Fractional charge and zero modes for planar systems in a magnetic field

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Attention is drawn to a relation between the recently discovered three-dimensional chiral anomaly and fermion zero modes. Application to planar systems of electrons in an external magnetic field is suggested.

A novel axial anomaly has been found in gauge theories defined on three-dimensional space-time, which describe dynamics confined to a plane: fermions moving in an external gauge field and governed by the 2×2 matrix equation (massless Dirac equation)

$$\gamma^\mu(i\partial_\mu - eA_\mu)\Psi = 0 \tag{1}$$

induce a topologically nontrivial vacuum current of abnormal parity,[1,2]

$$\langle j^\mu\rangle = \pm c\frac{e}{8\pi}\epsilon^{\mu\alpha\beta}F_{\alpha\beta} + \cdots \ . \tag{2}$$

Here γ^μ are three 2×2 "Dirac" matrices (Pauli matrices) and A_μ is the external vector potential, leading to the field strength $F_{\alpha\beta}$. The proportionality constant in (2) is determined by the fermion representation: if the Lie-algebra generators are T^a, then $\operatorname{tr} T^aT^b = c\delta^{ab}$; thus for the Abelian Maxwell theory $c=1$, while SU(N) fermions in the fundamental representation give $c=\frac{1}{2}$. In formula (2) all omitted terms, indicated by dots, are of normal parity and the right-hand side is conserved by virtue of the Bianchi identity satisfied by the exhibited term. The topological interest in the result derives from the fact that although (2) is gauge covariant, it cannot be obtained by varying a gauge-invariant (effective) action: $\langle j^\mu\rangle$ is given by

$$i\frac{\delta}{e\delta A_\mu}\ln\det(i\partial\!\!\!/ - eA\!\!\!/) \ ;$$

integrating (2) shows that $-i\ln\det(i\partial\!\!\!/ - eA\!\!\!/)$ contains the topological quantity

$$I_{\rm CS} = \frac{e^2}{32\pi^2}\int d^3x\, \epsilon^{\mu\alpha\beta}\left[A^a_\mu F^a_{\alpha\beta} - \frac{e}{3}\epsilon^{abc}A^a_\mu A^b_\alpha A^c_\beta\right] .$$

This is the Chern-Simons secondary characteristic class, which has also arisen in investigations of three-dimensional gauge theories[3] (as well as in four-dimensional theories, in a Hamiltonian, fixed-time formulation which is defined on three-space[4]). It is known that $I_{\rm CS}$ is not gauge invariant; rather under a gauge transformation it changes by the integral winding number of the transformation.

The sign ambiguity in (2) arises from the necessary regularization procedure. To maintain gauge invariance and to control divergences, a regulator mass in inserted. Although by the end of the calculation the regulator has disappeared, it leaves behind a spur: the sign in (2) is determined by the sign of the mass. This kind of ambiguity has been previously encountered in studies of soliton-induced charge fractionization in even-dimensional space-time;[5] with physical application to quasi-one-dimensional systems like polyacetylene,[6] and to monopole physics, if that elusive particle indeed exists.[7]

Let me recall the relevant theory. In the charge-conjugation-symmetric case, one finds the fermion charge to be $\pm\frac{1}{2}$; the two values correspond to two states, degenerate in energy ("filled" and "empty").[5] A way of deriving this is to compute first the soliton-induced fermion-number vacuum current. The computation requires introducing, at an intermediate stage, a regulator which breaks the conjugation symmetry. When the regulator is removed, it leaves behind its sign.[8] Technically what happens is that the conjugation-symmetric Dirac Hamiltonian possesses an energy spectrum that is symmetric about zero, and in the presence of a soliton there is also an unpaired zero-energy state, which is self-conjugate.[5] The regulator lifts this state above or below zero; removing the regulator restores the state to zero, but the sign of the final answer depends on whether the limit is approached from above or from below.

