Screening of classical color

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Color screening of static, spherically symmetric sources by classical SU(2) Yang-Mills fields is studied. Owing to the presence of a source, the global color symmetry SO(3) is broken to U(1), and the function space of gauge potentials is partitioned into topologically inequivalent sectors. The extent and nature of screening of the external source is shown to be dependent upon the associated topological charge M of the source-field system. This is illustrated in the case of a δ-shell distribution of external color. For systems of unit topological charge, static, spherically symmetric, total screening solutions of the field equations are constructed, extending the work of Jackiw, Jacobs, and Rebbi. However, in the $M=0$ sector, self-consistent arguments are presented which suggest that the external charge is never totally screened and, when the external charge exceeds a certain critical value, the minimum of the energy functional breaks the rotational symmetry of the Lagrangian.

I. INTRODUCTION AND SUMMARY

An outstanding problem in the study of the classical gauge potentials generated by external sources is the question of color screening. Namely, how well does the classical Yang-Mills theory mimic the expected behavior of the quantum theory, wherein the energetically favored (if not the only) states have zero total color? An understanding of color screening in the simpler context of the classical theory may supplement prevalent notions based on lattice studies that color confinement is a property of the quantum theory.

Previous demonstrations of charge screening in classical chromodynamics have been based on stability analyses of the Abelian Coulomb solution.[1] The Coulomb solution consists of choosing all color components of the gauge potential parallel to that of the external source which, in some gauge frame, may be chosen to lie along a particular direction in color space. It is trivial to show that such a solution exists for any distribution of external color. The gauge field in this case carries no charge, so that the total color Q_{tot} of the system is simply equal to that of the source Q,

$$\alpha_{\text{tot}} \equiv \frac{gQ_{\text{tot}}}{4\pi} = \frac{gQ}{4\pi} \equiv \alpha \tag{1}$$

(here g is the gauge coupling).

It is known that for sufficiently weak α, the Coulomb solution is a minimum of the Yang-Mills energy functional.[2] However, once the characteristic strength α, defined by Eq. (1) exceeds a certain critical value α_c [$\frac{3}{2}$ in SU(2) for a spherically symmetric source[1]], the solution becomes energetically unstable. Color components of the gauge field perpendicular to that of the source become excited, resulting in the production of color which tends to screen the source and lower the energy. Unfortunately, stability analysis does not indicate what the final configuration of gauge fields at the new minimum will be. Furthermore we are left with the question whether this configuration totally screens the external color (as is the case, for instance, in massless scalar electrodynamics[3]) or whether the total charge will merely be lowered to some subcritical but nonzero value.

Recently two studies have appeared which attempt to address this issue.[4,5] Both exploit the fact that the instability modes causing the decay of the Coulomb solution in the supercritical region are axisymmetric (symmetric about a single axis) even when, in the same gauge frame, the source is spherically symmetric. Incorporating axial symmetry in an *Ansatz* of gauge potentials, a new minimum of the (restricted) energy functional is found for each value of supercritical α. Results for the total color α_{tot} are shown in Fig. 1 for the gauge group SU(2). These studies thus indicate that screening saturates once the total color becomes subcritical.

These results, however, are inconclusive. Clearly, out of sheer practical necessity, they are based on *Ansätze* which restrict the space of gauge potentials probed for a

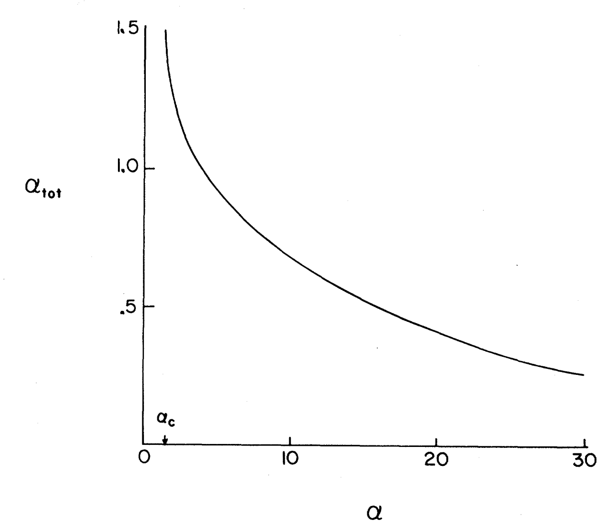

FIG. 1. Total color α_{tot} versus α for the partial-screening solutions of Ref. 4 [with source (2)].

minimum. A more complete exploration of potential space could reveal solutions which are total screening. Also, the *Ansätze* used are inherently nonspherical in the sense that local gauge-invariant quantities constructed from the potentials are rotationally noninvariant. The classical "ground-state" so constructed thus breaks the rotational symmetry of the Lagrangian, a peculiar and seemingly artificial situation.[6]

A more likely possibility is that, while the instability modes are asymmetric, the Coulomb solution will eventually decay to a minimum which is symmetric and total screening. In this regard, we are assisted by the fact that in SU(2) there exist only two classes of gauge fields which are spherically symmetric, namely, the Coulomb potential and the four-function *Ansatz* of gauge potentials given by Witten[7] some years ago. It has been determined that, for a spherical shell of external charge, the latter *Ansatz* solves the field equations and gives energy lower than that of the Coulomb solution.[8] The solutions are also found to totally screen the external charge.[9] One is then led to ask whether or not the Coulomb solution will, in the supercritical region, eventually decay to these total screening solutions.

The key to this puzzle lies in the existence of topological charge in Yang-Mills systems coupled to external charges. As first noted by Jacobs and Wudka,[9] the external charge plays the role of an adjoint Higgs field with a vacuum expectation value. The global color symmetry G is thus spontaneously broken to H, the residual set of gauge transformations which leave the source invariant. In the usual way, a conserved topological charge M emerges which may assume any value in $\pi_1(H)$ which is mapped to the identity in $\pi_1(G)$.[10] In our investigations below, the global SO(3) symmetry of SU(2) chromodynamics is broken by a color-vector external source to U(1), so that M may be any integer.

The crucial observation here is that while the Abelian Coulomb potential is topologically trivial ($M=0$), Witten's *Ansatz* carries unit topological charge ($M=1$). Therefore, the decay of the Coulomb solution, which represents a continuous deformation of the Coulomb potential, will never reach the $M=1$ total-screening solutions due to the conservation law. Since these two potentials exhaust all possible spherically symmetric SU(2) gauge potentials, this implies that, when the external charge goes supercritical, the minima of the energy functional in the $M=0$ sector must break the rotational invariance of the Lagrangian.

The absence of topological charge also has implications for the total charge of the system. It is possible to self-consistently argue that, for large enough values of the external charge, the spatial components of the gauge potential fall faster than r^{-1}, where r is the distance from the center of the source. Asymptotically, Gauss's law then reduces to its electromagnetic analog, with solutions of unscreened total charge, $0<\alpha_{\text{tot}}\leq\alpha_c$. Indeed, if the external charge is screened at all, that is if α_{tot} is strictly less than α, then an explicit demonstration of broken rotational symmetry is possible for large r, thus supporting our theoretical expectations above.

