Amplitude structure of off-shell processes

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(Received 2 June 1983)

The structure of M matrices, or scattering amplitudes, and of potentials for off-shell processes is discussed with the objective of determining how one can obtain information on off-shell amplitudes of a process in terms of the physical observables of a larger process in which the first process is embedded. The procedure found is inevitably model dependent, but within a particular model for embedding, a determination of the physically measurable amplitudes of the larger process is able to yield a determination of the off-shell amplitudes of the embedded process.

I. INTRODUCTION

Off-shell processes play an important role in many areas of atomic, nuclear, and particle physics. All of nuclear physics is based on the two- (and perhaps three-) nucleon interaction off shell. Similar off-shell situations abound in problems of atomic physics. To learn about such off-shell processes, various special reactions have been suggested, a prominent example for the nucleon-nucleon case being proton-proton bremsstrahlung. Some reactions cannot be directly reproduced in a laboratory for technical reasons, e.g., neutron-neutron scattering, and to study such reactions we use a composite target (e.g., deuteron in the above example), in which case the primary reaction we want to study is likely to be off shell. In very-high-energy physics, special dynamical processes proposed to explain the features of certain types of particle reactions are applied in off-shell contexts, and the study of inclusive reactions also implies off-shell processes.

In view of this widespread attention it is of importance to explore the amplitude structure of off-shell processes, especially since it has been suggested in some instances[1] that the difference between the on-shell and off-shell amplitude structure can be used to obtain off-shell information in an isolated manner, uninterfered with by on-shell dynamics. Although the exact way to carry out such a procedure is still in the process of formulation and in fact may present some difficulties,[2] this formulation can be greatly facilitated by a more detailed investigation of the difference between on-shell and off-shell amplitude structures.

In Sec. II we will therefore analyze in greater detail what is meant by "off shell." Armed with that knowledge, we then investigate, in Sec. III, the off-shell scattering amplitude or M matrix, and the off-shell potentials. In Sec. IV, we discuss some applications and draw some conclusions and in Sec. V we summarize the main results of the paper.

We will attempt to keep our discussion as general as possible. In particular the conclusions of Secs. II and III depend only on general considerations of spin structure and Lorentz, time-reversal, and parity invariance and not on any procedural assumptions such as the validity of perturbation theory. Such matters may become more relevant when specific ways of implementing the general prescriptions are chosen or in Sec. IV when specific models are considered.

II. WHAT DO WE MEAN BY "OFF SHELL"?

Consider the amplitude for a process $1+2\rightarrow 3+4$ where one of the particles, say number 3, is off shell. By "off shell" one means, roughly speaking, virtual, or with energy and momentum which do not satisfy the free particle relation $E^2=\vec{p}\,^2+m^2$. For physical processes one clearly has some other additional interaction amplitude which puts the particle back on shell, for example, a rescattering, a γ or π emission, or a weak vertex. It is not necessary at present to specify, or even consider, this latter reaction; instead we will discuss the off-shell amplitude in isolation, with the understanding that later it must be joined with some other amplitude to produce a measurable quantity.

There are two major approaches leading to off-shell amplitudes. One, starting from relativistic field theory, leads normally to off-mass-shell amplitudes while the other, originating in nonrelativistic scattering theory, leads to off-energy-shell amplitudes. Some of the confusion surrounding off-shell considerations arises because of the difference in these two approaches. Hence as a pedagogical introduction to the main part of the paper we want to discuss these two approaches and their differences.

A. Relativistic-field-theory approach

Consider again the process $1+2\to 3+4$ described by the four-momenta p_i which satisfy four-momentum conservation $p_1+p_2=p_3+p_4$. The on-shell particles satisfy

$$p_i^2=m_i^2,\;\; E_i=(\vec{p}_i^{\,2}+m_i^2)^{1/2}\,,$$

while the off-shell particle, say number 3, satisfies

$$p_3^2=m_3^{*2}\neq m_3^2,\;\; E_3=(\vec{p}_3^{\,2}+m_3^{*2})^{1/2}\,.$$

The magnitudes of the initial and final center-of-mass momenta are given by

$$|\vec{p}_i^{\,\text{c.m.}}|=\frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{2\sqrt{s}}\,,\qquad |\vec{p}_f^{\,\text{c.m.}}|=\frac{\lambda^{1/2}(s,m_3^{*2},m_4^2)}{2\sqrt{s}}\,, \tag{2.1}$$

where $s=(p_1+p_2)^2=(p_3+p_4)^2$ is the square of the total center-of-mass energy and

$$\lambda(x,y,z)=x^2+y^2+z^2-2xy-2xz-2yz\,.$$

Thus $|\vec{p}_i^{\,\text{c.m.}}|\neq|\vec{p}_f^{\,\text{c.m.}}|$ off shell even for elastic processes having $m_1=m_3$ and $m_2=m_4$. In this approach an off-shell process conserves three-momentum (though the magnitude of the center-of-mass three-momentum changes) and also conserves energy. However, the mass, defined essentially by the square of the four-momentum associated with the off-shell leg, changes, and so one sometimes says "mass" is not conserved and speaks of "off-mass-shell" amplitudes. In this approach the transition or M amplitude would normally be expressed in a manifestly covariant way in terms of four-vectors, the metric tensor $g^{\mu\nu}$, γ matrices in the fermion case, etc.

B. Nonrelativistic-scattering-theory approach

In this approach as applied to elastic scattering the off-shell amplitude is usually generated by starting with a nonrelativistic T operator $T(\mathscr{E})$ which is a function of an energy parameter $\mathscr{E}$. Matrix elements are then taken between plane waves of definite three-momenta $\vec{p}_i$ which satisfy three-momentum conservation $\vec{p}_1+\vec{p}_2=\vec{p}_3+\vec{p}_4$.

For the fully off-shell amplitude there is no particular relation between $\mathscr{E}$ and the initial and final center-of-mass momenta $\vec{p}_i^{\,\text{c.m.}}$ and $\vec{p}_f^{\,\text{c.m.}}$. For the half-off-shell amplitude (just one leg off shell) which we have been discussing, one would choose $\mathscr{E}$ as the center-of-mass energy corresponding to $\vec{p}_i^{\,\text{c.m.}}$ and the off-shell condition would be given by $|\vec{p}_i^{\,\text{c.m.}}|\neq|\vec{p}_f^{\,\text{c.m.}}|$. In this approach one defines an energy associated with each momentum state as $\widetilde{E}=(\vec{p}^{\,2}+m^2)^{1/2}$ or nonrelativistically $m+\vec{p}^{\,2}/2m$. This is a logical way to proceed since the T matrix contains the free Hamiltonian H_0 in terms involving $(\mathscr{E}-H_0)^{-1}$, and this choice of energy makes the wave functions which are used energy eigenstates of this free Hamiltonian. The energy defined in this way is not conserved, though now the masses do not change. Hence we get an "off-energy-shell" amplitude. This amplitude conserves three-momentum and has the same free masses, but does not conserve energy $\widetilde{E}$ as defined and has $|\vec{p}_i^{\,\text{c.m.}}|\neq|\vec{p}_f^{\,\text{c.m.}}|$. One could, however, still define an "energy" as in the relativistic approach via $E^2=\vec{p}^{\,2}+p^2=\vec{p}^{\,2}+m^{*2}$ which would be conserved. In the nonrelativistic approach one would normally construct the M amplitude from the available three-vectors, and in the fermion case from the Pauli spin operator $\vec{\sigma}$ as well.

III. OFF-SHELL M MATRICES

In this section we will investigate how the structure of the transition or M amplitude changes as we go off shell. Such changes might manifest themselves in three major ways.

(i) The amplitudes may depend on additional scalar variables or degrees of freedom which enter in the off-shell case.

(ii) There may be additional amplitudes, i.e., new structures which vanish on shell.

(iii) The various symmetry constraints such as parity and time reversal may lead to different restrictions in the on- and off-shell cases.

We want to consider each of these possibilities in turn in both the relativistic-field-theory and the nonrelativistic-scattering-theory approaches.

A. Scalar variables

In the relativistic approach an n-particle process will be described by $n-1$ independent four-vectors. These four-vectors and the independent scalars (or pseudoscalars when n is large enough) which can be formed from them are the same whether the amplitude is on or off shell. The only difference in the off-shell case is that for each off-shell leg there is now one variable, namely, the square of the four-momenta, which is not evaluated at m^2. This means in effect that there is one additional scalar variable for each off-shell leg, and the amplitudes can depend on this variable as well as on the usual ones necessary in the on-shell case. Thus for the case $1+2\to 3+4$ with particle 3 off shell which was discussed above, we could naturally choose as variables $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, and $m_3^{*2}\equiv p_3^2$ with m_3^* reducing to m_3 on shell.

In the nonrelativistic-scattering-theory approach there are also $n-1$ vectors, three-vectors in this case, and again we can form exactly the same scalar variables in the on- and off-shell cases, except for the addition of a single scalar variable in the off-shell case. The natural variables are different from those in the relativistic approach, i.e., for two-particle elastic scattering one would normally choose $|\vec{p}_i^{\,\text{c.m.}}|$, $\hat{p}_i^{\,\text{c.m.}}\cdot\hat{p}_f^{\,\text{c.m.}}$, and $|\vec{p}_f^{\,\text{c.m.}}|$ with $|\vec{p}_f^{\,\text{c.m.}}|$ reducing to $|\vec{p}_i^{\,\text{c.m.}}|$ on shell.

