PHYSICAL REVIEW D VOLUME 29, NUMBER 10 15 MAY 1984

Brief Reports

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Aharonov-Bohm effect as a scattering event

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(Received 2 November 1983)

The Aharonov-Bohm effect is reconsidered as a scattering event of an electron by a magnetic field confined in an infinite solenoid of finite radius both in the situation where the solenoid is penetrable as well as impenetrable. We next discuss the validity of the Born approximation for the partial-wave scattering amplitudes and explain why for the cylindrically symmetric ($m=0$) partial wave the first Born approximation fails in the long-wavelength limit or as the radius of the solenoid shrinks to zero.

In this note we reconsider the Ahronov-Bohm (AB) effect as a scattering event and attempt to clear up some points of interest not explicitly addressed in the original discussion.[1] Specifically, we examine the scattering of an electron of wave number k by a magnetic field in an infinite solenoid of finite radius R, looking both at the case where the solenoid is penetrable to the electron and the case where it is impenetrable. First, we show that in the limit $R \to 0$ the exact solutions lead to the original AB scattering cross section in both cases. Next we discuss the validity of the Born approximation for the partial-wave amplitudes. We show that for any nonzero magnetic flux $\Phi = \alpha\Phi_0$ in the solenoid, where $\Phi_0 = ch/e$ is the quantized unit of magnetic flux, the number of terms that must be retained in the Born series for the cylindrically symmetric ($m=0$) partial-wave scattering amplitude is of order $|\alpha \ln(kR)|$ when $kR \ll 1$. In the line-flux limit, of course, $|\ln(kR)| \to \infty$ so that all terms must be retained. Resummation then shows that the $m=0$ partial-wave scattering amplitude is proportional to α although the potential that gives rise to this scattering is of order α^2. This explains the apparent discrepancy[2,3] between the first-Born-approximation result and the exact AB result to first order in α for the $m=0$ partial-wave scattering amplitude.

For the case of electron scattering by an impenetrable solenoid of radius R confining a magnetic flux $\Phi = \alpha\Phi_0$, the electron wave function can be expanded as (we limit $|\alpha| < 1$ for convenience)

$$\psi = \sum_{m=-\infty}^{\infty} \phi_{mk}(r) e^{im\theta} , \tag{1}$$

and

$$\phi_{mk}(r) = A_m(k,R,\alpha)[J_{|m+\alpha|}(kr)N_{|m+\alpha|}(kR) - N_{|m+\alpha|}(kr)J_{|m+\alpha|}(kR)] , \tag{2}$$

where $A_m(k,R,\alpha)$ is a normalization factor. In the asymptotic region, the partial wave behaves like

$$\lim_{r\to\infty} \phi_{mk}(r) \sim (-i)^m J_m(kr) + f_m \frac{e^{ikr}}{\sqrt{r}} . \tag{3}$$

In Eqs. (2) and (3), J_m and N_m are the usual Bessel and Neumann functions. On comparing Eqs. (2) and (3), we have

$$A_m(k,R,\alpha) = \frac{(-i)^{|m+\alpha|-1}}{H^{(1)}_{|m+\alpha|}(kR)} , \tag{4}$$

where $H^{(1)}$ is the Hankel function of the first kind. Also,

$$f_m(k,\alpha,R) = \frac{(-1)^m}{\sqrt{2\pi k}} e^{-i\pi/4} \left(e^{2i\delta_m(\alpha)} - 1 - \frac{2e^{2i\delta_m(\alpha)} J_{|m+\alpha|}(kR)}{H^{(1)}_{|m+\alpha|}(kR)} \right) , \tag{5}$$

where the mth partial-wave phase shift is given[4] by

$$\delta_m(\alpha) = \frac{\pi}{2}(|m| - |m+\alpha|) = \begin{cases} -\pi\alpha/2 \text{ for } m \geq 0 , \\ \pi\alpha/2 \text{ for } m < 0 . \end{cases} \tag{6}$$

The total scattering amplitude is[5]

$$f(k,\alpha,R,\theta) = \sum_{m=-\infty}^{\infty} f_m(k,\alpha,R) e^{im\theta} . \tag{7}$$

