Quadratic constraints in amplitude analysis

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The quadratic constraints among bilinear products of reaction amplitudes for a reaction with arbitrary spins are discussed in terms of the experimental observables. While the specific form of these constraints may be quite lengthy, the optimal formalism of polarization phenomena exhibits the structure of these constraints transparently and gives a systematic procedure to calculate them. Some general features of these constraints are exhibited which are useful for planning experiments. Such features can be recognized without lengthy calculations, with the help of a diagrammatic analog combined with the transparent structure of the optimal formalism.

I. INTRODUCTION

Getting information about reaction amplitudes from experimental observables is a quadratic problem, independently of whether one uses as amplitude parameters phase shifts or real-imaginary or magnitude-phase amplitudes of the helicity kind, transversity kind, optimal kind, or any other kind. For a reaction described by n amplitudes, $2n-1$ appropriately chosen experimental observables completely determine the amplitudes, apart from discrete ambiguities and with an overall phase factor being arbitrary. Yet there are n^2 *linearly* independent experimental quantities, each being a linear combination of (some or all) of the n^2 bilinear products ("bicoms") of the n amplitudes. These n^2 bicoms or n^2 observables are therefore *nonlinearly dependent* on each other, and this is why not *any* $2n-1$ observables or bicoms picked out of the set of n^2 form a complete set, that is, a set which determines the n complex amplitudes completely apart from *discrete* ambiguities.

We see, then, that the problem of giving prescriptions for choosing complete sets of observables or complete sets of bicoms is closely linked with the analysis of the character of the nonlinear relationships among such observables or bicoms.

Since the choice of a set of observables giving a maximal amount of information about reaction amplitudes is a central problem in the fast increasing number of experimental programs for measuring various polarization quantities for various reactions in particle and nuclear physics, considerable attention has been directed toward prescriptions for complete sets of observables or bicoms. Some of these have been *ad hoc*,[1] pertaining to one specific reaction, and using straightforward "brute-force" methods. Some other work has aimed at solving the problem in complete generality,[2] in a form that is readily applicable to any specific case. For example, Ref. 3 devised a geometrical method to tell if a set of *bicoms* form a complete set for the determination of amplitudes, and gave a proof of the geometrical method in terms of an algebraic discussion of the nonlinear constraints among the bicoms. The same reference also suggested a test for whether a set of *observables* form a complete set for determining the amplitudes, but that prescription turned out to be incomplete. Reference 4 independently rediscovered the prescription of Ref. 3, using the algebraic language, and this recipe was applied to elastic *p-p* scattering by Ref. 5. In particular, Eqs. (3.8), (3.6)—(3.7), and (3.3)—(3.5) in Ref. 5 correspond to the rules in Ref. 3 pertaining to two-vertexed, triangular, and square diagrams, respectively.

The nature of the nonlinear constraints on the bicoms can be seen particularly vividly if we use a polar-coordinate description for the complex amplitudes. Then the n-by-n matrix of the bicoms composed of the n complex amplitudes a_i $(i=1,2,\ldots,n)$ is

$$\begin{bmatrix} |a_1|^2 & |a_1||a_2|\cos\phi_{21} & \cdots & \vdots \\ |a_1||a_2|\sin\phi_{21} & |a_2|^2 & \cdots & \\ |a_1||a_3|\sin\phi_{31} & |a_2||a_3|\cos\phi_{32} & & \\ \vdots & \vdots & & \\ \cdots & & & |a_n|^2 \end{bmatrix} . \tag{1.1}$$

We see from this that once the diagonal elements of this matrix are known (n quantities), then the $|a_i||a_j|$'s appearing in the off-diagonal elements are also known [corresponding to Eq. (2.9) or Eq. (3.8) of Ref. 5], and furthermore $\phi_{ik}=\phi_{ij}+\phi_{jk}$, so that any $n-1$ phases forming a connected set and containing all i's will determine all the ϕ_{ij}'s appearing in the off-diagonal elements [corresponding to some of Eq. (2.10) or to Eqs. (3.6)—(3.7) of Ref. 5].

Since the above prescriptions, embodied in Eqs. (2.9) and (2.10) of Ref. 5, or in the rules for the two-pronged and the triangular diagrams in Ref. 3, respectively, give the requisite number of constraints [namely, $n^2-(2n-1)$ of them], one wonders about the role of Eqs. (3.3)—(3.5) in Ref. 5, or of the rules for the square diagrams in Ref. 3. A closer inspection of the latter type of constraints reveals that they are just another way of expressing the phase constraints in the polar form of the amplitudes. Whether Eq. (2.10) or Eqs. (3.3)—(3.5) (or the corresponding triangular- or rectangular-diagram constraints) are more helpful depends on the particular set of bicoms we are considering.