Thus for both approaches the vectors available to form M amplitudes are the same in on- or off-shell cases and the number of scalar variables on which the M amplitudes depend is the same except for the addition of a single new scalar variable in the off-shell case for each off-shell leg. The additional variable can then be interpreted as a measure of the extent to which the amplitude is off shell.

B. New amplitude structures

While the vectors available to form amplitudes are the same in the off- as in the on-shell case, it is clear that there will be additional off-shell amplitudes because some of the structures one can form in general vanish on shell. A simple example of this occurs for spin $\frac{1}{2}$. An amplitude proportional to $(\not p - m)$ vanishes when the particle corresponding to the momentum p is on shell. Hence such an amplitude appears in the off-shell case but not in the on-shell case. In this section we want to describe how one counts the number of amplitudes, which is well known in the on-shell case, but apparently not in the off-shell case, and to give examples in both the relativistic-field-theory and the nonrelativistic-scattering-theory approaches.

Let us start with the relativistic approach and consider primarily the reaction $1+2 \to 3+4$ where the particles have spins S_1, S_2, S_3, S_4. In the on-shell case it is well known (see, e.g., Refs. 3 and 4) that the number of independent M amplitudes is just the number of helicity amplitudes, i.e., $N = \prod(2S_i + 1)$. These can be divided into two different parities, with N^+ and N^- amplitudes, respectively. For boson-fermion or fermion-fermion scattering $N^{\pm} = N/2$ while for boson-boson scattering $N^{\pm} = (N \pm 1)/2$ as can be determined by examining the behavior of helicity amplitudes under parity transformations.

Generalizing this counting procedure to the off-shell case involves a careful specification of what is meant by the wave function or field operator for the particle which is to be taken off shell. Recall that in the usual procedure[3] a wave function for a particle of spin S can be constructed from S four-vectors in the boson case or S_0 four-vectors coupled to a spin-$\frac{1}{2}$ spinor in the fermion case, where S_0 is the largest integer in S, i.e., $S = S_0 + \frac{1}{2}$. On shell, a number of subsidiary conditions must be imposed on such wave functions so as to remove extra degrees of freedom which are present. Thus, for example, for a spin-one wave function of momentum p and polarization vector ϵ_μ we have $\psi_\mu(p) \sim \epsilon_\mu$ and the subsidiary condition $\epsilon_\mu p^\mu = 0$. This condition reduces the four independent degrees of freedom of a four-vector to three, as is appropriate for a spin-one particle. In fact in general, at least for bosons, what the constraints actually do on shell is to project out the spin-S component of the wave function which initially contained components corresponding to spin S, $S-1$, $S-2, \ldots, 1, 0$. Thus the question which must be addressed is whether or not any or all of these subsidiary constraints apply off shell, and thus what should be taken as the spin content of an off-shell particle. Once the component spins have been determined one can obtain the number of independent amplitudes by calculating the number of helicity amplitudes separately for each spin component of the off-shell particle and adding up the results.

As long as the off-shell particle is considered to be elementary, in the sense that it is described by a field operator of well-defined spin, the specification of spin components is fairly straightforward and is a consequence of covariance which requires that the field operator transform according to an irreducible representation of the homogeneous Lorentz group either on or off shell. For composite systems, however, the transformation properties off shell involve the constituent field operators and thus the way in which the overall spin structure is most appropriately described depends on the context. As an example of the latter, the $\Delta(1236)$ field can be considered to transform as a pure spin-$\frac{3}{2}$ particle on shell at rest, but off shell it may be more appropriate to think of it as the product of π and nucleon fields making up a system with nonresonant spin-$\frac{3}{2}$ partial waves. Hence for the purpose here, namely, to write down the most general off-shell amplitude, we will always assume that the fields to be taken off shell are "elementary," i.e., that they are described in some approximation by a field operator of definite spin. We realize of course that in a physical sense what is elementary and what is composite is sometimes difficult to tell in an absolute way, since the result of the determination may depend on the state of our knowledge at that time, on the theoretical or phenomenological framework used and perhaps on other factors. For example, for low-energy pp bremsstrahlung one would usually consider the protons as elementary whereas at high energies for hard-photon-emission processes in large-transverse-momentum pp scattering, constituents of the proton become relevant and then the fields of consequence are the elementary parton fields, quarks, and gluons, in terms of which off-shell amplitudes could be defined. Our statements here therefore apply to whichever framework is being used in a given application as long as it allows in some approximation a description in terms of particles of definite spin structure.

To proceed further we first briefly review some of the elementary theory of representations of the Lorentz group. Irreducible representations of the full homogeneous Lorentz group O(3,1) can be classified according to representations of an $O(3)' \otimes O(3)''$ subgroup.[5] Such representations may be denoted in general by $[(j,l) \oplus (l,j)]$ where the first index in (j,l) refers to the "spin"-j representation of the $O(3)'$ subgroup and the second index to the "spin"-l representation of the $O(3)''$ subgroup. The doubling up is required for $j \neq l$ in order that the discrete operations of space and time inversion be represented. Since the dimension of the j representation of O(3) is $(2j+1)$ the $[(j,l) \oplus (l,j)]$ irreducible representation is $2(2j+1)(2l+1)$ dimensional. For $j = l$ no doubling is required and the representation (j,j) has dimension $(2j+1)^2$. Now ordinary space rotations correspond to a nontrivial O(3) subgroup of $O(3)' \otimes O(3)''$ such that (j,l) reduces to $[(j+l) \oplus (j+l-1) \oplus \cdots \oplus (|j-l|)]$ where (l) is the spin-l irreducible representation of O(3). To illustrate, the ordinary four-vector transforms according to $(\frac{1}{2}, \frac{1}{2})$ which has four components and reduces to $(1) \oplus (0)$ under space rotations. Similarly the Dirac spinor belongs to a $[(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})]$ four-dimensional irreducible representation of O(3,1) and reduces to $(\frac{1}{2}) \oplus (\frac{1}{2})$ under rotations. Using the four-vector and four-spinor representations all other representations can be formed. It is this approach which will be followed below.

Consider first the elementary boson field associated with a spin-S particle. Such a field, which we denote by $\Phi_{\mu_1, \mu_2, \ldots \mu_S}(p)$, can be constructed as a product of S four-vectors. On shell at least the conditions of symmetry under the interchange of any pair of indices and also tracelessness in any pair are imposed and ensure that the result transforms like an Sth-rank symmetric traceless four-tensor irreducible representation. The symmetry and tracelessness conditions reduce the original 4^S components of Φ to $(S+1)^2$ independent components corresponding to the ordinary spins S, $S-1$, $S-2,\ldots,1,0$ in the rest frame. This is precisely the dimensionality of the $(S/2,S/2)$ irreducible representation in the $\mathrm{O}(3)'\otimes\mathrm{O}(3)''$ notation. The Sth-rank symmetric and traceless four-tensor is equivalent to $(S/2,S/2)$, and is the appropriate representation for an off-shell particle.

For an on-shell particle there is an additional subsidiary condition

$$p^{\mu_1}\Phi_{\mu_1\mu_2\cdots\mu_S}(p)=0 \tag{3.1}$$

which projects out of the $(S/2,S/2)$ irreducible representation the (S) representation of O(3). This condition thus eliminates the S^2 components of the full representation which are associated with the spins $S-1,\ldots,1,0$ leaving just the $(2S+1)$ components of the free spin-S boson at rest. Hence in summary we expect that in a reaction in which a spin-S boson is off-mass shell the boson will be represented by a symmetric traceless tensor and will have spin components S, $S-1,\ldots,1,0$. This amounts to $(S+1)^2$ possible helicities or spin projections rather than the $(2S+1)$ values obtained for on-mass-shell particles.

An elementary fermion field associated with a spin $S=S_0+\frac{1}{2}$ particle is somewhat more complicated. Such an object can be constructed by the procedure of Rarita and Schwinger[3,6] as an outer product of a spin-$\frac{1}{2}$ Dirac four-spinor and an S_0th-rank symmetric and traceless four-tensor constructed from the product of S_0 four-vectors as was done for bosons and subject to some additional constraints. This generalized spinor, which we denote by $\Phi_{\alpha,\mu_1,\mu_2\cdots\mu_{S_0}}(p)$ with $\alpha=1,\ldots,4$ the spinor index and $\mu_1\ldots\mu_{S_0}$ the vector indices, is thus symmetric and traceless in any pair of the μ indices and satisfies the additional equation

$$(\gamma^{\mu_1})_{\alpha\beta}\Phi_{\beta,\mu_1,\cdots\mu_{S_0}}(p)=0\ . \tag{3.2}$$

Without this last equation there are $4(S_0+1)^2$ components which is the dimensionality of the reducible representation $[(\frac{1}{2},0)\oplus(0,\frac{1}{2})]\otimes(S_0/2,S_0/2)$ corresponding to the outer product of tensor and spinor used to construct Φ. Equation (3.2) reduces that number by $2S_0(S_0+1)$ conditions to $2(S_0+1)(S_0+2)$ independent components. (See appendix.) This latter number is the dimensionality of the equivalent irreducible representation

$$[(\tfrac{1}{2}(S+\tfrac{1}{2}),\tfrac{1}{2}(S-\tfrac{1}{2}))\oplus(\tfrac{1}{2}(S-\tfrac{1}{2}),\tfrac{1}{2}(S+\tfrac{1}{2}))]\ . \tag{3.3}$$

Under space rotations this reduces to the sequence $2[(S)\oplus(S-1)\oplus\cdots\oplus(1/2)]$. The doubling simply reflects the independence of particle and antiparticle degrees of freedom.

