A second lemma in hadron Compton scattering

S. Ragusa
Instituto de Física e Química de São Carlos, Caixa Postal 369, Universidade de São Paulo, São Carlos 13560, São Paulo, Brasil

(Received 15 July 1983; revised manuscript received 24 October 1983)

Using gauge invariance we show that the time-space part of the excited-state hadron Compton scattering amplitude can be written as $E_{0j}^{\alpha\beta}=k_i'\Gamma_{ij}^{\alpha\beta}$ where $\Gamma_{ij}^{\alpha\beta}$ is odd under crossing in the Breit frame. The usefulness of this result in obtaining higher-order low-energy theorems for the Compton amplitude is discussed.

More than ten years ago Singh[1] demonstrated a useful lemma for the study of low-energy theorems in hadron Compton scattering. Singh showed that the time-time component $E_{00}^{\alpha\beta}$ of the excited part of the scattering amplitude tensor $E_{\mu\nu}^{\alpha\beta}$ satisfies the relation

$$E_{00}^{\alpha\beta}(\vec{k},\omega\vec{k}',\omega')=k_i'k_j\Lambda_{ij}^{\alpha\beta}(\vec{k},\omega\vec{k}',\omega') \quad , \tag{1}$$

where α and β are "charge" labels pertaining to the initial and final photons, $k_\mu=(\vec{k},i\omega)=(\vec{k},ik_0)$ and k_ν' are the momenta of the incident and outgoing photon, respectively, and $\Lambda_{ij}^{\alpha\beta}$ is even under crossing:

$$\Lambda_{ij}^{\alpha\beta}(\vec{k},\omega,\vec{k}',\omega')=\Lambda_{ji}^{\beta\alpha}(-\vec{k}',-\omega',-\vec{k},-\omega) \quad . \tag{2}$$

As is well known, the property (1), Singh's lemma, has been the backbone of the derivation of low-energy theorems of order higher than the first in the frequency of the incident photon. The analysis of the results are based on the relation[2]

$$k_i'E_{ij}^{\alpha\beta}k_j-\omega\omega'E_{00}=\omega\omega'U_{00}^{\alpha\beta}-k_j'U_{ij}^{\alpha\beta}k_j+\frac{i}{2}(2\pi)^3(E_pE_{p'}/M^2)^{1/2}f^{\alpha\beta\gamma}\langle\vec{p}'|(\omega+\omega')J_0^\gamma(0)+(k_i'+k_i)J_i^\gamma(0)|\vec{p}\rangle \quad . \tag{3}$$

In this expression $U_{\mu\nu}^{\alpha\beta}$ is the unexcited part of the scattering amplitude tensor, $T_{\mu\nu}^{\alpha\beta}=E_{\mu\nu}^{\alpha\beta}+U_{\mu\nu}^{\alpha\beta}$. The last three terms of (3) are known quantities, and by using Singh's lemma one can calculate many of the terms of the excited part, which is expanded as

$$E_{ij}^{\alpha\beta}=\sum_N a_N^{\alpha\beta}B_{ij}^N \quad , \tag{4}$$

where B_{ij}^N are the minimal-basis components introduced by Pais:[3] $B_{ij}^1=\delta_{ij}$, $B_{ij}^2=\epsilon_{ijk}S_k$, etc. However, for all basis terms for which $k_i'B_{ij}^Nk_j=0$, that is, terms that are orthogonal to $k_i'k_j$, the procedure can give no information on the corresponding partial amplitude since it will not be present in Eq. (3).

When one considers the consequence of gauge invariance for each photon separately one has

$$k_\mu'T_{\mu\nu}^{\alpha\beta}=T_{\nu\mu}^{\alpha\beta}k_\mu+i(2\pi)^3(E_pE_{p'}/M^2)^{1/2}\langle\vec{p}'|J_\nu^\gamma|\vec{p}\rangle f^{\alpha\beta\gamma} \quad ,$$

from which one obtains the two independent relations

$$k_i'E_{ij}^{\alpha\beta}-\omega'E_{0j}^{\alpha\beta}=-k_i'U_{ij}^{\alpha\beta}+\omega'U_{0j}^{\alpha\beta}+i(2\pi)^3(E_pE_{p'}/M^2)^{1/2}f^{\alpha\beta\gamma}\langle\vec{p}'|J_j^\gamma(0)|\vec{p}\rangle \quad , \tag{5}$$

$$E_{0j}^{\alpha\beta}k_j-\omega E_{00}^{\alpha\beta}=-U_{0j}^{\alpha\beta}+\omega U_{00}^{\alpha\beta}-i(2\pi)^3(E_pE_{p'}/M^2)^{1/2}f^{\alpha\beta\gamma}\langle\vec{p}'|J_0^\gamma(0)|\vec{p}\rangle \quad . \tag{6}$$