The purpose of this paper is to derive similar results in the three-dimensional case under present discussion. We show that for static background fields in the $A_0=0$ Weyl gauge, the Dirac Hamiltonian corresponding to (1) possesses a conjugation-symmetric spectrum with zero modes, if the background field satisfies certain requirements. Although the topological interest is mainly in the non-Abelian theory, we shall concern ourselves with the Abelian Maxwell theory, which is of greater physical relevance, since it can describe the motion of charged fermions on a plane perpendicular to an external magnetic B field.

The demonstration is very simple. The Hamiltonian corresponding to (1) is

$$H = \vec{\alpha}\cdot(\vec{p} - e\vec{A}) \ , \tag{3}$$

where the "Dirac" $\vec{\alpha}$ matrices are the two Pauli matrices: $\alpha^1 = -\sigma^2$, $\alpha^2 = \sigma^1$. The β matrix, which would be present if there were a mass term, is taken to be σ^3. Since $\beta=\sigma^3$ anticommutes with H, it serves as a conjugation matrix, and the energy eigenmodes are symmetric about $E=0$,

$$\vec{\alpha}\cdot(\vec{p}-e\vec{A})\psi_E = E\psi_E \ ,$$
$$\sigma^3\psi_E = \psi_{-E} \ . \tag{4}$$

Of course in the presence of the mass term, the conjugation symmetry is broken.

To find the zero-energy modes we write the wave function as $\psi_0 = \binom{u}{v}$, and choose the Coulomb gauge for $\vec{A}$, which we assume to be single valued and well behaved at the origin,

$$A^i = \epsilon^{ij}\partial_j a \ , \tag{5}$$

$$B = -\nabla^2 a \ . \tag{6}$$

Then Eq. (4) reduces to the pair

$$(\partial_x + i\partial_y)u - e(\partial_x + i\partial_y)au = 0 \ ,$$
$$(\partial_x - i\partial_y)v + e(\partial_x - i\partial_y)av = 0 \ , \tag{7}$$

with the obvious solution

$$u = \exp(ea)f(x+iy) \ ,$$
$$v = \exp(-ea)g(x-iy) \ , \tag{8}$$

where f and g are arbitrary entire functions. Thus we can form self-conjugate solutions $\binom{u}{0}$ and $\binom{0}{v}$. Whether these are acceptable wave functions depends on the large-r behavior of a. If a grows sufficiently rapidly at large distance, then either u or v will be normalizable, and there exist one or more isolated zero-energy bound states, the multiplicity depending on how many different forms for f or g may be taken.

It is useful to classify the various possibilities in terms of the total flux, which is also proportional to the total induced charge:

$$\langle j^0 \rangle = \pm \frac{e}{4\pi} B \ , \tag{9}$$

$$Q = \int d^2\vec{r}\langle j^0\rangle = \pm\frac{e}{4\pi}\int d^2\vec{r}B = \pm\frac{e}{2}\Phi \ ,$$
$$\Phi = \frac{1}{2\pi}\int d^2\vec{r}B = -\frac{1}{2\pi}\int d\vec{r}\times\vec{\nabla}a \ . \tag{10}$$

[In electrodynamics, (9) is exact only for constant magnetic fields,[1,2] but Q is correctly given by (10) for arbitrary-strength magnetic fields.[2]] The last integral is over the circle at infinity. Since the potential is single valued, (10) may also be presented as

$$\Phi = -\frac{1}{2\pi}\int_0^{2\pi} d\theta\, r\frac{\partial}{\partial r}a\bigg|_{r=\infty} \ . \tag{11}$$

When Φ vanishes because a goes to zero at large distances, the modes (8) are not normalizable. On the other hand, nonvanishing flux, arising from the persistence of the vector potential at infinity will allow normalization of (8), and isolated zero-energy modes are present.