For on-mass-shell fermions there is an additional constraint, one of

$$(\not{p}\mp m)_{\alpha\beta}\Phi_{\beta,\mu_1\cdots\mu_{S_0}}(p)=0\ , \tag{3.4}$$

where $\not{p}=p_\mu\gamma^\mu$. This reduces the number of components to the requisite $2S+1$ of a true spin-S particle. Note that the further constraint equation

$$p^{\mu_1}\Phi_{\alpha,\mu_1\cdots\mu_{S_0}}(p)=0 \tag{3.5}$$

follows from Eqs. (3.2) and (3.4) and so does not lead to further independent conditions. The Dirac-type equation (3.4) projects out particle (or antiparticle) spinors, thereby eliminating the doubling. Furthermore it [actually the condition Eq. (3.5) which follows from it and Eq. (3.2)] eliminates all the auxiliary spins $(S-1)$, $(S-2),\ldots,\frac{1}{2}$. Thus the result is that an off-shell spin-S fermion corresponds to the irreducible representation of Eq. (3.3) obtained by imposing the symmetry, tracelessness, and γ_μ constraints on the wave function. It has spin components S, $S-1,\ldots,\frac{1}{2}$, each appearing twice which reflects the fact that an off-shell fermion may propagate either as a particle or an antiparticle.

To summarize then, for either bosons or fermions the number of independent amplitudes off shell can be calculated by (1) determining the spin content of the off-shell particles, which is just S, $S-1$, $S-2,\ldots$, (2) calculating the number of helicity amplitudes separately for each possible combination of the various spin components of the wave functions and adding the result, and (3) multiplying by a factor of 2 for each off-shell fermion, which comes from the particle/antiparticle degree of freedom. The number of amplitudes of the two different parities works out as well; simply calculate $N^\pm$ separately for each case using the on-shell formula and sum the results.

Before looking at some examples we digress to see what happens in the three-particle case, $1\to2+3$. There on shell one gets[3,4] the number of independent helicity amplitudes by coupling S_2 and S_3 to S_{23} in all possible ways and taking $N=\sum(2S_{\min}+1)$ when the sum is over all the S_{23} and where $S_{\min}$ is the minimum of S_1 and the particular S_{23}. The numbers of the different parities are given by[3]

$$N^\pm=(2S_2+1)(2S_3+1)/2-j(j+1)/2$$

for fermion-fermion-boson couplings and

$$N^\pm=[(2S_2+1)(2S_3+1)\pm1]/2-j(j+1)/2$$

for three-boson couplings where $j=S_2+S_3-S_1$ or zero if this is negative. The generalization to the off-shell case is the same as for the scattering situation, i.e., add together the number of amplitudes for each of the spin components of the off-shell particle, and multiply by 2 for each off-shell fermion.

To make these rules more concrete we now consider several examples of the rules, and see how we can construct the appropriate number of amplitudes from the available vectors in each case.

Example 1(a): $p_1(1/2)+p_2(0)\to p_3(1/2)+p_4(0)$, i.e., spin-zero—spin-$\frac{1}{2}$ scattering. According to the rules there are $2\times2=4$ helicity states for an off-shell spin-$\frac{1}{2}$ particle. Hence we should get $4\times4=16$ amplitudes if both fermions are off shell, $2\times4=8$ if only one is off shell, and $2\times2=4$ if both are on shell. To construct such amplitudes take p_1,p_2,p_3 as the independent vectors. We can then construct the following amplitudes:

$$A_1 1+A_2\not p_1+A_3\not p_2+A_4\not p_3+A_5\not p_2\not p_1 + A_6\not p_3\not p_2+A_7\not p_3\not p_1+A_8\not p_3\not p_2\not p_1 \,. \quad (3.6)$$

There are eight additional terms of the opposite parity obtained by multiplying each of the above by γ_5. Different orderings of the vectors can be reduced to these by commutation relations. Thus for the fully off-shell case there are 16 amplitudes as expected. For p_1 on shell the A_2,A_5,A_7,A_8 terms do not contribute, i.e., as actually written are not independent of the others, leaving a total of eight amplitudes when one spin-$\frac{1}{2}$ particle is off shell. Finally for p_3 on shell A_4,A_6 do not contribute, leaving a total of four amplitudes for the fully on-shell case. At each stage $N^{\pm}=N/2$ as appropriate for this case.

Example 2(a): $p_1(1)+p_2(0)\to p_3(1)+p_4(0)$, i.e., spin-zero—spin-one scattering. According to the rules the off-shell spin-one particle has components of spin 1 and spin 0. Hence we have to calculate the number of amplitudes by summing (in the fully off-shell case) the number of amplitudes for $0+0\to 0+0$, $1+0\to 1+0$, $1+0\to 0+0$, $0+0\to 1+0$. Thus we get $1+9+3+3=16$ amplitudes in the fully off-shell case, $9+3=12$ when only one of the spin-1 particles is off shell, and 9 when all particles are on shell. To see how this works take again p_1, p_2, and p_3 as independent vectors. The amplitude must be a two-index tensor to contract with the two spin vectors $(\epsilon_1)^{\mu}(\epsilon_3)^{\nu}$ corresponding to the two spin-one particles, and this tensor must be constructed from the available vectors, the metric tensor $g^{\mu\nu}$, and various contractions with the completely antisymmetric tensor $\epsilon_{\mu\nu\alpha\beta}$. The obvious candidates for such a tensor are

$$\begin{aligned}
&g^{\mu\nu},\ (p_1^{\mu},p_2^{\mu},p_3^{\mu})\otimes(p_1^{\nu},p_2^{\nu},p_3^{\nu})\,,\\
&\epsilon(\mu,\nu,p_1,p_2),\ \epsilon(\mu,\nu,p_1,p_3),\ \epsilon(\mu,\nu,p_2,p_3)\,,\\
&\epsilon(\mu,p_1,p_2,p_3)\otimes(p_1^{\nu},p_2^{\nu},p_3^{\nu})\,,\\
&\epsilon(\nu,p_1,p_2,p_3)\otimes(p_1^{\mu},p_2^{\mu},p_3^{\mu})\,,
\end{aligned} \quad (3.7)$$

where for example $\epsilon(\mu,\nu,p_1p_2)=\epsilon_{\mu\nu\alpha\beta}p_1^{\alpha}p_2^{\beta}$ and where $\otimes$ means the product of any pair of elements of two sets.

There are a total of 19 structures, three more than expected. This illustrates a problem which arises very regularly when the spins are other than 0 or $\frac{1}{2}$, namely, that it is very easy to construct more invariants than necessary. Scadron and Jones[4] have discussed this in detail. It turns out that there are highly nonobvious and nontrivial relations among some of the amplitudes, particularly those involving $\epsilon(\mu\nu\alpha\beta)$. In the present case one can show that the set $\epsilon(\nu,p_1,p_2,p_3)\otimes(p_1^{\mu},p_2^{\mu},p_3^{\mu})$ can, for example, be expressed in terms of the $\epsilon(\mu,\nu,p_i,p_j)$ and $\epsilon(\mu,p_1,p_2,p_3)\otimes(p_1^{\nu},p_2^{\nu},p_3^{\nu})$. Hence we can eliminate three of the above structures and obtain the fully off-shell amplitudes as a function of 16 independent structures:

$$\begin{aligned}
&A_1g^{\mu\nu}+A_2p_1^{\mu}p_1^{\nu}+A_3p_1^{\mu}p_2^{\nu}+A_4p_1^{\mu}p_3^{\nu}+A_5p_2^{\mu}p_1^{\nu}+A_6p_2^{\mu}p_2^{\nu}+A_7p_2^{\mu}p_3^{\nu}+A_8p_3^{\mu}p_1^{\nu}+A_9p_3^{\mu}p_2^{\nu}\\
&\quad +A_{10}p_3^{\mu}p_3^{\nu}+A_{11}\epsilon(\mu,\nu,p_1,p_2)+A_{12}\epsilon(\mu,\nu,p_1,p_3)+A_{13}\epsilon(\mu,\nu,p_2,p_3)+A_{14}\epsilon(\mu,p_1,p_2,p_3)p_2^{\nu}\\
&\quad +A_{15}\epsilon(\nu,p_1,p_2,p_3)p_1^{\mu}+A_{16}\epsilon(\mu,p_1,p_2,p_3)p_3^{\nu}\,.
\end{aligned} \quad (3.8)$$

If, say, particle 1 is on shell then using the subsidiary conditions the terms A_2,A_3,A_4,A_{15} do not contribute, leaving 12 amplitudes as expected. If particle 3 is also on shell then in addition A_7,A_{10},A_{16} do not contribute resulting in 9 amplitudes in the fully on-shell case. The prescription for $N^{\pm}$ leads to $(N^+,N^-)=(10,6)$, (7,5), and (5,4) for the two off-, one off-, and none off-shell cases, respectively, which agrees with the explicit amplitudes calculated.