Therefore our work suggests an intimate relation between α_{tot}, the total charge, and M, the topological charge, of a classical Yang-Mills source-field system. It appears that total screening ($\alpha_{\text{tot}}=0$) is possible only when $M\neq 0$, i.e., the source-field configuration is topologically nontrivial.

This paper is concerned with substantiating these claims, and with constructing total-screening solutions of the field equations utilizing Witten's *Ansatz* with a δ-shell source of external color. As a subset of our results, we shall reproduce the solutions of Ref. 8. These solutions may be categorized by the number of zeros, or nodes, in a certain function $a(z)$ parametrizing the spatial components of the gauge potential $\vec{A}^a$. We extend the work of Ref. 8 by showing that solutions may contain any number of nodes, and illustrate this by constructing configurations with up to three nodes in $a(z)$. All these solutions totally screen the external source and are manifestly time independent. In addition, as mentioned above, they contain a unit of topological charge.

We also find that solutions with a nonzero number n of nodes display bifurcating properties similar to those first discovered in Ref. 8 for one-node solutions. These solutions exist only when the external charge exceeds a certain critical value α_n^* depending on n, and bifurcate there into two different solutions with equal numbers of nodes. Gauge-invariant quantities, such as the total energy, are seen to bifurcate as well, indicating that bifurcation is a real physical effect in our theory, and is not associated with indeterminancies in the classical gauge field propagator.[11]

At this juncture, mention should be made of other total-screening solutions in the $M=0$ sector which have appeared in the literature.[12] All these studies couple classical Yang-Mills fields to *extended* sources, that is, sources distributed over finite spatial volumes, and thereby suffer from a problem of interpretation first pointed out by Hughes.[13] The difficulty stems from the fact that the integrand for the conserved total color [see, for example, Eq. (10)] does not transform covariantly, but picks up an inhomogeneous term which may be reinterpreted as an extended external source; it is impossible to separate the source and field contributions to Q_{tot}. Thus it is often the case that these so-called total-screening solutions may be gauge transformed to configurations where the gauge field color is locally zero everywhere and each color component of the source distribution is characterized by regions of color which alternate in sign and sum to zero. Such solutions are clearly uninteresting since they represent finite-energy configurations generated by sources of zero total strength. Hughes has indicated that this gauge ambiguity associated with extended sources is not present in the classical Yang-Mills theory when charge distributions of measure zero are considered.[13]

Point charge distributions have their own particular difficulties, one being the familiar linear divergence of the classical self-energy of the source. Ordinarily, as in classical electrodynamics, a well-defined theory results if the self-energy is simply subtracted. Unfortunately, as we shall see, the self-energies of sources coupled to Coulomb or Witten potentials differ by a factor of 3. Thus if we desire a meaningful comparison of energies with the Coulomb value, subtraction schemes are unacceptable.

This leaves us with the option of regulating the source by distributing it over a thin shell of radius r_0. In the radial gauge, the source we shall use is

$$gJ_0^a=\frac{\hat{r}^a}{{r_0}^3}q\left[\frac{r}{r_0}\right], \tag{2}$$

where

$$q(z)=\alpha\delta(z-1)$$

and α is given by Eq. (1). In view of the Hughes ambiguity, and our restriction to spherical symmetry, this choice of source is unique.

An outline of the paper follows. In Sec. II, we present Witten's *Ansatz* and review some of its properties. The potentials are coupled to source (2), revealing the residual U(1) local symmetry leaving the source invariant. A conserved topological current is thereby suggested and defined in Sec. III. Witten's *Ansatz* is shown to have unit topological charge, the Abelian Coulomb solution zero. The relation between M and α_{tot} is drawn out, and self-consistent arguments give that all $M=0$ solutions are (at best) only partial screening. In Sec. IV, we specialize to time-independent potentials and determine the equations of motion for Witten's *Ansatz*. Expressions for the energy ϵ and the total charge α_{tot} are obtained by which we shall, in addition to the external charge α be able to uniquely characterize all solutions. This is followed by a discussion of the boundary conditions leading automatically to the fact that all solutions are total screening. Next, in Sec. V, numerical solutions of the equations of motion are presented along with their values of ϵ as a function of α. Bifurcation is observed in solutions with nodes, and the critical values α_n^* at which bifurcation takes place are determined. Section VI gives our conclusions, while an appendix elaborates the connection betwen $M=0$ partial-screening solutions and rotational noninvariance.

II. The *Ansatz*

In the radial gauge, Witten's *Ansatz* of SU(2) gauge potentials[7] is given by

$$\begin{aligned} gA_0^a&=\hat{r}^a a_t ,\\ gA_j^a&=\epsilon^{jak}\frac{\hat{r}^k}{r}(1+\varphi_2)-\hat{r}^j\hat{r}^a a_r+(\delta^{ja}-\hat{r}^j\hat{r}^a)\frac{1}{r}\varphi_1 , \end{aligned} \tag{3}$$

where the quantities φ_1, φ_2, a_t, and a_r are functions of t and r. A nice feature of this *Ansatz* is that the action functional S may be written in a two-dimensional notation. With metric

$$\eta^{tt}=-\eta^{rr}=1 ,$$

$$\eta^{tr}=\eta^{rt}=0 ,$$

this is

$$\begin{aligned} S&=\int_{t_1}^{t_2}dt\int d^3x\left(\tfrac{1}{2}E_a^iE_a^i-\tfrac{1}{2}B_a^iB_a^i-J_0^aA_0^a\right)\\ &=\frac{4\pi}{g^2}\int_{t_1}^{t_2}dt\int_0^\infty dr\left[(\mathscr{D}_\mu\varphi)^*(\mathscr{D}^\mu\varphi)-\frac{r^2}{4}G_{\mu\nu}G^{\mu\nu}-\frac{1}{2r^2}(1-\varphi^*\varphi)^2-J_ta_t\right], \end{aligned} \tag{4}$$

where the Greek indices run over t and r and we have employed the definitions

$$\varphi=\varphi_1+i\varphi_2 ,$$

$$\mathscr{D}_\mu=\partial_\mu+ia_\mu,\quad \mu=t,r,$$

$$G_{\mu\nu}=\partial_\mu a_\nu-\partial_\nu a_\mu,\quad \mu,\nu=t,r,$$

and

$$J_t=\frac{1}{r_0}q\left[\frac{r}{r_0}\right].$$

As is easily seen,[7,14] our system describes an Abelian-Higgs model coupled to an external source J_t in a two-dimensional curved space of constant curvature with metric

$$g^{\mu\nu}=r^2\eta^{\mu\nu} .$$

Our ability to rewrite the action functional (4) in terms of variables on a two-dimensional manifold originates, of course, in the spherical symmetry of the *Ansatz* (3), providing perhaps the generic example of dimensional reduction.