In this note we want to show that the time-space part $E_{0j}^{\alpha\beta}$ of the excited-state scattering amplitude satisfies the property

$$E_{0j}=k_i'\Gamma_{ij}^{\alpha\beta}(\vec{k},\omega,\vec{k}',\omega') \quad , \tag{7}$$

where $\Gamma_{ij}^{\alpha\beta}$ is odd under crossing in the Breit frame:

$$\Gamma_{ij}^{\alpha\beta}(\vec{k},\omega,\vec{k}',\omega')=-\Gamma_{ji}^{\beta\alpha}(-\vec{k}',-\omega',-\vec{k},-\omega) \quad . \tag{8}$$

To show this we proceed first along the line of thought of Singh. T_{0j}, just like T_{00}, has no Schwinger terms,[4] and therefore E_{0j} can be expanded as

$$\frac{1}{(2\pi)^6}\frac{M^2}{E_pE_{p'}}E_{0j}=\sum_{n,\xi_n}\left\{\frac{\langle\vec{p}'|J_0^\alpha(0)|\vec{p}+\vec{k},E_n(\vec{p}+\vec{k}),\xi_n\rangle\langle\vec{p}+\vec{k},E_n(\vec{p}+\vec{k}),\xi_n|J_j^\beta(0)|\vec{p}\rangle}{E_n(\vec{p}+\vec{k})-\omega-E(\vec{p})}+\frac{\langle\vec{p}'|J_j^\beta(0)|\vec{p}-\vec{k}',E_n(\vec{p}-\vec{k}'),\xi_n\rangle\langle\vec{p}-\vec{k}',E_n(\vec{p}-\vec{k}'),\xi_n|J_0(0)|\vec{p}\rangle}{E_n(\vec{p}-\vec{k}')+\omega'-E(\vec{p})}\right\} \quad . \tag{9}$$

In this expression $E(\vec{p})$ and $E_n(\vec{p})$ are, respectively, the energy of the target and of the nth excited intermediate state with three-momentum $\vec{p}$, and ξ_n stands for other quantum numbers (such as spin) needed to specify the excited intermediate state. Using current conservation for the matrix elements of J_0^α we can write this expression in the desired form of Eq.

(7), where

$$\frac{1}{(2\pi)^6}\frac{M^2}{E_pE_{p'}}\Gamma_{ij}^{\alpha\beta}=\sum_{n,\xi_n}\left\{\frac{\langle\vec{p}'|J_i^\alpha(0)|\vec{p}+\vec{k},E_n(\vec{p}+\vec{k}),\xi_n\rangle\langle\vec{p}+\vec{k},E_n(\vec{p}+\vec{k}),\xi_n|J_j^\beta(0)|\vec{p}\rangle}{[E(\vec{p}')-E_n(\vec{p}+\vec{k})][E_n(\vec{p}+\vec{k})-\omega-E(\vec{p})]}\right.$$
$$\left.-\frac{\langle\vec{p}'|J_j^\beta(0)|\vec{p}-\vec{k}',E_n(\vec{p}-\vec{k}'),\xi_n\rangle\langle\vec{p}-\vec{k}',E_n(\vec{p}-\vec{k}'),\xi_n|J_i^\alpha(0)|\vec{p}\rangle}{[E(\vec{p})-E_n(\vec{p}-\vec{k}')][E_n(\vec{p}-\vec{k}')+\omega'-E(\vec{p})]}\right\} . \tag{10}$$

In the Breit frame we have $\vec{p}'=-\vec{p}$, $E(\vec{p}')=E(\vec{p})$, and the property (8) is therefore established.

The fact that the property (8) is satisfied only in the Breit frame is particularly gratifying since it is in this frame that the requirement of time reversal achieves its simplest form, as used by Pais to find the minimal basis $\{B_i^N\}$. Using the results (7) and (1), Eqs. (5) and (6) can be written as

$$k_i'E_{ij}^\alpha-\omega'k_i\Gamma_{ij}^{\alpha\beta}=-k_i'U_{ij}^{\alpha\beta}+\omega'U_{0j}^{\alpha\beta}+i(2\pi)^3(E_p/M)f^{\alpha\beta\gamma}\langle\vec{p}'|J_j^\gamma(0)|\vec{p}\rangle , \tag{11}$$

$$k_i'k_j\Gamma_{ij}^{\alpha\beta}-\omega k_i'k_j\Lambda_{ij}^{\alpha\beta}=U_{0j}^{\alpha\beta}k_j+\omega U_{00}^{\alpha\beta}+i(2\pi)^3(E_p/M)f^{\alpha\beta\gamma}\langle\vec{p}'|J_0^\gamma(0)|\vec{p}\rangle . \tag{12}$$

