Renormalization-scheme-invariant QCD and QED: The method of effective charges

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We review, extend, and give some further applications of a method recently suggested to solve the renormalization-scheme-dependence problem in perturbative field theories. The use of a coupling constant as a universal expansion parameter is abandoned. Instead, to each physical quantity depending on a single scale variable is associated an effective charge, whose corresponding Stückelberg—Peterman—Gell-Mann—Low function is identified as the proper object on which perturbation theory applies. Integration of the corresponding renormalization-group equations yields renormalization-scheme-invariant results free of any ambiguity related to the definition of the kinematical variable, or that of the scale parameter Λ, even though the theory is not solved to all orders. As a by-product, a renormalization-group improvement of the usual series is achieved. Extension of these methods to operators leads to the introduction of renormalization-group-invariant Green's function and Wilson coefficients, directly related to effective charges. The case of nonzero fermion masses is discussed, both for fixed masses and running masses in mass-independent renormalization schemes. The importance of the scale-invariant mass $\hat{m}$ is emphasized. Applications are given to deep-inelastic phenomena, where the use of renormalization-group-invariant coefficient functions allows to perform the factorization without having to introduce a factorization scale. The Sudakov form factor of the electron in QED is discussed as an example of an extension of the method to problems involving several momentum scales.

I. INTRODUCTION

The completion in the last few years of several next-to-leading-order perturbative calculations[1] in quantum chromodynamics (QCD) has focused much attention on the question of renormalization-scheme (RS) dependence, and the related issue of the convergence of the perturbative series. This problem has been extensively analyzed in the literature,[1–10] to which we refer the reader for a more complete discussion. We only recall here that *finite-order* perturbative results are arbitrary, to the extent that they depend on the RS, i.e., the choice of the expansion parameter,[9] although the exact result is independent of it. The purpose of this paper is to review, extend, and give some applications of a recently developed[8] approach to these questions, based on the renormalization group, which enjoys the following features.

(i) It provides a RS-invariant formulation of perturbation theory, even though the theory is not solved to all orders, which generalizes to all orders and all processes the well-known Λ_n scheme.[2]

(ii) It gives a RS-invariant, and experimentally testable, necessary condition for the validity of perturbation theory for each process, and identifies the proper objects on which perturbation theory applies as the Stückelberg—Peterman—Gell-Mann—Low functions associated with effective charges defined by each physical quantity.

(iii) The method is free of any ambiguity related to arbitrariness of parametrization. For instance, it yields results independent of the definition of the kinematic scale of the process, or of the definition of the scale parameter Λ of QCD.

(iv) It leads to a renormalization-group improvement of perturbation theory, giving quantitatively different predictions in those cases where large perturbative corrections occur.

It is often pointed out that, whereas the RS-dependence problem is in principle the same in QCD and in QED, its resolution appears to be a less pressing issue in the latter theory, because of the existence of a "natural" canonical RS, where perturbation theory appears to converge quite well. We would like to observe that there is another facet to this remark: While the problem is indeed less academic in QCD, it is also clearer how to solve it in this theory. Indeed, the solution suggested in Ref. 8 is based on a specific feature of renormalizable theories, viz., the *dimensional transmutation*[11] which also exists in QED, but is more conspicuous in QCD. The point is that in a renormalized field theory, the free parameter which corresponds to the coupling constant in the Lagrangian is a *scale* parameter. Consequently, the coupling constant "runs," and predictions for physical quantities σ in QCD are ultimately of the form $\sigma=F(Q^2/\Lambda^2)$, where no dimensionless coupling constant explicitly appears, and where the free parameter Λ cannot obviously be used at the same time as an expansion parameter (as is the case for the fine-structure constant α in QED). This fact makes the RS-dependence problem almost trivial, since it now reduces to the mere freedom of rescaling Λ (or Q). To the problem so drastically simplified, the approach of Ref. 8 brings an equally simple answer, which is based on this most basic feature of the renormalization-group analysis, rather than on general properties of "approximation theory."[9] Furthermore, we shall see that the RS problem can be solved on the same basis in low-energy

QED (high-energy QED is identical to QCD from the point of view of RS dependence), provided one makes the dimensional transmutation manifest by considering α as a function of the electron mass m.