The equations of motion corresponding to (4) are

$$\partial_r(r^2G^{rt})+2\,\text{Im}(\varphi^*\mathscr{D}^t\varphi)=J^t , \tag{5a}$$

$$\partial_t(r^2G^{tr})+2\,\text{Im}(\varphi^*\mathscr{D}^r\varphi)=0 , \tag{5b}$$

$$r^2\mathscr{D}_\mu\mathscr{D}^\mu\varphi-(1-\varphi^*\varphi)\varphi=0 , \tag{5c}$$

and the expression for the total energy of the source-field system is

$$\mathscr{E}=\frac{4\pi}{g^2}\int_0^\infty dr\left[|\mathscr{D}_t\varphi|^2+|\mathscr{D}_r\varphi|^2+\frac{r^2}{2}(G^{rt})^2+\frac{1}{2r^2}(1-\varphi^*\varphi)^2\right]. \tag{6}$$

As is evident from (4), the choice of radial gauge is not a complete specification of gauge. We have, as a residual U(1) gauge symmetry, invariance of the equations of motion (5) under the transformations

$$J_t\to J_t ,$$

$$a_\mu\to a_\mu-\partial_\mu\Lambda,\quad \mu=t,r,$$

$$\varphi\to e^{i\Lambda}\varphi ,$$

where Λ is an arbitrary function of r and t. In terms of the original SU(2) fields, this corresponds to the set of transformations U which leave the source term (2) invariant:

$$U(r,t)=\exp[-ig\Lambda(r,t)\hat{r}^aT^a] ,$$

$$J_0^aT^a=U(J_0^aT^a)U^{-1} ,$$

where T^a denote the basis vectors of the SU(2) Lie algebra. With this additional gauge degree of freedom, we shall choose to set φ_2 zero, i.e., φ real. The gauge is thereby completely specified save for the sign of φ. Also, without loss of generality, we may describe static systems by working with potentials which are manifestly time independent.[6,15]

The residual U(1) gauge symmetry illustrates another important facet of *Ansatz* (3), and is, in fact, a general feature of classical chromodynamics with external sources. By specifying a nonzero external color density, we have broken the global color symmetry of the theory to that subgroup of transformations that leave the source invariant; in our case, this is SO(3) broken to U(1). This has its obvious analog in Higgs-Yang-Mills theories when nonzero vacuum expectation values develop for scalar fields transforming nontrivially under the gauge group. As first shown by 't Hooft[16] and Polyakov,[17] such systems often admit the definition of a topologically conserved charge. The definition easily generalizes for chromodynamic systems coupled to colored external sources,[9] as we review in the next section.

III. TOPOLOGICAL CHARGE AND COLOR SCREENING

Let $\hat{\rho}^a$ denote the unit vector defining the direction in color space of the color density. Like the field-strength tensor $F^a_{\mu\nu}$, $\hat{\rho}^a$ transforms covariantly, permitting the introduction of the gauge-invariant "electromagnetic" tensor[16]

$$\mathscr{F}_{\mu\nu}=\hat{\rho}^a F^a_{\mu\nu}-\frac{1}{g}\epsilon^{abc}\hat{\rho}^a(D_\mu\hat{\rho})^b(D_\nu\hat{\rho})^c .$$

Here

$$(D_\mu\hat{\rho})^a\equiv\partial_\mu\hat{\rho}^a+g\epsilon^{abc}A^b_\mu\hat{\rho}^c$$

is the covariant derivative of $\hat{\rho}^a$. Alternatively, defining

$$a_\mu\equiv\hat{\rho}^a A^a_\mu \;, \tag{7}$$

we can express $\mathscr{F}_{\mu\nu}$ as

$$\mathscr{F}_{\mu\nu}=\partial_\mu a_\nu-\partial_\nu a_\mu-\frac{1}{g}\epsilon^{abc}\hat{\rho}^b\partial_\mu\hat{\rho}^b\partial_\nu\hat{\rho}^c .$$

From $\mathscr{F}_{\mu\nu}$, we may construct the conserved topological current

$$K^\alpha\equiv-\tfrac{1}{2}\epsilon^{\alpha\beta\mu\nu}\partial_\beta\mathscr{F}_{\mu\nu} ,$$

$$\partial_\alpha K^\alpha=0 ,$$

which in turn gives the conserved topological charge of our system,

$$\frac{M}{g}\equiv\frac{1}{4\pi}\int d^3x\,K_0$$
$$=\frac{1}{4\pi}\int_{|\vec{r}|=\infty}d\vec{S}\cdot\left[\vec{\nabla}\times\vec{a}+\frac{1}{2g}\epsilon^{abc}\hat{\rho}^a\vec{\nabla}\hat{\rho}^b\times\vec{\nabla}\hat{\rho}^c\right] . \tag{8}$$

(We have scaled out a factor of $1/g$ so that M is a pure number.) Applied to *Ansatz* (3), we have $\hat{\rho}^a=\hat{r}^a$ and

$$M=1,\quad \text{Witten's } Ansatz \;, \tag{9a}$$

whereas for the Abelian Coulomb solution, we have say, $\hat{\rho}^a=\delta^a_3$ and

$$M=0,\quad \text{Abelian Coulomb solution} . \tag{9b}$$

Homotopy theory for SO(3) broken to U(1) tells us that M may be any integer. However, it has been shown[18] that other values of M, besides those displayed in Eqs. (9), correspond to field configurations that are not spherically symmetric, and thus are excluded from consideration here.

In addition to a conserved topological charge, there is also an electric current given by

$$j^\nu\equiv\partial_\mu\mathscr{F}^{\mu\nu} ,$$

which is conserved by virtue of the antisymmetry of $\mathscr{F}^{\mu\nu}$ in its Lorentz indices. Thus the conserved total charge of the source-field system is

$$\frac{\alpha_{\rm tot}}{g}\equiv\frac{1}{4\pi}\int d^3x\,j_0$$
$$=\frac{1}{4\pi}\int_{|\vec{r}|=\infty}d\vec{S}\cdot\vec{E}^a\hat{\rho}^a$$
$$=\frac{1}{4\pi}\int d^3x\,[J^a_0\hat{\rho}^a+\vec{E}^a\cdot(\vec{D}\hat{\rho})^a] . \tag{10}$$

When applied, for static sources $(D_0\hat{\rho})^a=0$, to (3) this yields

$$\alpha_{\rm tot}=\lim_{r\to\infty}r^2G^{rt} . \tag{11}$$

It is now simple to present a self-consistent argument why $M=0$ static solutions must be partial screening. For this, it is convenient to work in the "Abelian" gauge, $\hat{\rho}^a=\delta^a_3$. Then the three color components of Gauss's law read

$$(-\nabla^2+V)a_0=\hat{\rho}^aJ^a_0 , \tag{12a}$$

$$\vec{\nabla}\cdot(a_0{}^2\vec{A}_1)=-ga_0{}^2\vec{A}_2\cdot\vec{a} , \tag{12b}$$

$$\vec{\nabla}\cdot(a_0{}^2\vec{A}_2)=ga_0{}^2\vec{A}_1\cdot\vec{a} , \tag{12c}$$

where the "screening potential" V is

$$V=(\vec{D}\hat{\rho})^a\cdot(\vec{D}\hat{\rho})^a=g^2(|\vec{A}_1|^2+|\vec{A}_2|^2)$$

and a_0 and $\vec{a}$ are given by (7). These components play a special role since, by (8) and (10) and by virtue of the Abelian gauge, they are directly related to the total electric and topological charges:

$$\alpha_{\rm tot}=\lim_{r\to\infty}r^2\int\frac{d\Omega}{4\pi}\left[-\frac{\partial}{\partial r}\right](ga_0) ,$$

$$M=\lim_{r\to\infty}r^2\int\frac{d\Omega}{4\pi}\hat{r}\cdot(\vec{\nabla}\times g\vec{a}) .$$

This means that, to leading order in r^{-1}, a_0 behaves like

$$ga_0=\frac{\alpha_{\rm tot}}{r}+\sum_{\substack{lm\\ l>0}}\frac{c_{lm}}{r}Y_{lm}(\Omega)+O(r^{-2}) ,\quad r\to\infty , \tag{13}$$

where c_{lm} are constants, and $\vec{a}$ possesses, for $M\neq0$, a Dirac string of the canonical form

$$g\vec{a}=\hat{\varphi}\frac{M}{r}\frac{(1-\cos\theta)}{\sin\theta}+\cdots, \quad r\to\infty .$$

Here the ellipses denote all other (nonsingular) terms of order r^{-1} and higher in the asymptotic expansion of $\vec{a}$.