Whether the algebraic or the geometric method is preferable for expressing the constraints is, to some extent, a matter of taste. There are, however, some objective advantages to the geometrical method. For example, Eqs. (3.6)—(3.7) give the constraints on three amplitudes directly always only in the form which corresponds to triangular diagrams in Ref. 3 in which the only loop in the diagram is flanked by two sides each of which has both a solid and a broken line (with the third side, opposite to the loop, having only a solid or only a broken line). But similar constraints also hold for triangular diagrams with the same line pattern but with the loop at one of the other two vertices. In the geometrical language this is obvious, but in the algebraic language this can be achieved only by further and possibly involved algebraic manipulations with Eqs. (3.6)–(3.7). Thus, in the geometrical language it is much easier to see if a given set of chosen bicoms is redundant or not. On the other hand, the exact quantitative relationships between the bicoms imposed by the constraints are exhibited only in the algebraic language.

We see then that the quadratic constraints among the amplitudes and the *bicoms* are completely understood and prescriptions for finding them are readily formulated, whether in a geometrical or in some algebraic language. What has not been generally analyzed are the constraints on the *observables,* because in the formalisms used so far the relationship between bicoms and observables is complicated and depends on the spins of the particles in the reaction. In this paper, therefore, we will formulate the constraints in terms of the observables. The analysis will be carried out in the optimal formalism,[6] since in it the relationship between bicoms and observables is simple, transparent, and structurally independent of the spins of the particles in the reaction. Furthermore, the analysis can be carried out for all the different optimal formalisms simultaneously. A completely general discussion is, therefore, possible, for all possible reactions no matter what the spins of the particles are.

II. QUADRATIC CONSTRAINTS ON THE OBSERVABLES

Beside the reasons mentioned at the end of Sec. I, the optimal formalism also has the advantage that in it it is easy to describe which observables depend on which bicoms, and what the coefficients are in the relationship between the observables and the bicoms. Furthermore, these features are independent of the spins of the particles in the reaction.

Let us therefore describe the consequences of Eqs. (2.9) and (2.10) of Ref. 5 on the optimal observables. The former equation is

$$|H_{ij}|^2=H_{ii}H_{jj}\ , \tag{2.1}$$

where H_{ij} is a bicom with one amplitude having index i and the other index j. In the optimal notation this can be written as

$$\begin{aligned}|D(\xi,u;\Xi U)D^*(\omega,v;\Omega,V)|^2&=[\mathrm{Re}D(\xi,u;\Xi,U)D^*(\omega,v;\Omega,V)]^2+[\mathrm{Im}D(\xi,u;\Xi,U)D^*(\omega,v;\Omega,V)]^2\\&=|D(\xi,u;\Xi,U)|^2\,|D(\omega,v;\Omega,V)|^2\ .\end{aligned}\tag{2.2}$$

The two bicoms in the last line of Eq. (2.2) appear in submatrix 1_M and each of them is uniquely connected to *one* polarization observable only which depends only on that bicom. Thus, we see that if we have determined all the magnitudes of the amplitudes (from submatrix 1_M), then in any other submatrix (be it 8_i, 4_i, 2_i, or 1_i), of the two adjoint submatrices (one containing real bicoms, the other imaginary ones), only one of the two provides new information. This is of interest because in a well-chosen optimal frame (which, in the case of parity-conserving reactions, is usually the transversity frame), one usually determines the magnitudes of the amplitudes first from a specially selected set of experiments of moderate size,[7] and then turns to the determination of the phases of the amplitudes from some additional set of experiments coming from submatrices other than 1_M. In choosing this additional set of experiments, the result obtained above is of importance.

Having now exhausted the content of Eq. (2.9) of Ref. 5, we turn to Eq. (2.10) of that reference which, in the optimal notation, reads

$$[D(\xi,u;\Xi,U)D^*(\omega,v;\Omega,V)][D(\omega,v;\Omega,V)D^*(\phi,w;\Phi,W)]=[D(\xi,u;\Xi,U)D^*(\varphi,w;\Phi,W)][D(\omega,v;\Omega,V)D^*(\omega,v;\Omega,V)]\ . \tag{2.3}$$

If we have already determined the magnitudes of the amplitudes from 1_M, then Eq. (2.3) means that, for example, of all the various 8_i submatrices, we have to use only a limited number, and the remaining ones can then be obtained from this limited number of 8_i submatrices by Eq. (2.3). As we know, the total number of 8_i submatrices (each with a real and an imaginary realization) is $\frac{1}{8}\prod_{i=1}^{4}(2s_i)(2s_i-1)$. Of these, we need to use only $\prod_{i=1}^{4}(2s_i-1)$. In particular, if we denote the values of one of the four indices in an amplitude by a dot, and we connect the pairs of values appearing in an 8_i by lines, we need to consider only sets of 8_i's which are represented by connected loopless diagrams. Thus, for an index with four possible values ($J=2$), we can have diagrams like in Figs. 1(a)–1(d). Each of the four indices in the amplitude has such a diagram.

The same situation prevails also for the 4_i submatrices, and the 2_i's and the 1_i's also.

