as relations involving $\bar{\psi}^A_{\dot{\alpha}}$ and $\bar{\psi}_{\dot{\alpha}A}$, which can be obtained from Eq. (5.7) with replacement $\phi_{\alpha A}\to\bar{\psi}_{\dot{\alpha}A}$, $\phi^A_\alpha\to\bar{\psi}^A_{\dot{\alpha}}$, and $\epsilon_{\alpha\beta}\to\epsilon_{\dot{\alpha}\dot{\beta}}$. Therefore, the pairs $(\phi^A_\alpha,\phi_{\alpha A})$ and $(\bar{\psi}^A_{\dot{\alpha}},\bar{\psi}_{\dot{\alpha}A})$ are tensor operators corresponding to the basic representation of C_n. We also have

$$\begin{aligned}
&\{\phi_{\alpha A},\bar{\psi}_{\dot{\beta}B}\}=\{\phi^A_\alpha,\bar{\psi}^B_{\dot{\beta}}\}=0\ ,\\
&\{\phi^A_\alpha,\bar{\psi}_{\dot{\beta}B}\}=-2\delta^A_B\sigma^m_{\alpha\dot{\beta}}P_m\ ,\\
&\{\phi_{\alpha A},\bar{\psi}^B_{\dot{\beta}}\}=2\delta^B_A\sigma^m_{\alpha\dot{\beta}}P_m\ .
\end{aligned} \tag{5.8}$$

We note that all these operators are closed either by commutation relations or by anticommutation relations. Therefore, they define a Lie superalgebra. Although this superalgebra is not of the type allowed by the theorem[22] of Haag *et al.*, this is not a contradiction since our algebra is defined only on the mass shell $P^2=-M^2$.

Next, we will construct a Casimir invariant other than P^2 for our original large superalgebra L. First, we define K_a by

$$K_a=\frac{1}{8}\sum_{A=1}^{n}(Q_A\sigma_a\bar{Q}^A-\bar{Q}^A\bar{\sigma}_aQ_A)\ , \tag{5.9}$$

which satisfies the commutation relation

$$[K_a,K_b]=-i\epsilon_{abcd}P^cK^d \tag{5.10}$$

with normalization $\epsilon_{0123}=1$.

We now define a modified Pauli-Lubanski operator W_a by

$$W_a=\frac{1}{2}\epsilon_{abcd}P^bJ^{cd}-\frac{1}{P^2}(\eta_{ab}P^2-P_aP_b)K^b\ . \tag{5.11}$$

Note that this definition differs from the conventional one[5,24,25] $\tilde{W}_a$ [see Eq. (5.17b) below] by the last term proportional to $(1/P^2)(PK)P_a$. However, the present form is perhaps more convenient since we have

$$P^aW_a=0\ , \tag{5.12}$$

$$[W_a,Q_{\alpha A}]=[W_a,\bar{Q}^A_{\dot{\alpha}}]=0\ . \tag{5.13}$$

In deriving Eq. (5.13), we utilized identities

$$\begin{aligned}
&\frac{i}{2}\epsilon_{abcd}(\sigma^c\bar{\sigma}^d-\sigma^d\bar{\sigma}^c)=\sigma_a\bar{\sigma}_b-\sigma_b\bar{\sigma}_a\ ,\\
&-\frac{i}{2}\epsilon_{abcd}(\bar{\sigma}^c\sigma^d-\bar{\sigma}^d\sigma^c)=\bar{\sigma}_a\sigma_b-\bar{\sigma}_b\sigma_a\ .
\end{aligned} \tag{5.14}$$

Because of Eq. (5.13),

$$I=W^aW_a \tag{5.15}$$

commutes with all elements of L, i.e., J_{ab}, Q_α, $\bar{Q}_{\dot{\alpha}}$, and P_m, so that it is a Casimir invariant of L. Although this form of I appears to be different from the standard formula, both are actually the same when we rewrite I as[5,24,25]

$$2P^2I=W_{ab}W^{ab}\ , \tag{5.16}$$

$$W_{ab}=P_aW_b-P_bW_a=P_a\tilde{W}_b-P_b\tilde{W}_a\ , \tag{5.17a}$$

$$\tilde{W}_a=\frac{1}{2}\epsilon_{abcd}P^bJ^{cd}-K_a\ . \tag{5.17b}$$

In passing, we note that W_a satisfies also the commutation relation

$$[W_a,W_b]=-i\epsilon_{abcd}P^cW^d\ . \tag{5.18}$$

For the case of $P^2=-M^2=0$, the situation is very different. We can readily find n^2 Casimir invariants of the form

$$F^A_B=\frac{1}{2}Q_B\sigma^m\bar{Q}^AP_m=-\frac{1}{2}\bar{Q}^A\bar{\sigma}^mQ_BP_m \tag{5.19}$$

for $A,B,=1,2,\dots,n$. Then,

$$J=K_aP^a=\frac{1}{2}\sum_{A=1}^{n}F^A_A \tag{5.20}$$

is also a Casimir invariant of L. We note that the Casimir invariant J cannot be obtained from $I=W^aW_a$ by the method of contraction. If we take the limit $P^2\to0$ in both sides of Eq. (5.16), it gives J^2 but not J itself. At any rate, the drastically different forms of Casimir invariants for the case $M=0$ are very much different from the situation of the Poincaré Lie algebra. Also, we note $(F_B^A)^2=0$ identically for $P^2=0$, so that F_B^A can assume only a zero eigenvalue in any irreducible representation. Therefore, these invariants are not useful to specify irreducible representations of L.