On substituting Eq. (5) into Eq. (7), we obtain

$$f(k,\alpha,R,\theta) = f^{\mathrm{AB}} - \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{\sqrt{2\pi k}} e^{-i\pi/4} \times \frac{2e^{2i\delta_m(\alpha)} J_{|m+\alpha|}(kR) e^{im\theta}}{H^{(1)}_{|m+\alpha|}(kR)} , \tag{8}$$

where f^{AB} is the Aharonov-Bohm amplitude for a line flux. The second term in Eq. (8) goes to zero as R goes to zero, giving the AB result. This is in harmony with the recent result of Ruijsenaars[6] from the S-matrix viewpoint.

In the case of a penetrable solenoid, the effective potential inside the solenoid is finite, and for the mth partial wave is given by

$$U_m = \frac{2\alpha m}{R^2} + \frac{\alpha^2 r^2}{R^4} \ . \tag{9}$$

The corresponding mth partial-wave function inside and outside the solenoid is

$$\phi_{mk}^{\rm in}(r) = a_m \xi_m(k,r,R,\alpha) \tag{10}$$

and

$$\phi_{mk}^{\rm out}(r) = a_m[\Delta_1 J_{|m+\alpha|}(kr) + \Delta_2 N_{|m+\alpha|}(kr)] \ , \tag{11}$$

where

$$\Delta_1 = \frac{\xi_m(k,R,R,\alpha)N'_{|m+\alpha|}(kR) - \xi'_m(k,R,R,\alpha)N_{|m+\alpha|}(kR)}{J_{|m+\alpha|}(kR)N'_{|m+\alpha|}(kR) - J'_{|m+\alpha|}(kR)N_{|m+\alpha|}(kR)} \ , \tag{12}$$

$$\Delta_2 = \frac{J_{|m+\alpha|}(kR)\xi'_m(k,R,R,\alpha) - \xi_m(k,R,R,\alpha)J'_{|m+\alpha|}(kR)}{J_{|m+\alpha|}(kR)N'_{|m+\alpha|}(kR) - J'_{|m+\alpha|}(kR)N_{|m+\alpha|}(kR)} \ , \tag{13}$$

$$\xi_m(k,r,R,\alpha) = \exp[-\alpha^2 r^2/(2R^2)](kr)^{|m|} \sum_{n=0}^{\infty} b_n (kr)^{2n} \ , \tag{14}$$

with

$$b_0 = 1 \ , \tag{15a}$$

$$b_n = \prod_{j=0}^{n-1} \left(\frac{\lambda}{2(n-j)} - \frac{1+\lambda}{4(n-j)(n-j+|m|)} \right) \text{ for } m \geq 0 \ , \tag{15b}$$

$$= \prod_{j=0}^{n-1} \left(\frac{\lambda}{2(n-j+|m|)} - \frac{1+\lambda}{4(n-j)(n-j+|m|)} \right) \text{ for } m < 0 \ , \tag{15c}$$

$$\lambda \equiv 2\alpha/(k^2R^2) \ , \tag{15d}$$

and

$$a_m = \frac{(-i)^{|m+\alpha|}}{\Delta_1 + i\Delta_2} \ . \tag{16}$$

In the asymptotic limit,

$$\lim_{r\to\infty} \phi_{mk} \sim \frac{(-i)^m}{\sqrt{2\pi kr}} \left\{ \exp\left[i\left(kr - \frac{m\pi}{2} - \frac{\pi}{4} \right) \right] + \exp\left[-i\left(kr - \frac{m\pi}{2} - \frac{\pi}{4} \right) \right] \right\} + f_m e^{ikr}/\sqrt{r} \ . \tag{17}$$

On comparing Eq. (17) with Eqs. (11)–(16), we find

$$f_m(k,R,\alpha) = f_m^{AB} + f'_m(k,\alpha,R) \ , \tag{18}$$

where

$$f_m^{AB} = \frac{(-1)^m e^{-i\pi/4}}{\sqrt{2\pi k}} (e^{2i\delta_m(\alpha)} - 1) \tag{19}$$

is the AB scattering amplitude and

$$f'_m = (-1)^{m+1} \frac{e^{-i\pi/4}}{\sqrt{2\pi k}} \frac{2i\Delta_2 e^{2i\delta_m(\alpha)}}{\Delta_1 + i\Delta_2} \ , \tag{20}$$

which goes to 0 as $R \to 0$, showing that the AB result is valid whether the solenoid is penetrable or not.