There are some interesting consequences of the above result. For example, we see that for particles with spins larger than $\frac{1}{2}$ the set of measurements consisting of "symmetric asymmetries" (that is, measurements of the sums and differences of the $+s_z$ and $-s_z$ states) alone do not suffice, since such a set does not form a diagram of the requisite type [see Fig. 1(e)], even if one other measurement of a different type is added. Since such a set is experimen-

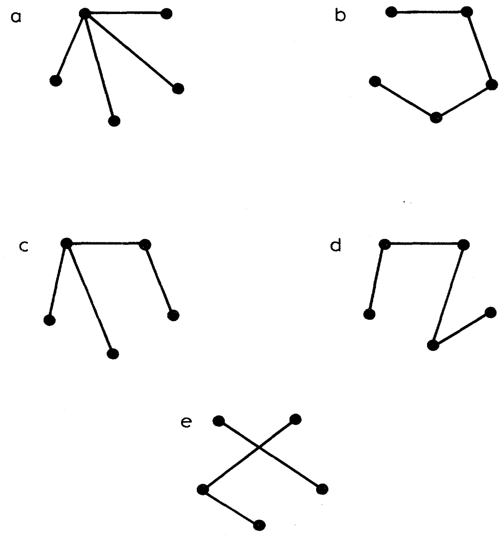

FIG. 1. Diagrams a—d represent sufficient sets of submatrices for $J=2$, while diagram e represents an insufficient set. For details, see the text.

tally simpler to obtain than other sets, it is regrettable but important to know that, in itself, it does not suffice.

The form of the constraints among observables can also be described easily though, in detail, these relationships can be fairly messy. From an 8_i submatrix (which contains only coefficients which are $+1$ or -1) we can form its inverse, which will also contain only $+1$'s and -1's and will give the bicoms in terms of observables. We denote by 8_{ij} the 8 submatrix containing bicom H_{ij}, where index i contains index values u, U, ξ, Ξ and j contains v, V, ω, Ω. Then we denote by $\{uv, UV; \xi\omega, \Xi\Omega\}$ some linear combination (with coefficients $+1$ or -1) of the eight observables with indices as in the curly brackets and with an even number (0, 2, or 4) of Re's associated with the indices if the generating bicom also has an Re in front of it. Similarly, for the bicom with Im in front of it, one takes the same combination of observables but with an odd number (1 or 3) of Re's. Then we get the constraint on the observables by replacing, in Eq. (2.3), each square bracket by the corresponding curly bracket. We see then that on the left-hand side we have, for the case of 8_i's, 128 terms and on the right-hand side 8 terms. If we start with a 4_i, the number of terms is, correspondingly, 32 and 4, etc. We see that these relationships are lengthy in detail, but their structure, in the optimal formalism, is simple and transparent.

The constraints of Eq. (2.2) translated into observables are similar: for the 8_i's, there are 128 terms on the left-hand side, and 1 on the right-hand side.

So far we considered only observables in which the polarization of all four particles is specified. We can deal, however, similarly with experiments in which some of the particles are unpolarized. Such observables can be thought of as pertaining to a set of $2s_i+1$ submatrices, where s_i is the spin of the unpolarized particle. Thus, such an averaged observable depends on $4(2s_i+1)$ bicoms if the original observable is in a submatrix 4_i, but the coefficients in this dependence of each bicom in a group of $2s_i+1$ are the same. Furthermore, since Eq. (2.3) also holds in a more general form, namely,

$$H_{ij}H_{km}=H_{im}H_{kj}$$

or (2.4)

$$\mathrm{Re}H_{ij}\mathrm{Re}H_{km}-\mathrm{Im}H_{ij}\mathrm{Im}H_{km}=\mathrm{Re}H_{im}\mathrm{Re}H_{kj}-\mathrm{Im}H_{im}\mathrm{Im}H_{kj}$$

and

$$\mathrm{Re}H_{ij}\mathrm{Im}H_{km}+\mathrm{Im}H_{ij}\mathrm{Re}H_{km}=\mathrm{Re}H_{im}\mathrm{Im}H_{kj}+\mathrm{Im}H_{im}\mathrm{Re}H_{kj}\ ,$$

or in the optimal notation

$$[D(\xi,u;\Xi,U)D^*(\omega,v;\Omega,V)][D(\phi,w;\Phi,W)][D^*(\psi,z;\Psi,Z)]=[D(\xi,u;\Xi,U)D^*(\psi,z;\Psi,Z)][D(\phi,w;\Phi,W)][D^*(\omega,v;\Omega,V)]\ , \quad (2.5)$$

we can simply extend Eq. (2.3) to hold for the $\sum_j H_{ij}$'s instead of just for the H_{ij}'s [the cross terms being taken care of by Eq. (2.4)]. Hence, the constraints on the averaged observables have the same structure as the ones on the unaveraged observables.

It should be recalled that the observables in the 8_i's cannot have unpolarized particles in them at all, those in the 4_i's can have at most one unpolarized particle, those in the 2_i's two, those in the 1_i's three, and 1_M contains the completely unpolarized cross section. Hence, the more particles are unpolarized in the observable, the simpler the structure of the constraints becomes.

To illustrate these results, let us give some examples from a reaction that is familiar, much used, and well documented, namely, the reaction $\frac{1}{2}+\frac{1}{2}\rightarrow\frac{1}{2}+\frac{1}{2}$. The observable-bicom relations for this reaction in the absence of any symmetry except Lorentz invariance (which is the case we are now considering) was given in Table III of Ref. 8. It should be recalled, however, that inasmuch as the 16-set, which constitutes the $\frac{1}{2}+\frac{1}{2}\rightarrow\frac{1}{2}+\frac{1}{2}$ reaction, is also the basic building block of *any* reaction,[9] the illustrations here go beyond the scope of one particular reaction.