Example 3(a): $p_1(\frac{3}{2})\to p_2(\frac{1}{2})+p_3(0)$, i.e., spin $\frac{3}{2}$ decay into spin $\frac{1}{2}$ plus spin 0. Here the off-shell spin-$\frac{3}{2}$ particle contains spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ components. Thus we need to consider the pairs $(\frac{3}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2})$. In each case $\frac{1}{2}$ is the minimum spin, so the fully off-shell case has $(2+2)\times 2\times 2=16$ amplitudes. If the spin-$\frac{3}{2}$ particle is on shell we get $(2)\times 2=4$ amplitudes. The spin $\frac{1}{2}$ on and spin $\frac{3}{2}$ off shell gives $(2+2)\times 2=8$ and both on shell give 2 amplitudes. To calculate the amplitudes explicitly we take p_1 and p_2 as independent vectors. The amplitudes can then be written as

$$A_1p_1^{\mu}+A_2p_2^{\mu}+A_3p_1^{\mu}\not p_1+A_4p_1^{\mu}\not p_2+A_5p_2^{\mu}\not p_2 + A_6p_2^{\mu}\not p_1+A_7p_1^{\mu}\not p_2\not p_1+A_8p_2^{\mu}\not p_2\not p_1\,. \quad (3.9)$$

Each of these can be multiplied also by γ_5, for a total of 16 amplitudes. If the spin-$\frac{3}{2}$ particle is on shell, then A_1, A_3, A_4, A_6, A_7, A_8 all do not contribute, leaving just A_2 and A_5 giving 4 amplitudes. For the spin-$\frac{1}{2}$ on shell, A_4, A_5, A_7, A_8 do not contribute, for a total of 8 amplitudes. Finally for both particles on shell only A_2 survives for a total of 2 amplitudes, all as predicted.

We now consider the same problem from the nonrelativistic scattering theory approach. As observed above, the vectors available for forming amplitude structures are the same in the on- and off-shell cases. The number of amplitudes is again given by the number of helicity amplitudes. There is a major difference here, however, from the relativistic-field-theory approach in that the context of nonrelativistic scattering theory does not admit the possibility of particles changing their internal quantum numbers. Thus a spin-J virtual particle in a nonrelativistic context is just spin J, not a mixture of particle and antiparticle. (These extra degrees of freedom are often included in nonrelativistic theories, but as separate contributions. An example would be the so-called virtual pair contribution to $pp\to\pi d$ or $pn\to d\gamma$.) This means that in effect in the nonrelativistic case the restrictions, which in the field theory case remove the extra degrees of freedom on shell, are present both on and off shell. Thus one concludes that in the nonrelativistic-scattering-theory approach the number of amplitudes in the off-shell case is the same as in the on-shell case (before imposing any symmetry restrictions) and is just the number of helicity amplitudes calculated in the usual on-shell fashion. The number of amplitudes of the two parities $N^{\pm}$ would then be calculated as in the relativistic on-shell case above.

To see how this works, consider the same three examples discussed above. It is natural in a nonrelativistic theory to work in the center of mass, and to construct the independent vectors from the center-of-mass momenta.

Example 1(b): $p_1(\frac{1}{2})+p_2(0)\rightarrow p_3(\frac{1}{2})+p_4(0)$. The number of helicity amplitudes in this case is $2\times2=4$. To see how this actually arises we take as independent unit vectors

$$\hat{p}=\frac{(\hat{p}_f^{\,\text{c.m.}}+\hat{p}_i^{\,\text{c.m.}})}{|\hat{p}_f^{\,\text{c.m.}}+\hat{p}_f^{\,\text{c.m.}}|},\quad \hat{k}=\frac{(\hat{p}_f^{\,\text{c.m.}}-\hat{p}_i^{\,\text{c.m.}})}{|\hat{p}_f^{\,\text{c.m.}}-\hat{p}_i^{\,\text{c.m.}}|},$$
$$\hat{n}=\frac{(\hat{p}_i^{\,\text{c.m.}}\times\hat{p}_f^{\,\text{c.m.}})\,|}{|\hat{p}_i^{\,\text{c.m.}}\times\hat{p}_f^{\,\text{c.m.}}|}\,.$$

The most general amplitude constructed from these vectors and the spin operator $\vec{\sigma}$ is then

$$A_1 1+A_2\vec{\sigma}\cdot\vec{\mathrm{n}}+A_3\vec{\sigma}\cdot\vec{\mathrm{p}}+A_4\vec{\sigma}\cdot\vec{\mathrm{k}}\,. \qquad (3.10)$$

This contains four terms with $N^{\pm}=2$ as expected. It holds on or off shell, the only difference being that off shell the A's are functions of the additional scalar variable.

Example 2(b): $p_1(1)+p_2(0)\rightarrow p_3(1)+p_4(0)$. Here, the number of helicity amplitudes is $3\times3=9$. The amplitude must be a tensor, contracted with two spin vectors $\vec{\epsilon}_1$ and $\vec{\epsilon}_3$. Using the same vectors as in the previous example and multiplying the $\vec{\epsilon}$'s in so as to avoid vector indices in the notation we find the following 13 simple structures which may be expected to enter the amplitude:

$$\vec{\epsilon}_1\cdot(\vec{\mathrm{p}},\vec{\mathrm{k}},\vec{\mathrm{n}})\otimes\vec{\epsilon}_3\cdot(\vec{\mathrm{p}},\vec{\mathrm{k}},\vec{\mathrm{n}}),\quad \vec{\epsilon}_1\cdot\vec{\epsilon}_3\,,$$
$$\vec{\epsilon}_1\times\vec{\epsilon}_3\cdot(\vec{\mathrm{p}},\vec{\mathrm{k}},\vec{\mathrm{n}}), \qquad (3.11)$$

where the notation means, for example, $\vec{\epsilon}_1$ dotted with any of the vectors in parentheses times $\vec{\epsilon}_3$ dotted also with any of the vectors in parentheses. As in the relativistic case there are too many simple structures, and hence they must not all be independent. Here it is clear, however, since $\vec{\mathrm{p}}$, $\vec{\mathrm{k}}$, and $\vec{\mathrm{n}}$ are mutually orthogonal unit vectors, that the nine pairs span the full space and hence that the $\vec{\epsilon}_1\cdot\vec{\epsilon}_2$ and $\vec{\epsilon}_1\times\vec{\epsilon}_3\cdot(\vec{\mathrm{p}},\vec{\mathrm{k}},\vec{\mathrm{n}})$ terms are not independent. (For example, $\vec{\epsilon}_1\cdot\vec{\epsilon}_3=\vec{\epsilon}_1\cdot\vec{\mathrm{p}}\,\vec{\epsilon}_3\cdot\vec{\mathrm{p}}+\vec{\epsilon}_1\cdot\vec{\mathrm{k}}\,\vec{\epsilon}_3\cdot\vec{\mathrm{k}}+\vec{\epsilon}_1\cdot\vec{\mathrm{n}}\,\vec{\epsilon}_3\cdot\vec{\mathrm{n}}$.) Therefore we can write the most general amplitude on or off shell as

$$A_1\vec{\epsilon}_1\cdot\vec{\mathrm{p}}\,\vec{\epsilon}_3\cdot\vec{\mathrm{p}}+A_2\vec{\epsilon}_1\cdot\vec{\mathrm{k}}\,\vec{\epsilon}_3\cdot\vec{\mathrm{k}}+A_3\vec{\epsilon}_1\cdot\vec{\mathrm{n}}\,\vec{\epsilon}_3\cdot\vec{\mathrm{n}}+A_4\vec{\epsilon}_1\cdot\vec{\mathrm{p}}\,\vec{\epsilon}_3\cdot\vec{\mathrm{k}}+A_5\vec{\epsilon}_1\cdot\vec{\mathrm{k}}\,\vec{\epsilon}_3\cdot\vec{\mathrm{p}}$$
$$+A_6\vec{\epsilon}_1\cdot\vec{\mathrm{p}}\,\vec{\epsilon}_3\cdot\vec{\mathrm{n}}+A_7\vec{\epsilon}_1\cdot\vec{\mathrm{n}}\,\vec{\epsilon}_3\cdot\vec{\mathrm{p}}+A_8\vec{\epsilon}_1\cdot\vec{\mathrm{k}}\,\vec{\epsilon}_3\cdot\vec{\mathrm{n}}+A_9\vec{\epsilon}_1\cdot\vec{\mathrm{n}}\,\vec{\epsilon}_3\cdot\vec{\mathrm{k}} \qquad (3.12)$$

which has the required 9 amplitudes with $N^{+}=5$ (terms A_1-A_5) and $N^{-}=4$ (terms A_6-A_9).

Example 3(b): $p_1(\frac{3}{2})\rightarrow p_2(\frac{1}{2})+p_3(0)$. Here the number of helicity amplitudes is just 2. There is just one vector available in the center of mass, say $\vec{\mathrm{q}}$, the relative momentum of particles 2 and 3. The amplitude must be a vector, contracted with the spin-one-vector $\vec{\epsilon}$ which is part of the spin-$\frac{3}{2}$ wave function. Thus we find

$$A_1\vec{\sigma}\cdot\vec{\epsilon}+A_2\vec{\mathrm{q}}\cdot\vec{\epsilon} \qquad (3.13)$$

which has $N=2$, $N^{\pm}=1$ as predicted.

Thus to summarize this section we have found (assuming always no constraints of symmetries) that in the relativistic field theory approach there are always additional amplitudes in the off-shell situation. These originate in the additional degrees of freedom in spin and particle-antiparticle nature which are allowed for a virtual particle in a relativistic theory. They vanish on shell by virtue of the subsidiary conditions imposed on on-shell wave functions. On the other hand, in the nonrelativistic-scattering-theory approach the number of amplitudes and their structure is the same in the off- and on-shell cases, since these additional degrees of freedom are precluded by the theoretical context, or more precisely, if put in, are put in by hand as separate contributions.

C. Restrictions arising from symmetries

So far amplitudes have been constructed without regard to constraints imposed by symmetry principles such as parity and time reversal. In this section we want to discuss what effect these constraints have and in particular any differences between the relativistic and nonrelativistic cases or between the on- and off-shell cases.