The paper is organized as follows. In Sec. II, the method of Ref. 8, for which we shall adopt the name[10] "method of effective charges," is reviewed, and its main features discussed. Section III gives its connection with ordinary perturbation theory, and shows that it represents a renormalization-group improvement of perturbation theory, such that the general Stückelberg-Peterman renormalization-group equations are automatically satisfied. In Sec. IV, we show that Green's functions and Wilson coefficient functions can also be related to effective charges, associated with renormalization-group (and RS-) invariant Green's functions and Wilson coefficient functions. Section V extends the method of effective charges to the case where fermion masses are present; both fixed masses and running masses in mass-independent RS's are considered. We emphasize the use of the scale-invariant mass as allowing to give a RS-invariant solution (within the class of mass-independent RS's) and, for zero-momentum observables, we introduce a new Callan-Symanzik function, which represents the variation with respect to the scale-invariant mass. Some applications are considered in Sec. VI. We show how the use of renormalization-group-invariant operators and Wilson coefficient functions allows us to get rid completely of the "factorization scale ambiguity" in deep-inelastic processes, and recover and generalize some simple formulas due to Bardeen and Buras,[12] which are naturally associated with effective charges. We discuss in some detail the longitudinal structure function in electroproduction. The Sudakov form factor of the electron in QED is discussed as an extension of the method to multiscales problems. Section VII contains our conclusions. The connection of the present method with Stevenson's approach[9] is discussed in Appendix A. For completeness, we also comment on the alternative procedure of Brodsky, Lepage, and Mackenzie[7] in Appendix B.

II. THE METHOD OF EFFECTIVE CHARGES: A GENERALIZED GELL-MANN–LOW APPROACH

In this section, we shall restrict ourselves for definiteness to the case of quantum chromodynamics (QCD) with massless quarks, although most of the following considerations apply equally well (with obvious changes) to any renormalizable field theory with a single dimensionless coupling constant in the Lagrangian.

Consider a dimensionless physical (or more generally, renormalization-group-invariant) quantity $\sigma(Q^2)$, depending upon a single kinematical scale variable Q^2, and calculable for large Q^2 in perturbation theory [as a standard example, one may think of $R^{e^+e^-}(Q^2) = \sigma(e^+e^-\to\text{hadrons})/\sigma(e^+e^-\to\mu^+\mu^-)$]. Assume the expansion of σ in powers of a renormalized coupling constant $\alpha_s(\mu^2)$ (μ is the renormalization point) takes the generic form

$$\sigma(Q^2)=A+B[\alpha_s(\mu^2)]^d[1+\sigma_1(Q^2/\mu^2)\alpha_s(\mu^2)+\sigma_2(Q^2/\mu^2)\alpha_s^2(\mu^2)+\cdots], \tag{2.1}$$

where d may be noninteger, and A,B are constants. It is well known that Eq. (2.1), when truncated to any finite order, is afflicted by ambiguities, related to the arbitrariness of the definition of $\alpha_s(\mu^2)$, which leads to the so-called renormalization-scheme-dependence problem.[1-10] This problem is greatly simplified when one considers, instead of Eq. (2.1), the asymptotic expansion of $\sigma(Q^2)\equiv F(Q^2/\Lambda^2)$ in powers of $1/\ln(Q^2/\Lambda^2)$ (where Λ is the scale parameter of QCD). Using the renormalization group one gets

$$\sigma(Q^2)=A+B\left[\frac{1}{\beta_1\ln(Q^2/\Lambda^2)}\right]^d \times\left[1+\frac{\sigma_1(1)+dC\beta_1^2-d(\beta_2/\beta_1)\ln\ln(Q^2/\Lambda^2)}{\beta_1\ln(Q^2/\Lambda^2)}\right]. \tag{2.2}$$