We may now implement the self-consistent argument. The crucial observation is that for $M\neq 0$ $\vec{a}$ must possess a Dirac string which falls like r^{-1}. From Eqs. (12b) and (12c), we see that to be consistent with this leading behavior of $\vec{a}$, $\vec{A}_1$ and $\vec{A}_2$ must also be of the same order (r^{-1}). It is then possible that the asymptotic expansion of the screening potential will possess a positive, spherically symmetric term of order r^{-2},

$$V=\frac{v(v+1)}{r^2}+\sum_{\substack{lm\\ l>0}}\frac{v_{lm}}{r^2}Y_{lm}(\Omega)+O(r^{-3}) ,$$

where v and v_{lm} are constants. For $v>0$ [as it is, for example, for Witten's *Ansatz* (3), where $v=1$], then a_0 must fall like r^{-v-1}, i.e., $\alpha_{\rm tot}=0$.

On the other hand, when $M=0$, $\vec{A}_a$ could fall faster than r^{-1} as r approaches infinity. If we assume this, then V decreases faster than r^{-2}, so that by (12a), $\alpha_{\rm tot}$ would be nonzero.

To check whether the behavior

$$\vec{A}_a\sim O(r^{-1-\delta}) , \tag{14}$$

$\delta>0$, is consistent, we must also verify that Ampere's law is asymptotically satisfied. In view of (13) with $\alpha_{\rm tot}$ nonzero and (14), it is sufficient to retain only terms linear in $\vec{A}_a$, obtaining

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}_\perp)-(ga_0)^2\vec{A}_\perp=0+O(r^{-3-2\delta}) , \tag{15a}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{a})=0+O(r^{-3-2\delta}) , \tag{15b}$$

where $\vec{A}_\perp$ denotes either $\vec{A}_1$ or $\vec{A}_2$. Clearly we may take $\vec{a}$ zero to this order and so, from (12b) and (12c), Eq. (15a) is supplemented by the condition

$$\vec{\nabla}\cdot[(ga_0)^2\vec{A}_\perp]=0+O(r^{-4-2\delta}) . \tag{16}$$

The consequence of Eqs. (15a) and (16) are considered in more detail in the Appendix. As a result of our analysis there, we find that

$$\vec{A}_\perp\sim\frac{1}{r^{\nu+1/2}} \tag{17}$$

with $\nu=[(\frac{3}{2})^2-\alpha_{\rm tot}{}^2]^{1/2}$. Consistency with our initial assumption (14) implies we must have $\nu>\frac{1}{2}$, or

$$\alpha_{\rm tot}<\sqrt{2} . \tag{18}$$

Thus $\alpha_{\rm tot}$ may be nonzero in the $M=0$ sector provided it lies below $\sqrt{2}$.

For most values of supercritical external charge $\alpha>\alpha_c=\frac{3}{2}$, we do indeed find $\alpha_{\rm tot}$ satisfying (18) (cf. Fig. 1), thus verifying our general line of argument. However, from Refs. 4 and 5, we find that $\sqrt{2}\leq\alpha_{\rm tot}\leq\alpha_c=1.5$ in the range $1.5\leq\alpha\lesssim 1.67$, so that the analysis given here is invalid in this region. Nevertheless, we conjecture that a more thorough study would confirm that all $M=0$ static solutions in the supercritical region are partial screening.

With a screening potential V falling faster than r^{-2}, we also see that the spherically asymmetric terms in the asymptotic expansion of a_0, described as coefficients in an expansion in spherical harmonics with angular momentum $l>0$, must fall like r^{-l-1}. This is due, of course, to the fact that the angular momentum barrier, for $l\neq 0$, overwhelms V in Eq. (12a) asymptotically [the c_{lm} in Eq. (13) are zero]. This fact, coupled with condition (16), implies that $\vec{A}_\perp$ is spherically asymmetric. An explicit construction is given in the Appendix. In particular, the gauge-invariant screening potential V is shown to be rotationally noninvariant, demonstrating that any static $M=0$ solution generated by supercritical charge breaks the rotational symmetry of the Lagrangian.

IV. THE EQUATIONS OF MOTION

Returning to *Ansatz* (3), we now seek static solutions of the equations of motion (5), where we work in a gauge where φ is real. The assumption of time independence simplifies the problem considerably, since Eq. (5b) reduces to the condition

$$\varphi^2 a_r=0 .$$

For $\varphi=0$, we see that Eqs. (5a) and (5c) reduce to those of the Abelian Coulomb *Ansatz,* but with an important difference: Owing to the inhomogeneous term

$$\epsilon^{jak}\frac{\hat{r}^k}{gr}$$

in A_j^a (3), there is an additional contribution to the energy (6) which is infinite. The corresponding solution describes an unscreened external charge with a (singular) unit magnetic monopole at the origin. However, because of its infinite energy, this solution will not be considered further here.

For $a_r=0$, Eqs. (5a) and (5c) become, respectively,

$$\left[-z^2\frac{d^2}{dz^2}+2a^2\right]f=zq , \tag{19a}$$

$$\left[z^2\frac{d^2}{dz^2}-a^2+1+f^2\right]a=0 , \tag{19b}$$

where, following the notation of Ref. 8, we express $a_t(r)$ and $\varphi(r)$ in terms of the dimensionless functions $f(z)$ and $a(z)$ of the rescaled radial coordinate $z=r/r_0$:

$$\begin{aligned} a_t(r)&=r^{-1}f\left[\frac{r}{r_0}\right] ,\\ a_r(r)&=0 ,\\ \varphi(r)&=a\left[\frac{r}{r_0}\right] . \end{aligned} \tag{20}$$

For the δ-shell source (2), the right-hand side of Eq. (19a) will be zero except at the shell $z=1$, where the external charge determines the discontinuity in the derivative of $f(z)$:

$$\frac{df}{dz}(1^-)-\frac{df}{dz}(1^+)=\alpha . \tag{21}$$

The two gauge-invariant quantities by which we shall characterize solutions are the total energy (6) and the total color (11). Values of the energy $\mathscr{E}$ shall be expressed in units ϵ of the Coulomb energy

$$\mathscr{E}_{\text{Coulomb}}=\frac{Q^2}{8\pi r_0},$$

for the δ-shell source (2). From Eqs. (6), and (20), we have

$$\epsilon=\frac{\mathscr{E}}{\mathscr{E}_{\text{Coulomb}}}=\frac{2}{\alpha^2}\int_0^\infty dz\left[\frac{1}{2}\left[z\frac{d}{dz}\frac{f}{z}\right]^2+\left[\frac{da}{dz}\right]^2+\frac{a^2f^2}{z^2}+\frac{(1-a^2)^2}{2z^2}\right]. \quad (22)$$

Using Gauss's law (19a), or alternatively Eqs. (11) and (20), the total charge is given by

$$\alpha_{\text{tot}}=\lim_{z\to\infty}\left[-z^2\frac{d}{dz}\frac{f}{z}\right]. \quad (23)$$

If $f(z)$ approaches or falls faster than a constant as z approaches infinity, then α_{tot} is simply given by the asymptotic value of f. Thus total screening is signaled by functions f which decrease monotonically to zero.