As $\Gamma_{ij}^{\alpha\beta}$ and $\Lambda_{ij}^{\alpha\beta}$ are three-tensors as is E_{ij}, they can be expanded in the same minimal basis,

$$\Gamma_{ij}^{\alpha\beta}=\sum_N b_N^{\alpha\beta}B_{ij}^N , \tag{13}$$

$$\Lambda_{ij}^{\alpha\beta}=\sum_N c_N^{\alpha\beta}B_{ij}^N . \tag{14}$$

Substitution of these expressions and of (4) in (11) and (12) will give

$$\sum_N(a_N^{\alpha\beta}-\omega'b_N^{\alpha\beta})k_i'B_{ij}^N=-k_i'U_{ij}^{\alpha\beta}+\omega'U_{0j}^{\alpha\beta}+i(2\pi)^3(E_p/M)f^{\alpha\beta\gamma}\langle\vec{p}'|J_j^\gamma(0)|\vec{p}\rangle , \tag{15}$$

$$\sum_N(b_N^{\alpha\beta}-\omega c_N^{\alpha\beta})k_i'k_jB_{ij}^N=-k_jU_{0j}^{\alpha\beta}+\omega U_{00}^{\alpha\beta}-i(2\pi)^3(E_p/M)f^{\alpha\beta\gamma}\langle\vec{p}'|J_0^\gamma(0)|\vec{p}\rangle . \tag{16}$$

To illustrate the usefulness of these relations consider the basis element number eight of Pais,[3]

$$B_{ij}^8=\epsilon_{ijm}(k_m\vec{S}\cdot\vec{k}+k_m'\vec{S}\cdot\vec{k}') . \tag{17}$$

As this basis element is orthogonal to $k_i'k_j$ it will not be present in Eq. (3) and this equation can give no information on the partial amplitude $a_8^{\alpha\beta}$. However, this term will be present in (15) given the following contribution to its left side:

$$(a_8^{\alpha\beta}-\omega'b_8^{\alpha\beta})k_i'B_{ij}^8=(a_8^{\alpha\beta}-\omega'b_8^{\alpha\beta})(\vec{k}\times\vec{k}')_j\vec{S}\cdot\vec{k} . \tag{18}$$

As the basis element in Eq. (17) is odd under crossing it follows from Eqs. (4) and (13), together with the fact that $E_{ij}^{\alpha\beta}$ is even under crossing and $\Gamma_{ij}^{\alpha\beta}$ is odd, that the isospin-symmetric amplitude $a_8^{\{\alpha\beta\}}$ and the isospin-antisymmetric $b_8^{[\alpha\beta]}$ are odd under crossing and $a_8^{[\alpha\beta]}$ and $b_8^{\{\alpha\beta\}}$ are even. Therefore, for the isospin-antisymmetric part of (18), we have the following expansion in powers of $\omega=\omega'$ (Breit frame):

$$(a_8^{[\alpha\beta]}-\omega'b_8^{[\alpha\beta]})(\vec{k}\times\vec{k}')_j\vec{S}\cdot\vec{k}$$
$$=[a_8^{[\alpha\beta]}(0)-\omega^2\bar{b}_8^{[\alpha\beta]}](\vec{k}\times\vec{k}')_j\vec{S}\cdot\vec{k} , \tag{19}$$

which is a known quantity in terms of the corresponding term on the right-hand side of (15). From (19) we see that the unknown term $\bar{b}_8^{[\alpha\beta]}$ cannot compete for the determination of $a_8^{[\alpha\beta]}(0)$, which will be given by the corresponding term of order zero on the right-hand side of Eq. (15). Therefore, we shall have a low-energy theorem for $a_8^{[\alpha\beta]}(0)$.

A systematic study of the low-energy theorems to a given order will be presented in the near future.

Before closing our discussion we remark that the property (7) can be demonstrated along the same lines that Bell[5] has presented for the derivation of Singh's lemma. Bell's argument avoids the expansion over intermediate states, using only gauge invariance and the notion that the excited-state part of the scattering amplitude is a smooth function.

Bell considered only neutral currents but his argument can be immediately generalized for the derivation of both Singh's lemma and of Eq. (7). The main point for the first case is that Bell's considerations applied to $U_{00}^{\alpha\beta}$ show that when $\vec{k}$, for instance, approaches zero this quantity is proportional to $-\langle\vec{p}'|J_0^\gamma|\vec{p}\rangle/\omega$. On the other hand, using current conservation the term containing the matrix element of J_i^γ on the right-hand side of Eq. (3) will be proportional to $(E_p-E_{p'})\langle\vec{p}'|J_0^\gamma|\vec{p}\rangle$ when $\vec{k}\to 0$, and all of the right-hand side of Eq. (3) will vanish. Therefore, this equation tells us that $E_{00}^{\alpha\beta}$ must vanish for $\vec{k}=0$ or, by the same reasoning, for $\vec{k}'=0$, and Eq. (1) follows. Likewise in our case the main point is that when $\vec{k}'$ approaches zero the unexcited term $U_{0j}^{\alpha\beta}$ coincides with the opposite of the last term of Eq. (5) divided by ω'. Therefore this equation tells us that $E_{0j}^{\alpha\beta}$ must vanish for $\vec{k}'=0$, and Eq. (7) follows.

[1]V. Singh, Phys. Rev. Lett. 19, 730 (1967).
[2]V. Singh, Phys Rev. 165, 1532 (1968).
[3]A. Pais, Nuovo Cimento 53A, 433 (1968).
[4]L. S. Brown, Phys. Rev. 150, 1338 (1966); M. A. Bég, Phys. Rev. Lett. 17, 333 (1966).
[5]J. S. Bell, Nuovo Cimento 53, 635 (1967).