Parity. In the discussion above the amplitudes were always separated into N^{+} and N^{-} amplitudes of different parities. If the scalar variables on which the A_i depend are all scalars rather than pseudoscalars, then the A_i will be scalars and parity conservation will eliminate either N^{+} or N^{-} amplitudes depending on the intrinsic parities of the particles involved. In more complicated cases when there are enough vectors to form pseudoscalar variables then the A_i will have both scalar and pseudoscalar parts. Then parity conservation will not reduce the number of amplitudes but will enforce particular transformation properties (either scalar or pseudoscalar) for the A_i. These restrictions are essentially the same for relativistic or nonrelativistic approaches and for on- or off-shell cases, though the resulting number of amplitudes will vary, since the unconstrained number also varies.

Time reversal. In the usual on-shell case and for elastic scattering time-reversal invariance of the interaction imposes certain constraints on the scattering amplitude which reduce the number of independent amplitudes. It is appropriate then to ask if there are analogous constraints on the off-shell amplitudes arising from time-reversal invariance.

We suppose throughout that the physical system is invariant under the time-reversal operation $\mathcal{T}$, that is, the Hamiltonian H satisfies $\mathcal{T}H\mathcal{T}^{-1}=H$. As a consequence[7,8] in either field-theory or nonrelativistic-scattering-theory approaches the T-matrix operator T or the S-matrix operator S satisfies $\mathcal{T}T\mathcal{T}^{-1}=T^{\dagger}$ or $\mathcal{T}S\mathcal{T}^{-1}=S^{\dagger}$, respectively. Now consider the process $p_1+p_2\to p_3+p_4$ with p_3 off shell and take matrix elements of, say, the T-matrix relation in a helicity basis $|p_i,\lambda_i\rangle$ with λ_i the helicity. This gives, as the consequence of time-reversal invariance of the interaction (see, e.g., Refs. 4, 7, or 8),

$$\langle p_3,\lambda_3,p_4,\lambda_4\,|\,T\,|\,p_1,\lambda_1,p_2,\lambda_2\rangle = \eta_T\langle -p_1,\lambda_1,-p_2,\lambda_2\,|\,T\,|\,-p_3,\lambda_3,-p_4,\lambda_4\rangle\ , \tag{3.14}$$

where we may set the phase $\eta_T=1$. Thus time reversal changes the signs of the momenta and spins (but not helicities) in the matrix elements and interchanges initial and final states. (If parity is also conserved one can reverse the signs of momenta again and also reverse the helicities.)

For elastic processes on shell the initial and final states are the same so that the interchange of initial and final states is unimportant and thus there are relations among various matrix elements which in effect reduce the number of independent amplitudes. In particular time reversal in this case requires that the T matrix be symmetric.

For an off-shell process, however, the interchange of initial and final states is crucial since the two are not the same, i.e., incoming on-shell scattering to outgoing off shell is not the same as incoming off-shell scattering to outgoing on shell. Thus time-reversal invariance of the interaction relates matrix elements of essentially different processes and so does not reduce the number of off-shell amplitudes. It will, however, force some of the amplitudes to be proportional to something like p^2-m^2 which vanishes on shell since on shell there will be relations which will eliminate some of the amplitudes.

This result is quite analogous to that for on-shell reactions (i.e., nonelastic processes) where the time-reversal relation relates matrix elements for one process to that for a different process. This is as expected, at least in the relativistic-field-theory approach, since there an off-shell particle is essentially like an on-shell particle with a new mass $m^*\neq m$, and so an off-shell amplitude can be considered as an on-shell reaction amplitude involving a change from a particle of mass m to one of mass m^*.

To make these ideas more explicit consider the example of spin-0, spin-$\frac{1}{2}$ elastic scattering which was discussed previously. In the nonrelativistic-scattering-theory approach the most general T matrix, corresponding to the left-hand side (LHS) of Eq. (3.14) above, can be written as

$$\begin{aligned}\text{LHS}=&A_1(\vec{p}_i^{\,2},\vec{p}_f^{\,2},\cos\theta)1+A_2(\vec{p}_i^{\,2},\vec{p}_f^{\,2},\cos\theta)\sigma\cdot\hat{n}\\&+A_3(\vec{p}_i^{\,2},\vec{p}_f^{\,2},\cos\theta)\sigma\cdot\hat{p}\\&+A_4(\vec{p}_i^{\,2},\vec{p}_f^{\,2},\cos\theta)\sigma\cdot\hat{k}\ ,\end{aligned} \tag{3.15}$$

where we have used the unit vectors $\hat{p}$, $\hat{k}$, $\hat{n}$ defined above and have taken $\vec{p}_i\equiv\vec{p}_i^{\,\text{c.m.}}$, $\vec{p}_f\equiv\vec{p}_f^{\,\text{c.m.}}$ and $\cos\theta=\hat{p}_i\cdot\hat{p}_f$ and have suppressed helicity or spin indices on the $\vec{\sigma}$'s. The right-hand side (RHS) of the equation can be obtained from the above by reversing initial and final states and changing signs of momentum and spin operators and is

$$\begin{aligned}\text{RHS}=&A_1(\vec{p}_f^{\,2},\vec{p}_i^{\,2},\cos\theta)1+A_2(\vec{p}_f^{\,2},\vec{p}_i^{\,2},\cos\theta)\sigma\cdot\hat{n}\\&+A_3(\vec{p}_f^{\,2},\vec{p}_i^{\,2},\cos\theta)\sigma\cdot\hat{p}\\&-A_4(\vec{p}_f^{\,2},\vec{p}_i^{\,2},\cos\theta)\sigma\cdot\hat{k}\ .\end{aligned} \tag{3.16}$$

Time-reversal invariance thus implies for the off-shell process that

$$A_i(\vec{p}_i^{\,2},\vec{p}_f^{\,2},\cos\theta)=\pm A_i(\vec{p}_f^{\,2},\vec{p}_i^{\,2},\cos\theta)$$

with the plus sign for $i=1,2,3$ and the minus sign for $i=4$. If we define

$$\begin{aligned}A_i^{(\pm)}&\equiv A_i^{(\pm)}(\vec{p}_i^{\,2},\vec{p}_f^{\,2}\cos\theta)\\&=\tfrac{1}{2}[A_i(\vec{p}_i^{\,2},\vec{p}_f^{\,2},\cos\theta)\pm A_i(\vec{p}_f^{\,2},\vec{p}_i^{\,2},\cos\theta)]\end{aligned} \tag{3.17}$$

we can write the most general off-shell amplitude which explicitly satisfies time-reversal invariance as

$$A_1^{(+)}1+A_2^{(+)}\sigma\cdot\hat{n}+A_3^{(+)}\sigma\cdot\hat{p}+A_4^{(-)}\sigma\cdot\hat{k}\ . \tag{3.18}$$

On shell $\vec{p}_i^{\,2}=\vec{p}_f^{\,2}$ and $A_4^{(-)}=0$ so that on-shell time-reversal invariance reduces the number of amplitudes from four to three. In this particular example parity conservation eliminates $A_3^{(+)}$ and $A_4^{(-)}$ terms both on and off shell so one gets, for example, for π-nucleon scattering just the usual two amplitudes on or off shell, but in general parity and time reversal would eliminate different sets of amplitudes.

One additional restriction on the amplitude can be obtained if the interaction being considered is sufficiently weak to be treated in first order, i.e., in Born approximation. In that case T is Hermitian. For a Hermitian T operator there is the additional constraint

$$\langle p_3,\lambda_3,p_4,\lambda_4\,|\,T\,|\,p_1,\lambda_1,p_2,\lambda_2\rangle = \langle -p_3,\lambda_3,-p_4,\lambda_4\,|\,T\,|\,-p_1,\lambda_1,-p_2,\lambda_2\rangle^* \tag{3.19}$$

which now relates the T matrix to itself even in the off-shell or reaction cases and thus puts constraints on the independent amplitudes. In the example discussed above this constraint requires that $A_1^{(+)}$, $A_3^{(+)}$, $A_4^{(-)}$ be real and $A_2^{(+)}$ pure imaginary. In the general case then deviations from this must come from second- or higher-order interactions.

Thus the situation as regards time-reversal constraints and matrix elements can be summarized as follows. In the general case of elastic scattering with one leg off shell, just as for on-shell reactions other than elastic scattering, time reversal relates matrix elements of different reactions and thus does not reduce the number of independent amplitudes. Constraints which reduce the number of amplitudes are obtained in the on-shell limit and additional constraints which determine the phases of the amplitudes are obtained in the approximation of a Hermitian T matrix as is obtained in Born approximation. All of these considerations hold both in the relativistic-field-theory and nonrelativistic-scattering-theory approaches.

A somewhat related question deals with the form of the potential and the constraints on that potential imposed by symmetries, particularly time reversal. Since the potential is part of the Hamiltonian and is Hermitian we have $\mathcal{T}V\mathcal{T}^{-1}=V$. The off- or on-shell matrix elements of the potential are constrained just as for the T matrix by both the general condition and the one appropriate for Hermitian operators as given above. Thus in general there can be terms in the potential which are present only in off-shell matrix elements.