Let us first take an example from the 1_i's. We will use[8] the notation

$$\begin{smallmatrix}++\\++\end{smallmatrix}=i,\ \begin{smallmatrix}++\\-+\end{smallmatrix}=j,\ \begin{smallmatrix}-+\\++\end{smallmatrix}=k\ ,$$

and use the real parts of the bicoms. Then

$$H_{ij}=(++,++;++,\{\begin{smallmatrix}R\\I\end{smallmatrix}\})$$

from 1_1,

$$H_{ik}=(++,++;\{{R \atop I}\},++)$$

from 1_2,

$$H_{jk}=(++,++;\{{R \atop I}\},R)+(++,++;\{{I \atop R}\},I)$$

from 2_1 (here the upper and lower terms in curly brackets refer to the real and imaginary parts of H_{xy}, respectively), and of course

$$H_{ii}=(++,++;++)$$

from 1_M. So the quadratic relationship among the observables in this case is

$$(++,++;++,R)[(++,++;R,R)+(++,++;I,I)]-(++,++;++,I)[-(++,++;R,I)+(++,++;I,R)]$$
$$=(++,++;R,++)(++,++;++,++) . \quad (2.6)$$

We see from this example that in a constraint of the type given by Eq. (2.3) the observables will come from different submatrices so that the types of submatrices "balance" in the obvious way exhibited by this example, and that the observable notation of the optimal formalism is such that the indices within the observables also "balance" in the obvious way shown in the example.

For a second example, let us take some averaged quantities from the same submatrices. For this we also need

$$\frac{-+}{-+}=l .$$

Then we have from submatrix 1_1

$$\begin{aligned}(++,++;A,R)&=\mathrm{Re}(H_{ij}+H_{kl}) ,\\ -(++,++;A,I)&=\mathrm{Im}(H_{ij}+H_{kl}) ,\\ (++,++;R,A)&=\mathrm{Re}(H_{ik}+H_{jl}) ,\\ -(++,++;I,A)&=\mathrm{Im}(H_{ik}+H_{jl}) .\end{aligned} \quad (2.7)$$

Consider then the combination

$$(++,++;A,R)(++,++;R,A)$$
$$+(++,++;\Delta,I)(++,++;I,\Delta) , \quad (2.8)$$

where $\Delta=(++)-(--)$, i.e., the asymmetry, in accordance with the notation of the first paper in Ref. 7, Table V. From the subtables for 1_1 and 1_2 we can express these observables in terms of H_{xy}'s and get

$$(\mathrm{Re}H_{ij}+\mathrm{Re}H_{kl})(\mathrm{Re}H_{ik}+\mathrm{Re}H_{jl})$$
$$+(\mathrm{Im}H_{ij}-\mathrm{Im}H_{kl})(\mathrm{Im}H_{ik}-\mathrm{Im}H_{jl}) . \quad (2.9)$$

Applying to these products Eq. (2.3), and then obtaining the corresponding observables for the resulting H_{xy}'s, using submatrices 2_1 and 1_M, we get finally

$$\begin{aligned}&(++,++;A,R)(++,++;R,A)+(++,++;\Delta,I)(++,++;I,\Delta)\\ &=[(++,++;R,R)-(++,++;I,I)][(++,++;++,++)+(++,++;--,--)]\\ &\quad+[(++,++;R,R)+(++,++;I,I)][(++,++;++,--)+(++,++;--,++)]\\ &=(++,++;R,R)(++,++;A,A)-(++,++;I,I)(++,++;\Delta,\Delta) .\end{aligned} \quad (2.10)$$

Similarly, we can work A's and Δ's into the other observables also.

So far we discussed the constraints in the case when only Lorentz invariance was imposed on the reaction matrix. When additional symmetries also hold, such as parity conservation, time-reversal invariance, or identical-particle restrictions, the nonlinear constraints listed above simply add to these other constraints. To give an example, consider Table VI of the first paper in Ref. 7, columns 1, 6, and 7 of which give some observables in the transversity optimal frame for elastic p-p scattering with all the relevant symmetries imposed. In it, let us take $i=\alpha-\beta$, $j=\delta$, and $k=\epsilon$. We see then that

$$\begin{aligned}\mathrm{Re}H_{ij}&=\mathrm{Re}(\alpha-\beta)\delta^*=-\tfrac{1}{2}H_{SSN} ,\\ \mathrm{Im}H_{ij}&=\mathrm{Im}(\alpha-\beta)\delta^*=\tfrac{1}{2}C_{SL} ,\\ \mathrm{Re}H_{jk}&=\mathrm{Re}\delta\epsilon^*=\tfrac{1}{8}(D_{SS}-D_{LL}) ,\\ \mathrm{Im}H_{jk}&=\mathrm{Im}\delta\epsilon^*=\tfrac{1}{8}(-H_{NSL}-H_{NLS}) ,\\ \mathrm{Re}H_{ik}&=\mathrm{Re}(\alpha-\beta)\epsilon^*=-\tfrac{1}{2}H_{SNS} ,\\ H_{jj}&=|\epsilon|^2=\tfrac{1}{4}(\sigma_0-C_{NN}-D_{NN}+K_{NN})\end{aligned} \quad (2.11)$$

and so our quadratic constraints yield

$$H_{SSN}(D_{SS}-D_{LL})-C_{SL}(H_{NSL}+H_{NLS})$$
$$=2H_{SNS}(\sigma_0-C_{NN}-D_{NN}+K_{NN}) . \quad (2.12)$$