Returning to Eq. (5.5), we multiply suitable powers of M and let $M\to 0$. In this way, the contracted algebra for the massless case is now obtained as

$$\begin{aligned}&[F_A^B,G^{CD}]=[F_A^B,G_{CD}]=0\ ,\\&[G^{CD},G_{AB}]=\delta_B^C F_A^D+\delta_A^C F_B^D+\delta_B^D F_A^C+\delta_A^D F_B^C\ ,\\&[F_B^A,F_C^D]=0\ ,\\&[G_{AB},G_{CD}]=[G^{AB},G^{CD}]=0\ ,\end{aligned} \tag{5.21}$$

where we have set

$$\begin{aligned}&F_A^B=\tfrac{1}{4}(\bar{D}^B\bar{\sigma}^m D_A-D_A\sigma^m\bar{D}^B)P_m\ ,\\&G_{AB}=G_{BA}=\tfrac{1}{2}D_A D_B\ ,\\&G^{AB}=G^{BA}=\tfrac{1}{2}\bar{D}^A\bar{D}^B\ .\end{aligned} \tag{5.22}$$

We note that F_A^B given in Eq. (5.22) is the same operator as is defined by Eq. (5.19) because of Eqs. (5.2) and (5.3). The Lie algebra defined by Eq. (5.21) is again solvable.

Finally, if we define

$$\begin{aligned}&F_a^A=\tfrac{1}{2}P_m\sigma^m_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}A}=-\tfrac{1}{2}P_m\sigma^m_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}A}\ ,\\&\bar{F}_{\dot{\alpha}A}=\tfrac{1}{2}Q_A^\alpha\sigma^m_{\alpha\dot{\alpha}}P_m=-\tfrac{1}{2}D_A^\alpha\sigma^m_{\alpha\dot{\alpha}}P_m\ ,\end{aligned} \tag{5.23}$$

then the physical space $\tilde{V}$ may be defined to satisfy

$$F_a^A\tilde{V}=\bar{F}_{\dot{\alpha}A}\tilde{V}=F_B^A\tilde{V}=0 \tag{5.24}$$

by the same reason as in Sec. IV.

Note added in proof. The Casimir invariant $I=W^aW_a$ defined by Eq. (5.15) can be further rewritten as follows. We first note the validity of the identities

$$\begin{aligned}(K_aP^a)^2=&\sum_{A,B=1}^{n}\{T^a_{AB},\bar{T}_a^{AB}\}_+\\&+\tfrac{1}{16}P^2[4K_aK^a-3n(n-2)P^2]\ ,\\K_aK^a=&-\tfrac{1}{16}\sum_{A,B=1}^{n}\{(Q_AQ_B),(\bar{Q}^A\bar{Q}^B)\}_+\\&-\tfrac{1}{4}n(n+2)P^2\ ,\end{aligned}$$

where we have set

$$\begin{aligned}&T^a_{AB}=-T^a_{BA}=\tfrac{1}{16}Q_A[\sigma^a(\bar{\sigma}P)-(\sigma P)\bar{\sigma}^a]Q_B\ ,\\&\bar{T}_a^{AB}=-\bar{T}_a^{BA}=\tfrac{1}{16}\bar{Q}^A[\bar{\sigma}_a(\sigma P)-(\bar{\sigma}P)\sigma_a]\bar{Q}^B\ .\end{aligned}$$

Then, I can be rewritten as

$$\begin{aligned}I&=\tilde{W}_a\tilde{W}^a-\frac{1}{P^2}(K_aP^a)^2\\&=\tilde{W}_a\tilde{W}^a-\tfrac{1}{4}K_aK^a+\tfrac{3}{16}n(n-2)P^2-\frac{1}{P^2}\sum_{A,B=1}^{n}\{T^a_{AB},\bar{T}_a^{AB}\}_+\\&=\tilde{W}_a\tilde{W}^a+\tfrac{1}{4}n(n-1)P^2+\tfrac{1}{64}\sum_{A,B=1}^{n}\{(Q_AQ_B),(\bar{Q}^A\bar{Q}^B)\}_+-\frac{1}{P^2}\sum_{A,B=1}^{n}\{T^a_{AB},\bar{T}_a^{AB}\}_+\ .\end{aligned}$$

For $n=1$, both T^a_{AB} and $\bar{T}_a^{AB}$ are identically zero, since they are antisymmetric in interchange of indices A and B. Therefore, the Casimir invariant I has no pole at $P^2=0$ for the special case $n=1$, so that it remains also to be a Casimir invariant of L even for the massless case $P^2=0$. However, for $n\geq 2$, T^a_{AB} and $\bar{T}_a^{AB}$ are no longer zero, and I has a singularity at $P^2=0$. The contraction of the product P^2I when we let $P^2\to 0$ leads to $(K_aP^a)^2=J^2$ as we have already observed in Sec. V. In conclusion, there appears to be an important difference between two cases of $n=1$ and $n\geq 2$ for existence of nontrivial Casimir invariants for the massless case.

After this paper was written, it came to the author's attention that the following references also deal with projection operators of Sec. II for the decomposition of the massive superspace: (a) J. G. Taylor and S. Ferrara, in *Super-Symmetry and Super-Gravity 1981,* edited by S. Ferrara, J. G. Taylor, and P. van Nieuwenhuizen (Cambridge University Press, London, 1982); (b) S. J. Gates, M. T. Grisaru, M. Roček, and W. Siegel, *Super-Space Frontiers in Physics, Lecture Notes Series No. 58* (Benjamin/Cummings, Reading, Massachusetts, 1983). The author would like to express his gratitude to Professor J. G. Taylor for calling his attention to these references.

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