To understand this it is easiest to look at our example of spin-0—spin-$\frac{1}{2}$ scattering. The most general potential which satisfies time-reversal invariance can be written, in exact analogy with the most general T matrix, as

$$V_1^{(+)}1+iV_2^{(+)}\vec{\sigma}\cdot\hat{n}+V_3^{(+)}\vec{\sigma}\cdot\hat{p}+V_4^{(-)}\vec{\sigma}\cdot\hat{k}\ , \qquad (3.20)$$

where the V_i are real functions of p_i and p_f with V_1, V_2, V_3 symmetric and V_4 antisymmetric in the interchange $p_i \leftrightarrow p_f$. Thus V_4 vanishes on shell and the usual nonrelativistic potential would consist just of the V_1, V_2, and V_3 terms.

To first order the usual nonrelativistic potential will contribute only to the A_1, A_2, and A_3 terms in the T matrix, i.e., to first order one cannot generate the extra term in the off-shell T matrix which vanishes on shell from the part of the potential which survives on shell. However, to second order, the V_1, V_2, and V_3 terms are sufficient in general to produce a contribution to the $A_4^{(-)}$ term of the T matrix. This can be easily seen by straightforward, though tedious, calculation, in particular in the special case when the V_i depend only on the magnitudes of p_i and p_f. An analogous situation was well known in early attempts to look for time reversal violating correlations in reactions.[9] There a second-order process, e.g., Coulomb corrections to a weak interaction or other final-state interaction, can generate a nonzero coefficient of a correlation which in first order vanishes because of time-reversal invariance. This occurs even when both interactions (and the overall process) are time-reversal invariant and satisfy, for example, detailed balance. On the other hand, if the V_4 term is nonzero then there will be a first-order contribution to A_4 off shell. There will also be second-order off-shell contributions to A_4 *and* to A_1, A_2, A_3 from V_4.

The situation illustrates an ambiguity in the usual nonrelativistic considerations of off-shell amplitudes. Given a potential one can calculate the off-shell T matrix and in general one will get new, but calculable terms as well as variation in the amplitudes present on shell. However, in general there can be additional contributions to the potential which vanish on shell and thus which cannot be determined from on-shell processes alone. These contributions affect the off-shell behavior of the usual on-shell amplitudes as well as the new amplitudes which are present off shell. In particular situations one may be able to rule out such additional contributions to the potential for physical or simplicity reasons. Thus, for example, the required antisymmetry of V_4 under the interchange $p_i \leftrightarrow p_f$ means that this potential term cannot depend only on $|p_i-p_f|$ and thus is nonlocal. Also it cannot take the usual separable form which is symmetric in $p_i \leftrightarrow p_f$.

IV. HOW TO MEASURE OFF-SHELL AMPLITUDES?

In this section we will discuss the ways in which information about off-shell amplitudes can be obtained from actual laboratory experiments. Such experiments, of course, always deal with some on-shell process, so that off-shell processes cannot be directly measured. They can, however, be embedded into an on-shell process which is larger than the process we want to study off shell, and then this larger process is measured on shell. The question then is how we can conclude something about the off-shell process by looking, on shell, at the larger process.

There are many examples of this situation. The study of nucleon-nucleon bremsstrahlung[10] has been done with the aim of learning about off-shell elastic nucleon-nucleon scattering. The study of pion production in pion-nucleon or nucleon-nucleon scattering involves off-shell elements of pion-nucleon or nucleon-nucleon elastic scattering.[11] Radiative capture (into a d) in neutron-proton collisions[12] contains the off-shell union of a neutron and a proton into a deuteron.

The procedure most often used in the past was to use some model for the off-shell contribution of the embedded reaction and join this model with the treatment of the rest of the larger reaction to make predictions for the measurements of that larger reaction. There is, however, considerable advantage in reversing this procedure, and trying to determine, in a phenomenological way, the off-shell amplitudes of the embedded reaction from the measured observables for the larger reaction. This is the procedure that we will discuss in particular in this section.

The conclusions can be summarized as follows.

(1) It is not possible to make a phenomenological determination of the off-shell amplitudes of the embedded process from the measured observables of the larger process in a way which would be model independent. In particular, what enters is the model we have to assume for the process of embedding itself, and the relationship between the observables of the larger process and the off-shell amplitudes of the embedded process will depend on the details of this model. For example, if we assume that the embedded process is described by a purely local potential we are specifying in part the model to be used for the embedding, e.g., such an assumption excludes particle emission from the interior of the potential. In the usual nonrelativistic potential approach the model assumed is normally some variation of the two-potential formula, i.e., the interaction which puts the off-shell leg back on shell is taken in first order only. Hence the off-shell amplitude separates out and appears as a factor of the overall amplitude or in the worst case appears only twice, in a "double scattering term," in such a way that it can be clearly separated from the second interaction which brings the process back on shell.

(2) If we assume a process of embedding which satisfies a simple factorizability, that is, if we can write the M matrix of the larger process as a product of the off-shell M matrix of the embedded process times an off-shell M matrix of the remainder of the larger process, which is assumed known, then, if we carry out a sufficient number of experiments to determine completely the amplitudes of the larger process, we can also determine completely the off-shell amplitudes of the embedded process, since the latter are well defined though complicated functions of the former.

(3) If the remainder of the large process is a photon emission, that is, if we consider bremsstrahlung processes, then gauge invariance and the usual soft photon theorems indicate that all off-shell effects appear, in general, only in the third-order term in the photon momentum k in the observables.[13] With a judicious choice of the observables to be measured we can change this to the second order by suppressing the leading term, but in any case the off-shell effects will be down^2 from the on-shell effects by an order in k.

(4) If the remainder involves a weak interaction instead of an electromagnetic one as in (3), then conserved vector current will have the same effect on the relative suppression of the off-shell contributions as gauge invariance had in the electromagnetic case.

(5) If the remainder involves some strong interaction, for example, pion emission, then such suppression mechanisms are not in effect. In that case, however, the model for the process of embedding (namely, the off-shell process followed by one-pion emission) is likely to be less reliable.

These conclusions as well as the results of the previous sections will now be illustrated using a spin-$\frac{1}{2}$—spin-zero scattering ($\frac{1}{2}+0\rightarrow\frac{1}{2}+0$) with the final spin-$\frac{1}{2}$ particle off shell which will be embedded in a larger process by tacking onto the final spin-$\frac{1}{2}$ leg an emission of a scalar particle.

First we will write down the terms, in the relativistic formalism, which the off-shell process will have in the most general case when both spin-$\frac{1}{2}$ particles are off shell. For later use we will write the terms in such a form that they have definite time-reversal properties and their behavior on shell is easily traceable. There will be 16 terms, which are simply constructed of linear combinations of the amplitude originally written down in Eq. (3.6), and which are given as

$$T\sim a_1^{(+)}1+a_2^{(+)}\not Q+a_3^{(+)}[\Lambda(p_1)+\Lambda(p_3)]+a_4^{(-)}[\Lambda(p_1)-\Lambda(p_3)]+a_5^{(+)}[\Lambda(p_3)\not Q+\not Q\Lambda(p_1)]$$
$$+a_6^{(-)}[\Lambda(p_3)\not Q-\not Q\Lambda(p_1)]+a_7^{(+)}\Lambda(p_3)\Lambda(p_1)+a_8^{(+)}\Lambda(p_3)\not Q\Lambda(p_1)$$
$$+\gamma_5\times\text{similar terms with coefficients } b_i\ . \tag{4.1}$$

Here we have taken the momenta to be $p_1(1/2)+p_2(0)=p_3(1/2)+p_4(0)$ and defined $Q\equiv p_2+p_4$, $\Lambda(p)\equiv\not p-m$. The coefficients a_i are functions of, say, the scalar variables $(p_1+p_2)^2$, $(p_1-p_3)^2$, $p_1{}^2,p_3{}^2$, and so can be written as $a_i(p_3,p_1)$. On shell this would be taken between free spinors $\bar u(p_3)$ and $u(p_1)$. These spinors satisfy $\Lambda(p_1)u(p_1)=\bar u(p_3)\Lambda(p_3)=0$, so for an on-shell particle $\Lambda(p)\rightarrow 0$ and the term does not contribute.

The amplitude has also been written so that it is invariant under time reversal. The time-reversal condition, as used earlier, is $\mathscr{T}T\mathscr{T}^{-1}=T^\dagger$. The operator $\mathscr{T}$ takes complex conjugate of all c numbers, changes $t\rightarrow -t$ and multiplies Dirac operators by $i\gamma^1\gamma^3$, using the notation of Ref. 14. Thus

$$\mathscr{T}\gamma^\mu\mathscr{T}^{-1}=(i\gamma^1\gamma^3)\gamma^{\mu *}(-i\gamma^3\gamma^1)=\gamma^0 \text{ or } -\vec\gamma\ ,$$
$$\mathscr{T}i\partial_\mu\mathscr{T}^{-1}=i\partial_0 \text{ or } -i\vec\partial$$

and hence $\mathscr{T}\not p\mathscr{T}^{-1}=\not p$ so that $\mathscr{T}T\mathscr{T}^{-1}$ is equal to T with $a_i\rightarrow a_i^*$. The additional adjoint in $T=(\mathscr{T}T\mathscr{T}^{-1})^\dagger$ then reverses the order of noncommuting operators, changes $a_i^*\rightarrow a_i$ and in effect interchanges initial and final momenta. Thus under the full transformation the coefficients of a_4 and a_6 change sign while the others do not. Hence $a_4^{(-)}$ and $a_6^{(-)}$ must be antisymmetric under the interchange $p_1\leftrightarrow p_3$, $p_2\leftrightarrow p_4$ whereas $a_i^{(+)}$ must be symmetric for invariance under time reversal. The transformation properties of the b_i's can be obtained in a similar way.