We also see from this example that different optimal frames readily yield quadratic constraints in very different forms. While Eq. (2.12) could be obtained very easily in the transversity frame, it would have been extremely laborious to derive this particular quadratic constraint in the helicity formalism.

III. CONCLUSION AND SUMMARY

We have seen that the quadratic constraints on the bilinear products of reaction amplitudes (bicoms) can be converted into constraints on observables relatively easily when one uses the optimal formalism for describing the spin structure of reactions. In particular, the following general features can be established for reactions involving particles with arbitrary spins:

(1) Once the magnitudes of the amplitudes have been determined from observables in the submatrix 1_M, then of

TABLE I. The bicom-observable relationship for the reaction $0+\frac{1}{2}\rightarrow 0+\frac{1}{2}$ in the presence of Lorentz invariance only, in the optimal formalism.

1_M

U	V	Ξ	Ω	$\|a\|^2$	$\|b\|^2$	$\|c\|^2$	$\|d\|^2$
+	+	+	+	+			
−	−	+	+		+		
+	+	−	−			+	
−	−	−	−				+
A		+	+	+	+		
A		−	−			+	+
+	+	A		+		+	
−	−	A			+		+
Δ		+	+	+	−		
Δ		−	−			+	−
+	+	Δ		+		−	
−	−	Δ			+		−
A		Δ		+	+	−	−
Δ		A		+	−	+	−
A		A		+	+	+	+
Δ		Δ		+	−	−	+

1_1 Re Re

U	V	Ξ	Ω	ac^*	bd^*
+	+	R		+	
−	−	R			+
A		R		+	+
Δ		R		+	−

1_1 Im Im

U	V	Ξ	Ω	ac^*	bd^*
+	+	I			-
−	−	I			
A		I		−	−
Δ		I		−	+

1_2 Re Re

UV	Ξ	Ω	ab^*	cd^*
R	+	+	+	
R	−	−		+
R	A		+	+
R	Δ		+	−

1_2 Im Im

UV	Ξ	Ω	ab^*	cd^*
I	+	+	+	
I	−	−		+
I	A		+	+
I	Δ		+	−

2_1 Re Re

UV	$\Xi\,\Omega$	ad^*	bc^*
R	R	+	+
I	I	+	−

2_1 Im Im

UV	$\Xi\,\Omega$	ad^*	bc^*
R	I	−	−
I	R	+	−

the two "twin" submatrices of any kind other than 1_M (one of the twins containing the real parts of amplitude products, the other the imaginary parts), only one of the two provides new information.

(2) Of the many submatrices of a given type N_i (e.g., 8_i, 4_i, etc.) only a limited number of them give independent information. Sets of such independent submatrices correspond to connected loopless diagrams depicting the transitions in the four arguments of the amplitudes. This fact has experimental consequences. For example, for particles

TABLE II. Same as Table I, except with time-reversal invariance in addition to Lorentz invariance. The upper and lower signs hold for the planar and transversity formalisms, respectively.

1_M

U	V	Ξ	Ω	$\|a\|^2$	$\|b\|^2$	$\|c\|^2$
+	+	+	+	+		
−	−	+	+		+	
−	−	−	−			+
A		+	+	+	+	
A		−	−		+	+
Δ		+	+	+	−	
Δ		−	−		+	−
A		Δ		+		−
A		A		+	+2	+
Δ		Δ		+	−2	+

$(++--)=(--++)$
$(++A)=(A++)$
$(++\Delta)=(\Delta++)$
$(--\Delta)=(\Delta--)$
$(\Delta A)=(A\Delta)$

$1_1=1_2$ Re Re

U	V	Ξ	Ω	ab^*	bd^*
+	+	R		∓	
−	−	R			+
A		R		∓	+
Δ		R		∓	−

$(R++)=\mp(++R)$
$(R--)=\mp(--R)$
$(RA)=\mp(AR)$
$(RD)=\mp(\Delta R)$

$1_1=1_2$ Im Im

U	V	Ξ	Ω	ab^*	bd^*
+	+	I		±	
−	−	I			+
A		I		±	+
Δ		I		±	−

$(I++)=\pm(++I)$
$(--I)=\pm(--I)$
$(IA)=\pm(AI)$
$(I\Delta)=\pm(\Delta I)$

2_1 Re

UV	$\Xi\,\Omega$	ad^*	$\|b\|^2$
R	R	+	∓
I	I	+	±

2_1 Im

UV	$\Xi\,\Omega$	ad^*
R	I	−

$(IR)=-(RI)$

with spins larger than $\frac{1}{2}$, measurements involving only pairwise the $+s_z$ and $-s_z$ substates of a particle do not suffice for a complete determination of the amplitudes.