An example is a constant magnetic field, giving rise to infinite total flux and induced charge. The relevant vector potential may be chosen in various ways, since there still remains gauge freedom within the Coulomb gauge: gauge transformations with harmonic gauge functions may be performed. Interesting choices are

$$a^{\mathrm{I}} = -\tfrac{1}{4}r^2 B \ , \tag{12a}$$

$$a^{\mathrm{II}} = -\tfrac{1}{2}x^2 B \ . \tag{12b}$$

In the former case $\exp(ea)$ is square integrable, and f may be any integer power. Thus the zero-energy states are infinitely degenerate:

$$\psi^{\mathrm{I}}_{0(n)} = e^{-eBr^2/4}r^n e^{in\theta}, \quad n = 0,1,\ldots \ . \tag{13a}$$

Making the gauge-equivalent choice (12b), we find $\exp(ea)$ to be square integrable in x, and f must be chosen to be continuum normalizable in y. Thus again we find infinite degeneracy, parametrized by a continuous variable k:

$$\psi^{\mathrm{II}}_{O(k)} = e^{-eBx^2/2}e^{k(x+iy)} \ . \tag{13b}$$

These are of course the familiar Landau states;[9] with (12a) and (13a) angular momentum is diagonalized, while linear momentum in the y direction is diagonal with (12b) and (13b).

Finite flux is obtained with a solenoid or vortex magnetic field,

$$a \underset{r\to\infty}{\sim} -\Phi \ln r \ . \tag{14}$$

Assuming $\Phi > 0$, $u \sim_{r\to\infty} r^{-e\Phi}$ is normalizable for $e\Phi > 1$. Also f may be any integer power less than $e\Phi - 1$. Thus there are $[e\Phi - 1]$ normalized zero-energy states:

$$\psi_{O(n)} = \exp(ea)r^n e^{in\theta}, \quad n = 0,1,\cdots,[e\Phi-1] \ . \tag{15}$$

Here $[\nu]$ denotes the largest integer less than ν. When flux is quantized $e\Phi = N$, the charge is integral or half-integral and there are $N-1$ states.[10]

There is a mismatch between the value of quantized flux and the number of zero modes. The reason is that the $N=1$ mode is asymptotic to r^{-1}, which cannot be normalized on the plane with measure $r\,dr\,d\theta$; rather the norm is logarithmically divergent. This discrepancy may be removed when the R_2 manifold is compactified to S_2 by stereographic projection. The eigenvalue equation for nonzero eigenvalues acquires a weight, but the zero eigenvalues are unchanged. The measure becomes $r\,dr\,d\theta 2R^2/(R^2+r^2)$, where R is the radius of the sphere whose surface is S_2. With this measure the $N=1$ mode is normalizable, giving N zero-eigenvalue states when $e\Phi = N$. What is being done here is to take cognizance of the circumstance that mathematical index theorems, which relate the number of zero eigenvalues to topological properties of the background field, take their simplest form on closed compact manifolds like S_2.[11]

Another way to understand our zero-energy eigenstates is to consider the square of H which coincides with the Pauli Hamiltonian:

$$H^2=2m\left[\frac{1}{2m}(\vec{p}-e\vec{A})^2-\frac{\sigma^3}{2m}B\right]. \tag{16}$$

The latter is an example of supersymmetric quantum mechanics.[12] When supersymmetry is not broken, necessarily zero eigenvalues exist.

Thus we have closed the circle between zero modes, unexpected quantum numbers, and vacuum currents in three dimensions. The situation is quite analogous to two and four dimensions. The signal for topologically interesting effects is nonvanishing flux, and its magnitude measures the degeneracy of the zero modes.