We will now embed this process into a physical (on-shell) one in which the outgoing fermion line in the above process (the p_3 line) emits a scalar meson of mass m_0 with coupling g_0 and thereby gets itself back on shell. We will assume that the off-shell process is parity conserving, so that all of the b_i terms vanish. With p_1 on shell $\Lambda(p_1)\rightarrow 0$ so that there are only four independent terms surviving corresponding to $a_1^{(+)}$, $a_2^{(+)}$, $a_3'\equiv a_3^{(+)}-a_4^{(-)}$, and $a_5'\equiv a_5^{(+)}+a_6^{(-)}$. Let us reserve p_3 for the momentum of the outgoing (on-shell) fermion in the larger, actually observed process and replace p_3 in the above off-shell amplitude by $p_3^*=p_3+k$, where k is the momentum of the outgoing scalar boson that is tacked on to the outgoing fermion line. We have then for the M matrix of the larger, actually observed reaction

$$M\sim\bar u(p_3)g_0\frac{(\not p_3^*+m)}{p_3^{*2}-m^2}[a_1^{(+)}+a_2^{(+)}\not Q+\Lambda(p_3^*)(a_3'+a_5'\not Q)]u(p_1)$$
$$=g_0\bar u(p_3)\left[\left[\frac{2m}{2k\cdot p_3+m_0{}^2}a_1^{(+)}+a_3'\right]+\left[\frac{2m}{2k\cdot p_3+m_0{}^2}a_2^{(+)}+a_5'\right]\not Q+\frac{\not k}{2k\cdot p_3+m_0{}^2}\left[a_1^{(+)}+a_2^{(+)}\not Q\right]\right]u(p_1)\ . \tag{4.2}$$

Since the larger reaction is a five-particle reaction, it will have four independent amplitudes even if time-reversal invariance and parity conservation hold. Hence we have four measurable amplitudes at our disposal which depend on four off-shell amplitudes as given in Eq. (4.2). Thus measuring a sufficient number of observables to yield the four measurable amplitudes will also yield, in principle, the four off-shell amplitudes.

In arriving at the above result, we have made a number of assumptions.

(1) The embedded process is embedded into the larger process in only one way, namely, the one we considered.

(2) The embedding process is such that we can consider the M matrix of the larger process as a product of the off-shell M matrix of the embedded process and the propagator of the external final fermion line (the vertex function in this case being 1). This restriction excludes, for example, processes in which the extra scalar boson is emitted from somewhere inside the embedded process rather than from one of its external lines.

(3) We definitely know the M matrix of the remaining part of the larger process (in this case, the emission of the extra scalar boson).

These assumptions make the extraction of the off-shell amplitudes from the observables for the larger process definitely model dependent.

As another example, let us consider the case when the extra emitted boson has spin 1 rather than 0 and is massless, i.e., we are considering the prototype of the bremsstrahlung process. In that case the M matrix is given by an expression like the first part of Eq. (4.2) with the coupling $g_0 \to \not{\epsilon}$ where ϵ_μ is the polarization vector of the spin-one particle. After some algebra the M matrix reduces to

$$\bar{u}(p_3)\left[\frac{2\epsilon\cdot p_3+\not{\epsilon}\not{k}}{2k\cdot p_3}(a_1^{(+)}+a_2^{(+)}\not{Q}) + \not{\epsilon}(a_3'+a_5'\not{Q})\right]u(p_1)\ . \tag{4.3}$$

In this case the larger process has eight amplitudes, since the "photon" has two polarization states also, so again in principle a complete measurement of the larger process gives within the assumed model a determination of the off-shell amplitudes. For a real-photon process additional terms involving radiation from other charged legs would be necessary for gauge invariance, so that the formulas would be much more complicated. The general result would be the same, however.

The above procedure rests on carrying out a sufficient number of experiments on the larger reaction so that all of its amplitudes can be observed. Such a determination of the amplitude need not be carried out in terms of the particular set of amplitudes which appear in the second half of Eq. (4.2) or in Eq. (4.3). Once the amplitudes are known in terms of one basis set of amplitudes, any other set of amplitudes can be obtained from that by a simple mathematical procedure. Thus the actual determination of the amplitudes in the larger reaction should be done in terms of a set of amplitudes which make such a determination the easiest from an experimental point of view. The amplitudes suitable for this purpose are an optimal set of amplitudes in the sense of Ref. 15. Which optimal set is to be used depends on many factors, including the repertoire of the experimentalist who does the actual measurements. This problem has been amply discussed in connection with various reactions in Refs. 15 and 16.

To carry out an experimental program to determine completely the whole set of amplitudes is, however, in most cases a substantial undertaking. The question arises, therefore, whether one could get at least some partial useful information about the off-shell amplitudes from just a few measurements of observables, from a set that is smaller than what is needed to determine all amplitudes completely. We see from the above considerations that one cannot give a model-independent answer to this question. For any particular way of embedding, one can calculate the relationship between any particular observable and the off-shell amplitudes, using the above connections together with the connections between the amplitudes and the observables of an on-shell process, something that has been discussed in Refs. 15 and 16. In general, each observable will contain a complicated mixture of off-shell amplitudes, since these amplitudes are not optimal in the sense of Ref. 15 and hence have no simple relationships to observables.

As discussed earlier, there are two types of off-shell amplitudes, those which are present in the off-shell limit and simply change numerically off shell, and those which vanish in the on-shell limit and hence are not included in usual on-shell descriptions. We turn now to a study of the relative possibilities of getting useful information from these two types. In our previous example $a_1^{(+)}$ and $a_2^{(+)}$ belong to the first group which are present on shell and a_3' and a_5' belong to the second which vanish in the on-shell limit. In order to get some information on off-shell behavior, it may be preferable to deal with the second group, since if we find an observable that depends on those amplitudes linearly, the experiment detecting them will be a "null experiment" in that on shell that observable vanishes. Hence to measure such off-shell effects we measure a deviation from zero rather than a change in a quantity that has a finite on-shell value. Whether this is really a practical advantage depends, of course, on how fast the observable changes off shell, e.g., a large deviation from a finite value could still be easier to measure than a small deviation from zero.

We see, however, from Eqs. (4.2) and (4.3) that the particular combinations appearing make our task difficult. In Eq. (4.2), a_3' and a_5' appear only in the first two terms and there only together with $a_1^{(+)}$ and $a_2^{(+)}$ which in turn are multiplied by a factor containing a pole. For sufficiently small $m_0{}^2$ this pole and hence the $a_1^{(+)}$ and $a_2^{(+)}$ terms may dominate near the on-shell point making it hard to extract a_3' and a_5'. This is in fact a fairly general situation since the terms which vanish on shell will always have an inverse propagator factor $\Lambda(p)$ which will cancel the pole coming from the propagator. For the spin-1 process of Eq. (4.3) the same situation prevails though here the two groups of amplitudes appear in different terms of the M matrix and hence one might hope for a more distinguishable effect from the a_3' and a_5' amplitudes.

For a real bremsstrahlung process with all terms included so that the result is gauge invariant one can make a somewhat stronger statement, as has been previously emphasized.[2] In such a situation one can generalize[17] the Burnett-Kroll theorem[18] to spin observables and thus show that a spin correlation in a bremsstrahlung process can be given through $O(1/k)$ and $O(k^0)$ by a simple operator acting on the same spin correlation in the elastic on-shell process.[17] Since the first term of this operator is just a constant factor, a spin correlation which vanishes in the on-shell elastic process will vanish in $O(1/k)$ for the bremsstrahlung process. The second term, however, does not vanish necessarily, since it depends on derivatives of the correlation. The off-shell terms enter only in the third term which is of $O(k)$. Thus for such a bremsstrahlung process one can choose a particular spin observable which vanishes on shell and thus suppress the leading term and ensure that the interesting off-shell contributions such as those arising from amplitudes like a_3' and a_5' are suppressed by only one, rather than two, powers of k. For bremsstrahlung processes at least this seems to be about the best one can do in terms of choosing a particular observable so as to isolate or emphasize the contribution of the off-shell amplitudes.

V. SUMMARY OF RESULTS

It might be helpful to the reader if, after this somewhat extensive discussion, we summarized the principal results of this paper.

In this paper we have discussed the spin structure of off-shell processes and the related question of how such a structure can be determined from actual laboratory experiments. The first task was to analyze the meaning of "off shell." In general, "off shell" refers to a process in which energy and momentum do not satisfy the usual relations which hold for a reaction with free particles. In relativistic field theory such a situation is often referred to as "off-mass shell." In that case the four-momenta of a four-particle process satisfy four-momentum conservation, but the relationship between the momentum and the energy of at least one particle involves not the free mass of that particle but an "effective" mass. Both momentum and energy are conserved, but the magnitude of the initial center-of-mass momentum is not equal to the magnitude of the final center-of-mass momentum. In contrast, in nonrelativistic scattering theory, we talk about "off energy shell." There the relationship between energy and momentum is defined using the real free mass of each particle, but the energy thus defined is not conserved, though three-momentum is conserved. As in the previous case, the magnitude of the initial center-of-mass momentum is not equal to the magnitude of the final center-of-mass momentum.