(3) Although the structure of the quadratic constraints is transparent, the actual constraints may be quite lengthy, since in the worst case the product of two bicoms can involve, in terms of the *observables*, 128 terms. In most cases, however, especially when a number of symmetries constrain the reaction matrix, the constraints are less lengthy. In terms of experimental quantities, the form of the constraints is different depending on which optimal formalism one uses. The constraints can also be formulated in terms of experimental quantities involving unpolarized particles.

Since the specific forms of the constraints are so tedious and depend on the formalisms, the more useful content of these quadratic constraints is their general features as explained above in (1) and (2).

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APPENDIX

In this appendix the results of this paper will be illustrated on a very simple reaction, namely, $0+\frac{1}{2}\to 0+\frac{1}{2}$, which is amply represented among the reactions frequently dealt with such as pion-nucleon scattering.

When only Lorentz invariance is imposed, this reaction has four amplitudes. They are, using the notation $D(\Lambda L)$ of the optimal formalism

$$D(++)\equiv a,\ \ D(+-)\equiv b\ ,\qquad D(-+)\equiv C,\ \ D(--)\equiv d\ . \tag{A1}$$

The relationship between the observables and the bilinear products of these amplitudes (the bicoms) in the optimal formalism is given in Table I.

In this case the four amplitudes represent $2\times 4-1=7$ real parameters but there are $4^2=16$ bicoms, so there are 9 linearly independent nonlinear relationships among the 16 bicoms. The geometrical technique of Ref. 3, or, correspondingly, the equations in Ref. 4 or 5, give many more such relationships which therefore are linearly not independent.

Although it is straightforward to enumerate the very large number of these relationships, it is tedious; hence, we will now consider the same reaction but with time-reversal invariance imposed on it in addition to Lorentz invariance (as is the case, for example, in elastic pion-nucleon scattering). The observable-bicom structure for this case, in the optimal formalism, is shown in Table II. In this case we have only three amplitudes, that is, five real parameters, but nine bicoms, and hence four linearly independent relationships among the nine bicoms. There are, however, again many more ways of expressing these nonlinear relationships among the bicoms. Let us enumerate them. In the language of Ref. 3, there will be two-vertexed diagrams (see Fig. 2) as well as four types of three-vertexed diagrams. There are three of the former type (since there are three ways of selecting two vertices out of three).

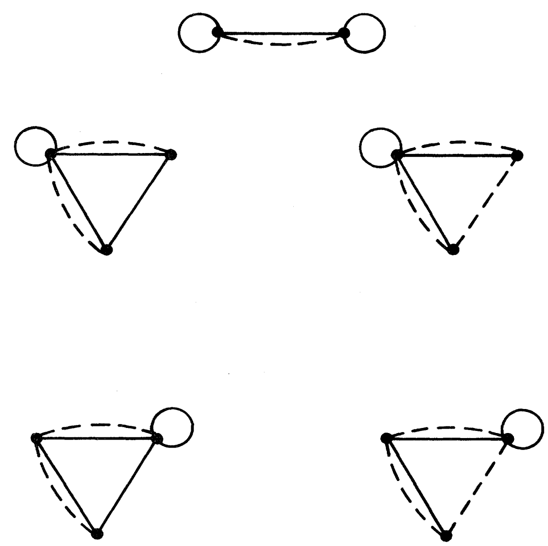

FIG. 2. Constraint-diagram types with two and three vertices, in the characterization of Ref. 3. For details, see the text.

Each of the first two of the three-vertexed types also have three different diagrams, since the bubble can be at any of the three vertices. Finally, each of the second two three-vertexed types gives six different diagrams, since there again the bubble can be at any of the three vertices, and, in addition, for a given location of the bubble we have two different diagrams, obtainable from each other by interchanging the other two vertices. Thus, we have $3+3+3+6+6=21$ different nonlinear relationships among the bicoms, of which only four are linearly independent.

Let us now write out these relationships. The ones corresponding to the two-vertexed diagrams are those of Eq. (2.9) of Ref. 4, or Eq. (3.8) of Ref. 5. They state that

$$(\mathrm{Re}ab^*)^2+(\mathrm{Im}ab^*)^2=|a|^2|b|^2\ , \tag{A2}$$

$$(\mathrm{Re}ad^*)^2+(\mathrm{Im}ad^*)^2=|a|^2|d|^2\ , \tag{A3}$$

$$(\mathrm{Re}bd^*)^2+(\mathrm{Im}bd^*)^2=|b|^2|d|^2\ . \tag{A4}$$

The first of the three-vertexed diagrams in Fig. 2 correspond to Eq. (3.6) in Ref. 5. The relationships are

$$|a|^2(\mathrm{Re}bd^*)=(\mathrm{Re}ab^*)(\mathrm{Re}ad^*)+(\mathrm{Im}ab^*)(\mathrm{Im}ad^*)\ , \tag{A5}$$

$$|b|^2(\mathrm{Re}ad^*)=(\mathrm{Re}ab^*)(\mathrm{Re}bd^*)-(\mathrm{Im}ab^*)(\mathrm{Im}bd^*)\ , \tag{A6}$$

$$|d|^2(\mathrm{Re}ab^*)=(\mathrm{Re}ad^*)(\mathrm{Re}bd^*)+(\mathrm{Im}ad^*)(\mathrm{Im}bd^*)\ . \tag{A7}$$