Having established that the penetrability of the solenoid does not affect the AB result in the $R \to 0$ limit, we next examine the partial-wave scattering amplitude of the electron due to the magnetic field confined in an impenetrable solenoid of radius R. The mth partial wave satisfies the integral equation

$$\phi_{mk}(r) = a_m[J_m(kr)N_m(kR) - N_m(kr)J_m(kR)] + \frac{\pi}{2} \int_R^r g_m(r,r')U_m(r')\phi_{mk}(r')r'dr' \ , \tag{21}$$

where

$$g_m(r,r') \equiv N_m(kr)J_m(kr') - J_m(kr)N_m(kr') \tag{22}$$

is the Green's function for the mth partial wave, and

$$U_m(r) = \frac{2m\alpha}{r^2} + \frac{\alpha^2}{r^2} \tag{23}$$

is the corresponding effective scattering potential. In the limit $kR \to 0$, the terms proportional to $J_m(kR)$ are negligible compared with those proportional to $N_m(kR)$. On comparing Eq. (21) with Eq. (3), we have

$$\lim_{kR\to 0} f_m(k,\alpha,R) = \left(\frac{\pi}{2k} \right)^{1/2} (-i)^{m+1} e^{-i\pi/4} \int_R^\infty J_m(kr)U_m(r)(-i)^{|m+\alpha|} J_{|m+\alpha|}(kr)dr \ . \tag{24}$$

For $m \neq 0$, we find

$$\lim_{kR\to 0} f_m(k,\alpha,R) = f_m^{AB}[1 + O((kR)^{2|m|+\alpha\,\mathrm{sgn}(m)})] \ , \tag{25}$$

where $\mathrm{sgn}(m)$ stands for the sign of m. Since α is restricted to be between 0 and 1, we see from Eq. (25) that in the long-wavelength limit the finite radius of the solenoid has little effect on the AB scattering amplitude for the $m \neq 0$ partial waves. The story is, however, very different for the cylindrically symmetric ($m = 0$) partial wave. For $m = 0$, from Eq. (23), it appears that the scattering potential is of order α^2, and so the $m = 0$ partial-wave scattering amplitude should vanish to first order in α.[2,3] On the other hand, the exact AB result in the line-flux limit for the $m = 0$ partial wave does not vanish to first order in α. As we shall show, there is a subtle reason behind this apparent discrepancy. According to Eq. (24), the $m = 0$ partial-wave scattering amplitude is given by

$$\lim_{kR\to 0} f_0(k,\alpha,R) = (-i)^{1+\alpha}\left(\frac{\pi}{2k}\right)^{1/2} e^{-i\pi/4} \times \int_R^\infty J_0(kr)\frac{\alpha^2}{r}J_\alpha(kr)\,dr \quad . \tag{26}$$

We replace the integral in Eq. (26) by

$$\int_R^\infty = \int_0^\infty - \int_0^R \quad . \tag{27}$$

The first integral on the right-hand side (RHS) of Eq. (27) can be evaluated exactly,[7] and is

$$\alpha^2 \int_0^\infty J_0 J_\alpha \frac{dr}{r} = \frac{2\sin\pi\alpha/2}{\pi} \quad . \tag{28}$$

The second integral can be evaluated by making a power-series expansion on the Bessel functions:

$$\alpha^2 \int_0^R J_0(kr)J_\alpha(kr)\frac{dr}{r} = \frac{\alpha(kR/2)^\alpha}{\Gamma(1+\alpha)}[1+O((kr)^2)] \quad . \tag{29}$$