Solutions of the equations of motion require a specification of the boundary conditions. This is simple to do since Eqs. (19) possess regular singular points at the origin and at infinity. Regularity of the functions f and a at the end points plus the requirement that the total energy be finite delimits the possible asymptotic behaviors. At the origin we have

$$\begin{aligned} &f\to f_2z^2+O(z^4)\,,\\ &a\to 1+a_2z^2+O(z^4)\,, \end{aligned} \quad (24a)$$

while at infinity, $a(z)$ may have either one of two asymptotic forms:

$$\begin{aligned} &f\to\frac{f_{-1}}{z}+O(z^{-2})\,,\\ &a\to\pm1+\frac{a_{-1}}{z}+O(z^{-2})\,, \end{aligned} \quad (24b)$$

where f_2, a_2, f_{-1}, and a_{-1} are constants to be self-consistently determined by a solution of the equations of motion. By setting $a(0)=1$, we eliminate the remaining gauge degree of freedom associated with the sign of φ. Thus, provided solutions of (19) with boundary conditions (24) do exist, the asymptotic behavior of $a(z)$ as z approaches infinity determines two types of solutions. Defining the quantity,

$$\Delta a\equiv\tfrac{1}{2}[a(0)-a(\infty)]=\tfrac{1}{2}[1-a(\infty)]\,, \quad (25)$$

we shall, following Ref. 8, call these two types of solutions type I and type II, depending on whether Δa is zero or one, respectively.

As an immediate consequence of boundary condition (24b), the total charge of the source-field system (23) is zero. Therefore we infer that all spherically symmetric static solutions with unit color magnetic charge ($M=1$) are total screening, since all other candidates have infinite energy and are divorced from the spectrum of the theory.

The two types of solutions are characteristic of one (space)-dimensional field theories with soliton solutions (kinks and antikinks) stabilized by topological charge conservation. In our case the quantity Δa is the relevant topological charge (not to be confused with the color magnetic charge M discussed above). If we confine ourselves to spherically symmetric, static potentials, it is absolutely conserved, since any continuous deformation of $a(z)$, from one value of Δa to another must pass through a configuration of infinite energy. Unfortunately, it has not proven possible to generalize, in a U(1) gauge-invariant fashion, the topological charge Δa in terms of the original variables φ and a_μ.

An explicit example of a soliton solution of Eqs. (19) may be obtained when the source is tuned to zero ($Q=0$). This is given by the type-II solution

$$\begin{aligned} &f=F_{\text{II}}(z)=0,\\ &a=A_{\text{II}}(z)=\frac{1-\tau(z)}{1+\tau(z)}, \end{aligned} \quad \text{soliton} \quad (26)$$

where

$$\tau(z)=\left[\frac{z}{\lambda}\right]^{\sqrt{2}}.$$

The emergence of an arbitrary scale λ as a parameter in this solution is of course a consequence of the scale invariance of classical Yang-Mills theory without external sources. Thus, we have a whole one-parameter family of solitons with energies equal to λ^{-1} times a fixed constant. There is also the trivial, zero energy, type-I solution,

$$\begin{aligned} &f=F_{\text{I}}(z)=0\,,\\ &a=A_{\text{I}}(z)=1\,, \end{aligned} \quad \text{vacuum} \quad (27)$$

which, for obvious reasons, we shall call the vacuum solution. This exhausts all gauge-inequivalent sourceless solutions of Eq. (19).

The existence of the sourceless solutions (26) and (27) permit an analysis of the behavior of solutions as the external charge is increased from zero. Denoting the sourceless solutions by capital letters (F,A), the problem is to expand $f(z)$, $a(z)$, and $q(z)$ in small fluctuations,

$$\begin{aligned} &f=F+\delta f\,,\\ &a=A+\delta a\,,\\ &q=0+\delta q\,, \end{aligned} \quad (28)$$

linearize the equations of motion, and solve for δf and δa for a given increase δq of the source strength. Boundary conditions for δf and δa must be such that the asymptotic behaviors of f and a in (28) are consistent with (24):

$$\begin{aligned} &\delta f\sim\begin{cases}\delta f_2z^2+O(z^4), & z\to0\,,\\ \delta f_{-1}z^{-1}+O(z^{-2}), & z\to\infty\,,\end{cases}\\ &\delta a\sim\begin{cases}\delta a_2z^2+O(z^4), & z\to0\,,\\ \delta a_{-1}z^{-1}+O(z^{-2}), & z\to\infty\,,\end{cases} \end{aligned} \quad (29)$$

where δf_2, δf_{-1}, δa_2, and δa_{-1} are constants. The linearized field equations can be succinctly stated in matrix form as

$$K(F,A)\begin{bmatrix}\delta f\\ \delta a\end{bmatrix}=\left[\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}\frac{d^2}{dz^2}+\mathscr{V}(F,A)\right]\begin{bmatrix}\delta f\\ \delta a\end{bmatrix}=\begin{bmatrix}\delta q/z^3\\ 0\end{bmatrix}, \tag{30}$$

where

$$\mathscr{V}(F,A)=\frac{2}{z^2}\begin{bmatrix}A^2 & 2AF\\ 2AF & F^2+1-3A^2\end{bmatrix}$$

and $\delta q=\alpha\delta(z-1)$ (α small). However, because the various soliton solutions (26) can be related by continuous scale transformations, $K(F_{\rm II},A_{\rm II})$ possesses a nontrivial normalizable zero mode given by

$$\psi=\begin{bmatrix}0\\ z\dfrac{d}{dz}A_{\rm II}(z)\end{bmatrix},$$

$$K(F_{\rm II},A_{\rm II})\psi=0\ ,$$

and thus is noninvertible. Therefore, we see that, except precisely at $Q=0$, there are no type-II solutions for an external charge arbitrarily close to zero. If type-II solutions exist at all, they must arise at some finite critical value α^* of the external source strength.