Similarly, the second of the three-vertexed diagrams in

TABLE III. Observables for $0+\frac{1}{2}\to 0+\frac{1}{2}$ with Lorentz invariance and time-reversal invariance in the two types of optimal formalisms. The eight columns represent the following: Column 1, the traditional name of the observable; because of differing conventions there might be a sign ambiguity for some of these observables. Column 2, notation for the observable analogous to the Argonne c.m. notation; see first paper in Ref. 8, Table VI. Column 3, name of the observable in the planar formalism. Column 4 the planar submatrix in Table II in which the observable appears. Column 5, the expression for the planar observables in terms of the planar amplitudes a, b, and d. Column 6, same as 3, except in the transversity formalism. Column 7, same as 4, except in the transversity formalism. Column 8, same as 5, except in the transversity formalism; here the three amplitudes are α, β, and δ.

1	2	3	4	5	6	7	8
σ_0	(0,0)	(A,A)	1_M	$\|a\|^2+2\|b\|^2+\|d\|^2$	(A,A)	1_M	$\|\alpha\|^2+2\|\beta\|^2+\|\delta\|^2$
P_N	$(0,y)$	(A,I)	1_1	$\text{Im}ab^*+\text{Im}bd^*$	(A,Δ)	1_M	$\|\alpha\|^2-\|\delta\|^2$
P_L	$(0,z)$	(A,Δ)	1_M	$\|a\|^2-\|d\|^2$	(A,R)	1_1	$\text{Re}\alpha\beta^*+\text{Re}\beta\delta^*$
P_S	$(0,x)$	(A,R)	1_1	$-\text{Re}ab^*+\text{Re}bd^*$	(A,I)	1_1	$-\text{Im}\alpha\beta^*+\text{Im}\beta\delta^*$
D_{NN}	(y,y)	(I,I)	2_1	$\text{Re}ad^*+\|b\|^2$	(Δ,Δ)	1_M	$\|\alpha\|^2-2\|\beta\|^2+\|\delta\|^2$
D_{LL}	(z,z)	(Δ,Δ)	1_M	$\|a\|^2-2\|b\|^2+\|d\|^2$	(R,R)	2_1	$\text{Re}\alpha\delta^*+\|\beta\|^2$
D_{SS}	(x,x)	(R,R)	2_1	$\text{Re}ad^*-\|b\|^2$	(I,I)	2_1	$\text{Re}\alpha\delta^*-\|\beta\|^2$
D_{NL}	(y,z)	(I,Δ)	1_1	$\text{Im}ab^*-\text{Im}bd^*$	(Δ,R)	1_1	$\text{Re}\alpha\beta^*-\text{Re}\beta\delta^*$
D_{LS}	(z,x)	(Δ,R)	1_1	$\text{Re}ab^*+\text{Re}bd^*$	(R,I)	2_1	$-\text{Im}\alpha\delta^*$
D_{NS}	(y,x)	(I,R)	2_1	$\text{Im}ad^*$	(Δ,I)	1_1	$-\text{Im}\alpha\beta^*-\text{Im}\beta\delta^*$

Fig. 2 corresponds to Eq. (3.7) in Ref. 5. The relationships are

$$|a|^2\text{Im}(bd^*)=(\text{Re}ab^*)(\text{Im}ad^*)-(\text{Re}Ad^*)(\text{Im}ab^*)\,, \tag{A8}$$

$$|b|^2\text{Im}(ad^*)=(\text{Re}ab^*)(\text{Im}bd^*)+(\text{Re}bd^*)(\text{Im}ab^*)\,, \tag{A9}$$

$$|d|^2\text{Im}(ab^*)=-(\text{Re}ad^*)(\text{Im}bd^*)+(\text{Re}bd^*)(\text{Im}ad^*)\,. \tag{A10}$$

Now we turn to the third three-vertexed diagram in Fig. 2. As mentioned in passing in the main body of the text, of the four different types only two correspond directly to the algebraic equation in Ref. 5, namely, the first corresponds to Eq. (3.6) of Ref. 5 while the second to Eq. (3.7) of Ref. 5. The remaining two diagrams can presumably be obtained from Eqs. (3.6) and (3.7) of Ref. 5 by combining the two equations so that, using the language of Ref. 3, one of the solid (or one of the broken) lines next to the loop in the first of the two three-vertexed diagrams "cancel."

So far we have dealt only with bicoms. Of practical interest, however, is the conversion of these two constraints into relationships among observables rather than bicoms, and it is here that the optimal formalism is advantageous since in it one can easily tell which bicoms are connected to which observables, no matter what the spins of the particles participating in the reaction are.