If one retains just the first integral on the RHS of Eq. (27) in Eq. (26), one recovers the AB result in the line-flux limit, which does not vanish to first order in α, as is obvious from the small-α limit for Eq. (28), namely,

$$\lim_{\alpha\to 0}\frac{2\sin\pi\alpha/2}{\pi} = \alpha \quad . \tag{30}$$

In the line-flux limit ($R\to 0$), the second integral on the RHS of Eq. (27) goes to zero. However, for any finite R, this integral goes like α to lowest order in α, as is evident by writing the RHS of Eq. (29) in the form[8]

$$\frac{\alpha(kR/2)^\alpha}{\Gamma(1+\alpha)} = \frac{\alpha\exp[\alpha\ln(kR/2)]}{\Gamma(1+\alpha)} \tag{31}$$

$$= \frac{\alpha}{\Gamma(1+\alpha)}\sum_{n=0}^{\infty}\frac{[\alpha\ln(kR/2)]^n}{n!} \quad . \tag{32}$$

For any finite R, the expressions on both sides of Eq. (31) are analytic in α. Thus in the small-α limit, for any finite R, the $m=0$ partial-wave scattering amplitude in the long-wavelength limit is

$$f_0(k,\alpha,R) = (-i)^{1+\alpha}\left(\frac{\pi}{2k}\right)^{1/2} e^{-i\pi/4} g(\alpha,k,R) \quad , \tag{33}$$

where

$$g(\alpha,k,R) = \alpha - \alpha\sum_{n,j=0}^{\infty}\alpha^{j+n}\frac{[\ln(kR/2)]^n}{n!}\frac{d^j}{dx^j}[\Gamma(x)^{-1}]_1 \tag{34}$$

$$\equiv \alpha - \alpha\sum_{n,j=0}^{\infty} T_{n,j}(kR)\alpha^{j+n} \tag{35}$$

$$= \alpha - \alpha\sum_{\lambda=0}^{\infty}\alpha^\lambda\sum_{j=0}^{\lambda} T_{(\lambda-j),j}(kR) \quad . \tag{36}$$

It is obvious that

$$T_{00} = 1 \quad , \tag{37}$$

so that

$$g(\alpha,kR) = \alpha\sum_{\lambda=1}^{\infty}\alpha^\lambda\sum_{j=0}^{\lambda} T_{(\lambda-j),j}(kR) \quad , \tag{38}$$

showing that indeed for finite R the leading term in α in the $m=0$ partial-wave amplitude goes like α^2. However, as $R\to 0$ $|\ln(kR/2)|\to\infty$, so more and more terms must be retained in the Born series (the series expansion in α). In fact, it is not difficult to see from the structure of Eq. (34) that the number of terms that must be retained is of order $\alpha|\ln(kR)|$. As $R\to 0$, we need to retain every term in the Born series because each term in the Born series, though of different order in α, is comparable in magnitude. On resummation, the correction to the AB scattering amplitude due to the finite size of the solenoid can be written as

$$f^{\rm AB}(\alpha,k,R) = f^{\rm AB}_{\rm line}(\alpha,k) - \frac{\alpha(kR/2)^\alpha}{\Gamma(1+\alpha)} \quad , \tag{39}$$

where the $f^{\rm AB}_{\rm line}(\alpha,k)$ is the original AB scattering amplitude due to a line flux.

This work is supported in part by the National Science Foundation under Grant No. ISP-80-11451.

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[5]The difference between using e^{ikx} and $e^{ikx-i\alpha\theta}$ as the incident wave function is immaterial when we visualize the physical situation where the incident particle should be described by a wave packet in which case the variation over the possible range of values of θ is practically zero, hence making $e^{-i\alpha\theta}$ a multiplicative constant. In a recent paper, Frolov and Skarzhinsky consider the AB scattering as due to the adiabatic switching on of the line flux, in which case the incident wave may be chosen to be a plane wave. Similar results for the scattering wave function as in the present paper are obtained. [See P. Frolov and V. D. Skarzhinsky, Nuovo Cimento 76, 35 (1983).] However, in this paper, these authors are not concerned with the question of the validity of the Born approximation in AB scattering.

[6]S. N. M. Ruijsenaars, Ann. Phys. (N.Y.) 146, 1 (1983).

[7]See, for example, I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965), p. 692.

[8]Here we first use the property of the logarithm $\exp(\ln z) = z$ and then the series expansion for $\exp(z)$ which are both valid in the entire complex plane.