The same conclusion does not apply to type-I solutions since, in the "vacuum" background field (27), $K(F,A)$ is invertible, with explicit form

$$K^{-1}(F_{\rm I},A_{\rm I})=\begin{bmatrix}1 & 0\\ 0 & -\frac{1}{2}\end{bmatrix}G(z,z')\ ,$$

where

$$G(z,z')=\frac{z'}{3}\begin{cases}\left[\dfrac{z}{z'}\right]^2, & z<z'\ ,\\[2ex] \left[\dfrac{z'}{z}\right], & z>z'\ .\end{cases}$$

Thus, calculating to first order in small α, the fluctuation solution is

$$\delta f=\frac{\alpha}{3}\begin{cases}z^2, & z\le 1\ ,\\ z^{-1}, & z\ge 1\ ,\end{cases}$$

$$\delta a=0$$

with energy approaching the limit

$$\lim_{\alpha\to 0}\epsilon(\text{type I})=\tfrac{1}{3} \tag{31}$$

as Q goes to zero, as we shall subsequently verify.

It would be most interesting if we could generalize solutions of Eqs. (19) to the case of a point-source distribution. Unfortunately, we cannot if we wish to maintain a clear relation between the energies of the solutions obtained here with that of the Abelian Coulomb solution. This is due to the presence of the term $(fa/z)^2$ in the energy (22). For the point source

$$q(z)=\alpha\delta(z)\ ,$$

we must have $f(0)=\alpha$. To ensure that the contribution of the chromomagnetic field to the energy near the origin is still finite, we must retain the condition $a(0)=1$. As a result, the coefficient of the short-distance linear divergence

$$\frac{Q^2}{4\pi}\left[\tfrac{1}{2}+1\right]$$

is larger by a factor of 3 than that of the Coulomb energy, and no subtraction scheme can be consistently applied to both.

Before concluding this section, we posit an alternative expression for the energy[8]

$$\mathscr{E}=\int d^3x\, J_0^a\vec{r}\cdot\vec{E}^a\ ,$$

which holds whenever the chromoelectric and chromomagnetic fields fall faster than $r^{-3/2}$ as r approaches infinity. Using (24b), one may verify this is the case. Using (2) and taking due care for Eq. (21), we obtain

$$\epsilon=\frac{-2}{\alpha^2}\int_0^\infty dz\, q(z)\frac{d}{dz}\frac{f(z)}{z}=\frac{2}{\alpha}\left[f(1)-\frac{1}{2}\frac{df}{dz}(1^+)-\frac{1}{2}\frac{df}{dz}(1^-)\right]. \tag{32}$$

IV. NUMERICAL SOLUTIONS

Solutions of the coupled, nonlinear equations of motion (19) with boundary conditions (24) is a complex problem for which current analytical methods are inadequate. Thus we must resort to numerical evaluation of $f(z)$ and $a(z)$ for various values of α. The results of this analysis, as well as the determination of ϵ, are the subject of this section.

Our technique for integrating Eqs. (19) is a simple functional generalization of Newton's method for determining the zeros of a function:[4,19] The equations of motion are linearized about an initial guess $f^{(0)}$ and $a^{(0)}$, which incorporate the boundary conditions (24) (with initial values for f_2, a_2, f_{-1}, a_{-1}) and the discontinuity (21) of df/dz at $z=1$. The linearized equations are then solved for δf and δa utilizing standard techniques. In this regard, it is important that only the relevant difference boundary conditions (29), and not the full ones (24), are applied to δf and δa. New approximate solutions are constructed,

$$f^{(1)}=f^{(0)}+\delta f\ ,$$

$$a^{(1)}=a^{(0)}+\delta a\ ,$$

and the procedure is repeated once again, replacing ($f^{(0)}$, $a^{(0)}$) by ($f^{(1)}$, $a^{(1)}$) as the new starting point. An iterative algorithm is thus defined, yielding a sequence of approximate solutions, ($f^{(i)},a^{(i)}$), ideally converging to a limit. Convergence may be conveniently monitored using some functional of $f(z)$ and $a(z)$. In our numerical work, we have used the total energy, Eqs. (22) or (32), as a convergence criterion, requiring an absolute accuracy of six decimal places.

Sample plots of the functions $f(z)$ and $a(z)$ for $\alpha=30$

are presented in Figs. 2(a) and 2(b). In all cases $f(z)$ is a positive function with a peak at $z=1$ and a discontinuity in its derivative there, as given by (21). The behavior of $a(z)$ as z approaches infinity determines the type of solution. We have verified that all solutions respect the boundary conditions (24) we have determined on the basis of series analysis.

The most striking qualitative feature of the solutions is the nodal structure of $a(z)$. In addition to the $n=0,1$ solutions of Ref. 8, we have also been able to contain solutions with two and three nodes. We have searched, but have been unable to detect, solutions with higher numbers of nodes.

The $n=1,2,3$ node solutions share the property that they exist only when the external charge is sufficiently strong, i.e., $\alpha > \alpha_n^*$. The critical values are found to be

$$\alpha_1^* = 5.835 ,$$

$$\alpha_2^* = 13.119 ,$$

$$\alpha_3^* = 15.763 .$$

This behavior was predicted for the type-II ($n=1,3$) solutions in our analysis of the last section. Moreover, solutions with nodes always come in pairs in their supercritical regions. This is the phenomenon of bifurcation that was first discovered in Ref. 8 for the case $n=1$. As discussed in Ref. 14, bifurcation is related to the existence of zero modes in the operator K, Eq. (30), at the point of bifurcation.

The plot of ϵ versus α (Fig. 3) portrays the bifurcation of solutions in more graphic form. According to the analysis of Ref. 14, the energy separation $\Delta\epsilon$ between the two solutions near the critical point scales like

$$\alpha^2 \Delta\epsilon \sim (\alpha - \alpha^*)^{3/2}, \quad \alpha > \alpha^* ,$$

which we have verified. Figure 3 also shows that the type-I $n=0$ solution is the only one that exists for all values of the charge α, with ϵ approaching the limit (31) as α goes to zero.

All calculations have been performed up to $\alpha = 50$ with no qualitative changes in the results.

VI. CONCLUSION

The existence of a topologically conserved charge in classical Yang-Mills systems coupled to external color