In view of these definitions, the off-shell M matrices will differ from on-shell matrices in that each amplitude will depend on additional scalar variables, there will be additional amplitudes, and the amplitudes will show different simplifications under symmetries. In particular, in both the relativistic-field-theory approach and the nonrelativistic-scattering-theory approach the amplitudes will depend on one extra scalar variable for each off-shell particle. The number of additional amplitudes depends on the approach. In the relativistic-field-theory approach, if the particles involved are composite, the spin structure becomes somewhat ambiguous. If, however, the particles can be considered "elementary," the number of independent amplitudes can be calculated as follows. (1) determine the spin content (that is, S, $S-1$, $S-2$, etc.) for each off-shell particle in the reaction; (2) calculate the number of helicity amplitudes for each combination of spin components and add these numbers; (3) for fermions multiply the result in (2) by a factor of 2 for each off-shell particle. These additional amplitudes vanish on shell due to the subsidiary conditions. In the nonrelativistic approach, in contrast, the number of amplitudes off shell is the same as the number of amplitudes on shell, as long as there are no additional constraints from symmetries other than Lorentz or rotation invariance.

Finally, consider the restrictions arising from the imposition of additional symmetries. The constraints of parity conservation are imposed in the same way on off-shell amplitudes as on on-shell amplitudes. The constraints of time-reversal invariance, however, are qualitatively different for on-shell and off-shell processes. Both in the relativistic and nonrelativistic approaches, the time-reversed off-shell process is different from the original off-shell process (except for a tiny class of "pathological" cases). Hence time reversal imposes no constraints at all on the original off-shell process and thus does not reduce the number of amplitudes. On shell, when the time-reversed process is the same as the original process (which is the case for a large class of reactions), there are constraints on the M matrix which reduce the number of amplitudes. In addition, if one uses the approximation of a Hermitian T matrix, as, for example, in the Born approximation, one gets yet more constraints for the on-shell case.

After establishing the structure of the amplitudes for off-shell processes, the measurement of these amplitudes is discussed in the context of real (on-shell) laboratory experiments which "embed" the off-shell processes. We find the following. (1) It is not possible to make a phenomenological determination of the off-shell amplitudes embedded in a larger process without making some model-dependent assumptions. (2) If one assumes a factorizable embedding process and measures a sufficient number of polarization quantities in the larger process, one can completely determine the amplitudes of the embedded off-shell process. (3) If the remainder of the large process is soft photon emission, then the off-shell effects appear, in general, only in the third-order term in the photon momentum, though for a judicious choice of experimental observables this can be changed to the second order, but not beyond that. (4) The situation is similar when the remainder of the large process is a weak interaction. (5) If the remainder is a strong interaction, the above suppression is not in effect, but in that case our knowledge of the embedding process is less reliable.

The above results are not only established in the paper but also illustrated in a sizable assortment of examples.

ACKNOWLEDGMENTS

This work was supported in part (H.W.F.) by a grant from the Natural Sciences and Engineering Research Council of Canada and in part (G.R.G. and M.J.M.) by grants from the U.S. Department of Energy.

APPENDIX

The reduction of Rarita-Schwinger[6] fermion wave functions under various constraint equations is herein explained.

The wave function $\Phi_{\alpha,\mu_1,\ldots,\mu_{S_0}}(p)$ has Dirac index α ($=1,\ldots,4$) and S_0 four-vector indices μ_i ($=0,\ldots,3$). It is assumed to be symmetrized in the S_0 four-vector indices, thus forming an S_0th-rank symmetric four-tensor coupled to a Dirac spinor, and therefore has

$$4\times[(S_0+3)(S_0+2)(S_0+1)/6]$$

components. The traceless conditions eliminate

$$4\times[(S_0+1)S_0(S_0-1)/6]$$

components leaving $4(S_0+1)^2$ components, corresponding to the reducible representation

$$[(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})]\otimes\left[\frac{S_0}{2},\frac{S_0}{2}\right].$$

To form an irreducible representation the auxiliary conditions of Eq. (3.2) are imposed. The set of conditions,

$$(\gamma^{\mu_1})_{\alpha\beta}\Phi_{\beta,\mu_1,\ldots,\mu_{S_0}}=0\ , \tag{A1}$$

is equivalent to

$$\begin{aligned}\xi_{a;0,\mu_2,\ldots,\mu_{S_0}}&=+(\sigma_k)_{ab}\eta_{b;k,\mu_2,\ldots,\mu_{S_0}}\ ,\\ \eta_{a;0,\mu_2,\ldots,\mu_{S_0}}&=+(\sigma_k)_{ab}\xi_{b;k,\mu_2,\ldots,\mu_{S_0}}\ ,\end{aligned} \tag{A2}$$

where

$$\Phi_{\mu_1,\ldots,\mu_{S_0}}=\begin{bmatrix}\xi_{\mu_1,\ldots,\mu_{S_0}}\\ \eta_{\mu_1,\ldots,\mu_{S_0}}\end{bmatrix} \tag{A3}$$

with $\Phi\ldots$, a four-spinor, decomposed into the two-spinors $\xi\ldots$ and $\eta\ldots$, and $a,b=1$ or 2. Hence the time components $\Phi_{0,\mu_2,\ldots,\mu_{S_0}}$ are eliminated. Without the traceless conditions there are

$$4\times[(S_0+2)(S_0+1)S_0/6]$$

of these time components leaving

$$4\times[(S_0+2)(S_0+1)/2]$$

independent space components; the latter is precisely the number of components in the irreducible representation

$$\left[\left[\frac{S_0+1}{2},\frac{S_0}{2}\right]\oplus\left[\frac{S_0}{2},\frac{S_0+1}{2}\right]\right].$$

Hence (A2) imposes $2S_0(S_0+1)$ conditions on the $4(S_0+1)^2$ components of the *traceless* $\Phi_{\mu_1,\ldots,\mu_{S_0}}$, as stated following Eq. (3.2). The trace conditions need not be counted separately since the symmetry of $\Phi_{\mu_1,\ldots,\mu_{S_0}}$ under interchange of four-vector indices and condition (A1) imply the trace conditions, viz.,

$$\begin{aligned}g^{\mu_1\mu_2}\Phi_{\mu_1,\mu_2,\ldots,\mu_{S_0}}&=\tfrac{1}{2}\{\gamma^{\mu_1},\gamma^{\mu_2}\}\Phi_{\mu_1,\mu_2,\ldots,\mu_{S_0}}\\ &=\tfrac{1}{2}\gamma^{\mu_1}\gamma^{\mu_2}\Phi_{\mu_2,\mu_1,\ldots,\mu_{S_0}}\\ &\quad+\tfrac{1}{2}\gamma^{\mu_2}\gamma^{\mu_1}\Phi_{\mu_1,\mu_2,\ldots,\mu_{S_0}}\\ &=0\ .\end{aligned} \tag{A4}$$

The on-shell conditions are one or the other of

$$(\not p\mp m)\Phi_{\mu_1,\ldots,\mu_{S_0}}=0\ , \tag{A5}$$

which relate the upper and lower two-spinors in the manner well known from the Dirac equation

$$\eta_{\mu_1,\ldots,\mu_{S_0}}=\frac{\vec\sigma\cdot\vec{\mathrm p}}{(p_0\pm m)}\xi_{\mu_1,\ldots,\mu_{S_0}}\ . \tag{A6}$$

Then half of the $2(S_0+2)(S_0+1)$ space components are eliminated. However, (A6) implies further restrictions on the time components, and since the latter are already dependent on space components there are further restrictions on the remaining $(S_0+2)(S_0+1)$ space components. With (A2) and (A6) used to eliminate $\eta_{\mu_1,\ldots,\mu_{S_0}}$ altogether we obtain two relations

$$\begin{aligned}\xi_{0,\mu_2,\ldots,\mu_{S_0}}&=+\sigma_k\left[\frac{\vec\sigma\cdot\vec{\mathrm p}}{p_0\pm m}\right]\xi_{k,\mu_2,\ldots,\mu_{S_0}}\ ,\\ \left[\frac{\vec\sigma\cdot\vec{\mathrm p}}{(p_0\pm m)}\right]\xi_{0,\mu_2,\ldots,\mu_{S_0}}&=+\sigma_k\xi_{k,\mu_2,\ldots,\mu_{S_0}}\ .\end{aligned} \tag{A7}$$

Eliminating the time components leaves one relation

$$(\vec\sigma\cdot\vec{\mathrm p})p_k\xi_{k,\mu_2,\ldots,\mu_{S_0}}=p_0(p_0\pm m)\sigma_k\xi_{k,\mu_2,\ldots,\mu_{S_0}}\ . \tag{A8}$$

Calling $\hat z$ the direction of $\vec{\mathrm p}$ we see that (A8) relates the up (down) z or longitudinal component $\xi_{z,\mu_2,\ldots,\mu_{S_0}}$ to a combination of down (up) transverse components $\xi_{x,\mu_2,\ldots,\mu_{S_0}}\pm i\xi_{y,\mu_2,\ldots,\mu_{S_0}}$. Since there are $(S_0+2)(S_0+1)$ space components before (A8) is imposed, there are

$$[(S_0-1)+2][(S_0-1)+1]=(S_0+1)S_0$$

longitudinal components leaving

$$(S_0+2)(S_0+1)-(S_0+1)S_0=2(S_0+1)$$

or $[2(S_0+\tfrac{1}{2})+1]$ independent transverse components. This latter is precisely the number of spin-$(S_0+\tfrac{1}{2})$ states required for the on-shell fermion as claimed. The final condition, $p^{\mu_1}\Phi_{\mu_1,\ldots,\mu_{S_0}}=0$, of Eq. (3.5) gives no additional restrictions since $p^\mu=\tfrac{1}{2}[\gamma^\mu(\not p\pm m)+(\not p\mp m)\gamma^\mu]$ and so it is not independent of the conditions (A1) and (A5).

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