The present results may be relevant to condensed-matter situations where electrons move in a magnetic field which is constant in one direction. However, before applying our theory to actual phenomena, the relevance of the three-dimensional massless Dirac equation must be established. Certainly electrons are not massless three-dimensional particles. Rather one may expect that the two-by-two matrix equation emerges in a well-defined approximation to a one-component nonrelativistic theory, where the energy dispersion law is linearized and a two-component structure emerges kinematically. Examples of such constructions have been found in other dimensionalities. The continuum limit of the Su-Schrieffer-Heeger polyacetylene Hamiltonian[13] and its generalizations[14] yields a Dirac equation in two-dimensional space-time.[15] Similarly, an approximate description of electrons near degeneracy points (points where two electronic energy bands are in contact) in a hypothetical gapless, parity-nonvariant semiconductor gives rise to a two-component Dirac (Weyl) equation[16] in four-dimensional space-time. Thus one may hope that with planar systems, a physical role for Eq. (4) will also be found. In this connection, it is interesting to note that (2) implies that an external, constant electric field, produces a current perpendicular to it, and the conductivity is $e^2/4\pi$. Physical electrons possess two states, spin up and spin down. For these one should multiply by 2, yielding a conductivity of $e^2/2\pi$. Evidently there exists a quantum Hall effect in our system.[17]

Note added in proof: The quantized Hall effect has also been analyzed from the point of view advocated in this paper by K. Ishikawa, Hokkaido University Reports Nos. EPHOU 83 Dec 005 and EPHOV 84 Feb 002 (unpublished); Y. Srivastava and A. Widom, Lett. Nuovo Cimento (to be published); M. Friedman, J. Sokoloff, A. Widom and Y. Srivastava, Phys. Rev. Lett. (to be published).

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[1]A. N. Redlich, Phys. Rev. Lett. 52, 18 (1984); preceding paper, Phys. Rev. D 29, 2366 (1984); L. Avarez-Gaumé and E. Witten, Nucl. Phys. B234, 269 (1984). Earlier perturbative calculations on this are in S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) 140, 372 (1982); as well as in I. Affleck, J. Harvey, and E. Witten, Nucl. Phys. B206, 413 (1982).

[2]A. Niemi and G. Semenoff, Phys. Rev. Lett. 51, 2077 (1983).

[3]R. Jackiw and S. Templeton, Phys. Rev. D 23, 2291 (1981); J. Schonfeld, Nucl. Phys. B185, 157 (1981).

[4]R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976).

[5]R. Jackiw and C. Rebbi, Phys. Rev. D 13, 3398 (1976).

[6]R. Jackiw and J. R. Schrieffer, Nucl. Phys. B190 [FS3], 253 (1981).

[7]V. Rubakov, Pis'ma Zh. Eksp. Teor. Fiz. 33, 658 (1981) [JETP Lett. 33, 644 (1981)]; Nucl. Phys. B203, 311 (1982); C. Callan, Phys. Rev. D 25, 2141 (1982); 26, 2058 (1982).

[8]J. Goldstone and F. Wilczek, Phys. Rev. Lett. 47, 968 (1981).

[9]L. Landau and E. Lifshitz, *Quantum Mechanics*, 3rd ed. (Pergamon, Oxford, 1977).

[10]Massive fermion modes in the field of the vortex have been examined by C. Nohl, Phys. Rev. D 12, 1840 (1975); H. de Vega, *ibid.* 18, 2932 (1978). Zero modes for massless fermions interacting with a vortex and a scalar field are discussed by R. Jackiw and P. Rossi, Nucl. Phys. B190 [FS3], 681 (1981).

[11]M. Ansourian, Phys. Lett. 70B, 301 (1977).

[12]A. Barducci, R. Casalbuoni, and L. Lusanna, Nuovo Cimento 35A, 377 (1976).

[13]W.-P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. 42, 1698 (1979).

[14]M. J. Rice and E. J. Mele, Phys. Rev. Lett. 49, 1455 (1982).

[15]H. Takayama, Y. Lin-Liu, and K. Maki, Phys. Rev. B 21, 2388 (1980); R. Jackiw and G. Semenoff, Phys. Rev. Lett. 50, 439 (1983).

[16]H. Nielsen and M. Ninomiya, Phys. Lett. 130B, 389 (1983).

[17]K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45, 494 (1980).

[18]F. Wilczek (unpublished).