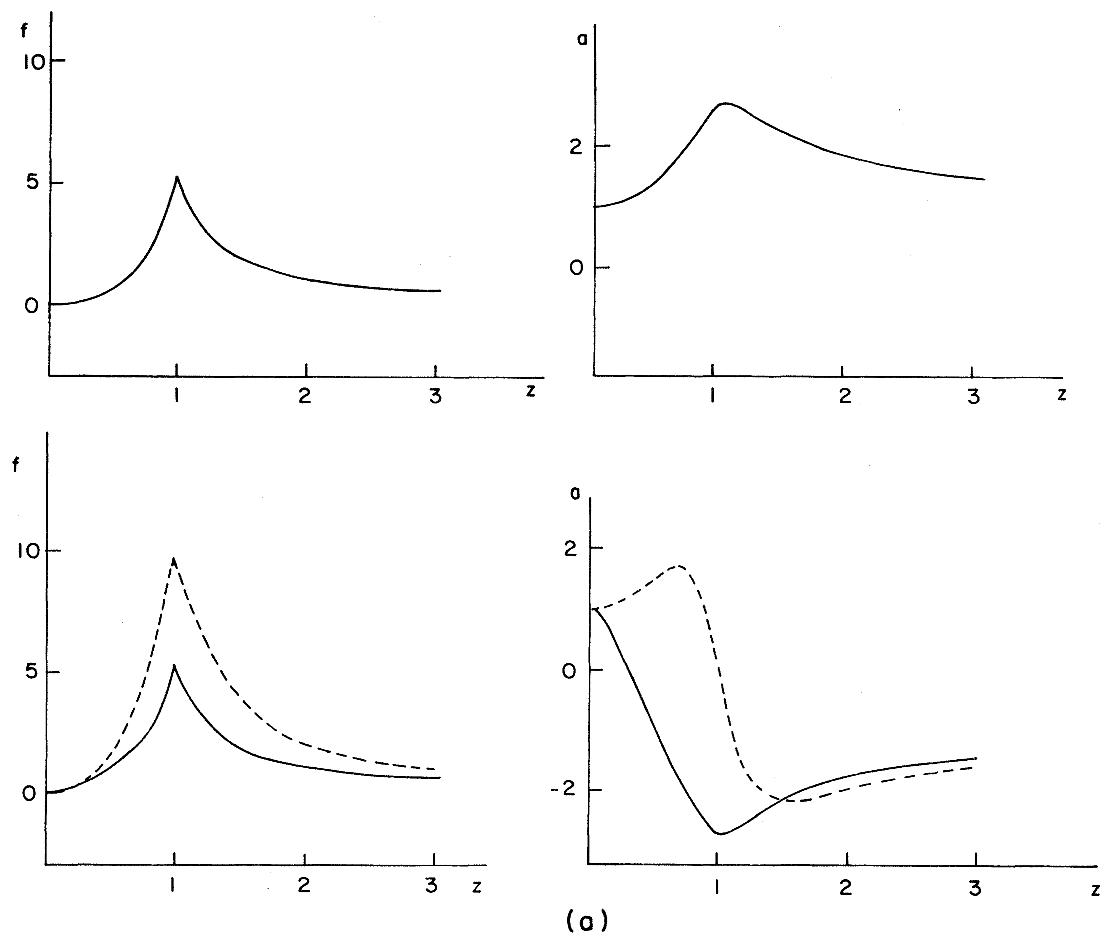

FIG. 2. (a) The $n=0$ and $n=1$ solutions for $f(z)$ and $a(z)$ at $\alpha=30$. Bifurcating solutions are superimposed, with solutions of lower energy indicated by solid lines. (b) The $n=2$ and $n=3$ solutions for $\alpha=30$.

charges offers special understanding of the nature of the classical ground state. The function space of gauge potentials is seen to partition into topologically inequivalent sectors, each labeled by the charge M and energetically inaccessible to one another. Characteristics of configurations in each sector, such as color screening of the external charge, may be quite different.

We have seen this explicitly in the case of SU(2) gauge potentials coupled to a well-defined δ-shell external source. The spherically symmetric subspace is seen to partition into two sectors, one topologically trivial (Abelian-Coulomb) and the other with unit topological charge [Witten's *Ansatz* (3)]. Color screening is complete in the $M=1$ case, while for $M=0$, there is no screening whatsoever.

We have also seen that topological charge has consequences for solutions of the static Yang-Mills equations even outside the spherically symmetric subspace. For example, in conjunction with information derived from perturbative studies[1-5] about the Coulomb potential, we can self-consistently argue that the classical minimum in the $M=0$ sector must, for sufficiently large coupling, be spherically asymmetric, thus breaking the rotational symmetry of the Lagrangian. More importantly, it appears that a spherically symmetric source is never totally screened in the absence of topological charge.

Total screening solutions in the $M=1$ sector were considered in detail, reproducing and extending the work of Jackiw, Jacobs, and Rebbi.[8] In particular, we find that bifurcation of solutions is associated with a finite number of nodes in the function $a(z)$ and exist only when the external color exceeds certain critical values. The existence of critical couplings for type-II solutions can be explained by the existence of soliton solutions at all scales when the external source is zero, but as bifurcation appears for type-I solutions as well, the latter is a more general phenomenon still in want of a more explicit dynamical explanation.

A number of questions are left unanswered by our investigations. To us, the more important issue is whether or not static, stable solutions with topological charge are always total screening. If true, then some of the elements necessary to invoke a dual-Meissner-type effect to explain color confinement may already be present in the classical theory.

Another question is one of stability. General considerations[14] show that the higher-energy branches of bifurcating solutions are unstable. However, the stability of the lower branches and of the $n=0$ solutions is unknown. Nevertheless, if we restrict destabilizing fluctuations to be purely radial, then the $n=0$ and $n=1$ solutions are stable due to the existence of the topological charge Δa (25).

It would be interesting to investigate time-dependent solutions of the equations of motion (5). We may speculate that, with the introduction of an additional scale, the frequency of time oscillations, partially screened (dyon)