For the reaction we are considering, the relationship between the standard observables and the amplitudes is

TABLE IV. Same as Table I, except with time-reversal invariance and parity conservation imposed in addition to Lorentz invariance. The product of the four intrinsic parities is taken to be $+1$. This table holds for the planar formalisms.

U	V	Ξ	Ω	$\|a\|^2$	$\|b\|^2$	$1_M,2_1$
+	+	+	+	+		$=(--,--)$
−	−	+	+		+	$=(++,--)$
A		+	+	+	+	$=(--,A)=(I,I)=(++,A)=(A,--)=\frac{1}{2}(A,A)$
Δ		+	+	+	−	$=(++,\Delta)=-(--,\Delta)=(R,R)=-(\Delta,--)=\frac{1}{2}(\Delta,\Delta)$

U	V	Ξ	Ω	Re ab^*	Im ab^*	$1_1,1_2$
+	+	R		−		$=-(--,R)=-(R,++)=(R,--)=-\frac{1}{2}(R,\Delta)=\frac{1}{2}(\Delta,R)$
+	+	I			+	$=(--,I)=(I,++)=(I,--)=\frac{1}{2}(I,A)=\frac{1}{2}(A,I)$

$(R,I)=(R,A)=(I,R)=(I,\Delta)=(A,R)=(A,\Delta)=(\Delta,I)=(\Delta,A)=0$

TABLE V. Same as Table IV, except for the transversity formalism.

1_M

U	V	Ξ	Ω	$\|a\|^2$	$\|d\|^2$	
+	+	+	+	+		$=(++,A)=(++,\Delta)=(A,++)=(\Delta,++)$
−	−	−	−		+	$=(--,A)=(--,\Delta)=(A,--)=(\Delta,--)$
A		A		+	+	$=(\Delta,\Delta)$
A		Δ		+	−	$=(\Delta,A)$

2_1

U	V	Ξ	Ω	Re $\|a\|^2$	Im $\|d\|^2$	
R		R		+		$=(I,I)$
R		I			−	$=-(I,R)$

$(++,--)=(++,R)=(++,I)=(--,++)=(--,R)=(--,I)=(R,++)=(R,--)=(R,A)$
$=(R,\Delta)=(I,++)=(I,--)=(I,A)=(I,\Delta)=(A,R)=(A,I)=(\Delta,R)=(\Delta,I)=0$

given in Table III. We can illustrate the results stated in this paper. For example, once we measured 1_M only the real or only the imaginary part of any other submatrix represents new information.

In particular, for the planar formalism, once σ_0, P_L, and D_{LL} have been measured, then in the sets $\{P_N, D_{LS}\}$, $\{P_S, D_{NL}\}$, and $\{D_{NN}, D_{NS}\}$, only one of the two observables represents new information. Similarly, in the transversity formalism, once σ_0, P_N, and D_{NN} have been measured, in the sets $\{P_L, D_{NS}\}$, $\{P_S, D_{NL}\}$, and $\{D_{SS}, D_{LS}\}$, only one of the two observables represents new information.

Since the two experiments in a given set represent types of experiments and degrees of difficulties which are quite different from each other, the above results can be helpful and in fact crucial in planning experimental programs.

Let us now turn to the constraints created by the three-vertexed (triangular) diagrams, namely, Eqs. (A5)—(A10). With the simple and systematic pattern of Table II, it is now easy to substitute observables for the bicoms. We get, in the planar formalism, for the first of these

$$\tfrac{1}{8}(\sigma_0+D_{LL}+2P_L)(P_S+D_{LS}) = \tfrac{1}{4}(-P_S+D_{LS})(D_{NN}+D_{SS})+\tfrac{1}{4}(P_N+D_{NL})D_{NS}\ , \quad \text{(A11)}$$

and the other five give similar kinds of relationships.

Such relationships are involved, but can be used in some cases for consistency checks in situations when one has at one's disposal more than $2n-1$ types of experimental data. This is the case for some reactions already, and is a desirable goal anyway since such apparently redundant sets help to resolve any discrete ambiguities in the amplitude solutions.

As mentioned in the main body of this paper, however, such an explicit quantitative relationship between the bicoms is often not needed, but it is enough to know that some observables are dependent on some others and hence need not be measured. For that purpose, the diagrammatic approach of Ref. 3, together with the transparent structure of the optimal formalism, suffice and provide the needed information in a few seconds that would be yielded by the algebraic method only after lengthy calculations.

For example, since a and d (or α and δ) are not adjacent but only on next-to-adjacent levels[8] of the amplitude structure, the bicoms containing a and d appear in submatrices 2_i. From the diagrams corresponding to Eqs. (A5) and (A8) we see immediately that in fact, once the observables in 1_M and 1_1 are known, those in 2_1 are superfluous. Since the latter are relatively complicated experiments, this is welcome news. In this manner a set of complete experiments, suited to a particular set of instrumental realities, can be planned easily and transparently.

The case for the same reaction when, in addition, parity conservation is also imposed is even simpler since there we have only two amplitudes. The bicom-observable relationship is given in Tables IV and V and with them the quadratic constraints can be obtained immediately.

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