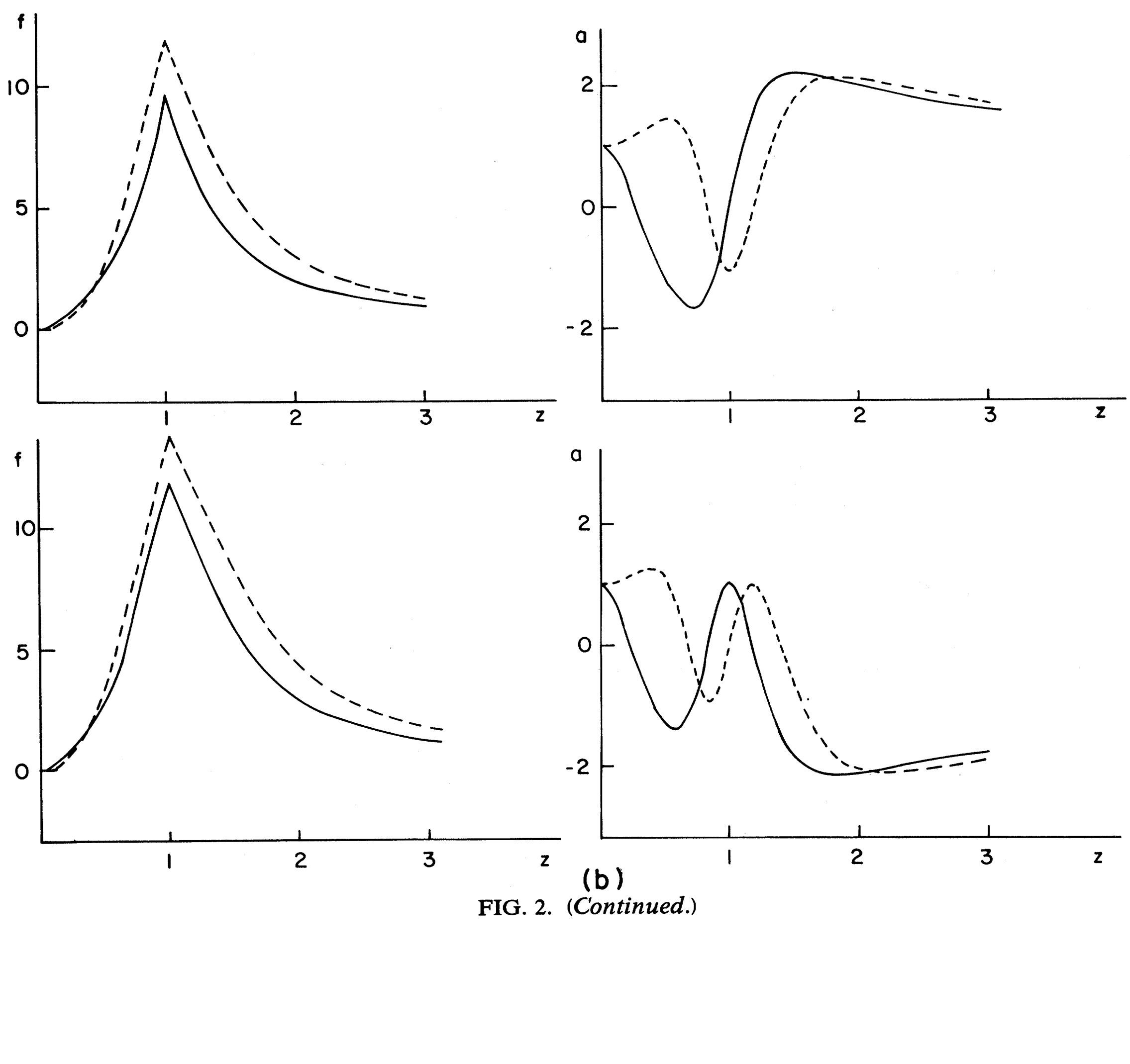

FIG. 2. (*Continued.*)

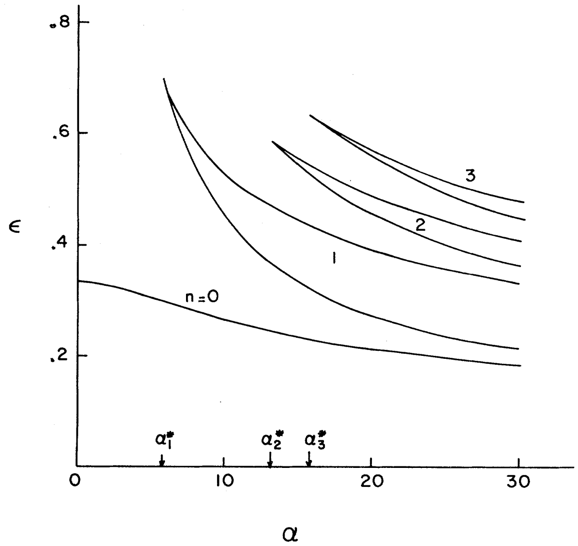

FIG. 3. Total energy ϵ [Eq. (22)] versus α.

solutions could be constructed in analogy to the nontopological, time-dependent solitons of Friedberg, Lee, and Sirlin.[20]

Witten's *Ansatz* represents the unique, nontrivial embedding of the rotation group SO(3) in color SU(2). At least two,[21] and possibly more,[22] inequivalent embeddings are possible in color SU(3), one of which is a simple generalization of Witten's *Ansatz*. The equations of motion in the latter case are identical to (5), and thus a δ-shell source in the appropriate SU(2) subgroup is also totally screened. It is an open question whether or not the alternative embeddings and/or fully SU(3) sources also admit total-screening solutions.

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APPENDIX: THE CONNECTION BETWEEN PARTIAL SCREENING AND ROTATIONAL NONINVARIANCE

In this appendix, we expand on comments made at the end of Sec. III regarding the asymptotic behavior of the gauge field components $\vec{A}_\perp$ color perpendicular to the source. This is determined by Eq. (15a) with the transversality condition (16), which we reproduce here for convenience:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}_\perp)-(ga_0)^2\vec{A}_\perp=0+O(r^{-3-2\delta}) , \qquad (15a)$$

$$\vec{\nabla}\cdot[(ga_0)^2\vec{A}_\perp]=0+O(r^{-4-2\delta}) , \quad r\to\infty . \qquad (16)$$

As a byproduct of the analysis of these equations, we shall also see, by explicit calculation, that the gauge-invariant screening potential

$$V=(\vec{D}\hat{\rho})^a\cdot(\vec{D}\hat{\rho})^a=g^2\vec{A}_\perp^a\cdot\vec{A}_\perp^a$$

has angular dependence at large distances from the source, proving that $M=0$ partial-screening solutions are inherently spherically asymmetric.

Equation (16) suggests that we define the vector field

$$\vec{C}=(ga_0)^2\vec{A}_\perp ,$$

which is purely transverse. We can then rewrite Eq. (15a) in terms of $\vec{C}$, exploiting the fact that, to leading order, ga_0 is simply α_{tot}/r. We obtain

$$-\nabla^2\vec{C}-\frac{4}{r}\frac{\partial}{\partial r}\vec{C}+\frac{2}{r}\vec{\nabla}(\hat{r}\cdot\vec{C})-\frac{4}{r^2}\vec{C}$$
$$-\frac{2}{r^2}(\vec{C}-\hat{r}\hat{r}\cdot\vec{C})-\frac{\alpha_{\text{tot}}^2}{r^2}\vec{C}=0+O(r^{-5-2\delta}) .$$

The procedure then is to expand $\vec{C}$ in a complete set of transverse vector spherical harmonics, determine the large-r behavior of the corresponding coefficients, and thereby deduce the asymptotic form of $\vec{A}_\perp$. The results of this straightforward analysis is that, as r approaches infinity,

$$\vec{A}_\perp\to\sum_{j\geq 1}\{H_j(r)\vec{T}_j+r^2\vec{\nabla}\times[K_j(r)\vec{T}_j]\} , \qquad (A1)$$

where

$$H_j(r)\sim\frac{h_j}{r^{\nu_j+1/2}} ,$$

$$K_j(r)\sim\frac{k_j}{r^{\nu_j+1/2}} ,$$

$$\nu_j=[(j+\tfrac{1}{2})^2-\alpha_{\text{tot}}^2]^{1/2} ,$$

$$\vec{T}_j=\frac{(-i\vec{r}\times\vec{\nabla})}{[j(j+1)]^{1/2}}Y_{j0}(\Omega) ,$$

and h_j and k_j are constants. Hence we see that the leading-order asymptotic behavior of $\vec{A}_\perp$ is given by the $j=1$ terms in the expansion, i.e.,

$$\vec{A}_\perp\sim\frac{1}{r^{\nu_1+1/2}} .$$

Using the full angular dependence of the $j=1$ modes, we may also construct the screening potential at large r:

$$V\to\frac{1}{r^{2\nu_1+1}}\{|h_1|^2\tfrac{1}{2}\sin^2\theta$$
$$+|k_1|^2[2\cos^2\theta+\tfrac{1}{2}\sin^2\theta(\nu_1+\tfrac{1}{2})^2]\}$$
$$+O(r^{-2\nu_2-1}) .$$

The summation over color perpendicular to the source ($a=1,2$) is suppressed. Note that for no choice of $|k_1|^2$ and $|h_1|^2$ can V be rendered spherically symmetric.

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