β_1 and β_2 are, respectively, the one- and two-loop β function coefficients:

$$\mu^2\frac{\partial\rho}{\partial\mu^2}=\beta(\rho)=-\beta_1\rho^2-\beta_2\rho^3+\cdots, \tag{2.3}$$

$$\beta_1=11-\tfrac{2}{3}f \qquad (f=\text{number of flavors}),$$

$$\beta_2=102-\tfrac{38}{3}f$$

where we put, for convenience $\rho\equiv\alpha_s/4\pi$ and C is a constant which depends solely upon the definition of the QCD scale Λ (or that of the external scale Q). The only[6] arbitrariness in Eq. (2.2) reduces to the freedom of redefining Λ (or Q): performing the rescaling $\Lambda\to\bar{\Lambda}=\lambda\Lambda$ [or $Q\to\bar{Q}=(1/\lambda)Q$], C changes to $\bar{C}=C-(\ln\lambda^2)/\beta_1$. These transformations do not leave Eq. (2.2), truncated to any finite order, invariant, but the scheme dependence can now be parametrized with just one number λ.

The solution to the RS problem proposed in Ref. 8 takes care in the simplest possible way of this remaining difficulty, since it yields, in effect, the expansion of *the inverse function* $Q^2/\Lambda^2=F^{-1}(\sigma)$. Clearly, dealing with $F^{-1}(\sigma)$ allows us to get rid of any ambiguity related to the definition of Λ (or of Q), a redefinition of one of these scales resulting only in a trivial, and controllable, overall rescaling of $F^{-1}(\sigma)$. Furthermore, the latter property remains true *even if the theory is not solved to all orders,* i.e., when the exact $F^{-1}(\sigma)$ is replaced by an approximation. Apart from this rescaling, $F^{-1}(\sigma)$ depends solely on the physical quantity considered, and is therefore a RS-invariant object. To obtain the expansion of $F^{-1}(\sigma)$, one introduces the coupling constant $\bar{\alpha}_s(Q^2)$ of the particular RS where *all* higher-order corrections to σ vanish, i.e., $\bar{\alpha}_s(Q^2)$ is defined through all orders by the identity

$$\sigma(Q^2)\equiv A+B[\bar{\alpha}_s(Q^2)]^d. \tag{2.4}$$

We call $\bar{\alpha}_s$ the *effective charge* associated to σ. It coincides with the strong coupling constant effectively extracted by experimentalists from a *leading-order* analysis of the experimental data for σ [since Eq. (2.4) looks like a leading-order result]. The QCD prediction for $\bar{\alpha}_s(Q^2)$ [from which that for $\sigma(Q^2)$ follows trivially knowing A, B, and d] is done in the standard way in terms of the generalized Gell-Mann—Low function $\bar{\beta}$ associated to σ, defined by the equation

$$Q^2\frac{\partial\bar{\rho}}{\partial Q^2}=\bar{\beta}(\bar{\rho})=-\beta_1\bar{\rho}^2-\beta_2\bar{\rho}^3-\bar{\beta}_3\bar{\rho}^4+\cdots\ , \tag{2.5}$$

where $\bar{\rho}\equiv\bar{\alpha}_s/4\pi$. $\bar{\beta}$ is associated to σ just in the same way as the Gell-Mann—Low function ψ is associated to the Fourier transform of the potential between two charges in QED, a particular example of $\sigma(Q^2)$. The first two coefficients β_1 and β_2 are universal and given in Eq. (2.3). Higher-order coefficients $\bar{\beta}_3$, $\bar{\beta}_4$, etc., are *process-dependent, parameter-free predictions of QCD,* and, together with the scale introduced below, summarize in a RS-invariant manner the information contained in the Feynman diagrams for σ. It is convenient to write the *exact* solution of Eq. (2.5) in the form

$$\beta_1\ln\frac{Q^2}{\Lambda^2}=\frac{1}{\bar{\rho}}+\frac{\beta_2}{\beta_1}\ln(\beta_1\bar{\rho})+K_1+\int_0^{\bar{\rho}}dx\left[\frac{1}{x^2}-\frac{\beta_2}{\beta_1}\frac{1}{x}+\frac{\beta_1}{\bar{\beta}(x)}\right]\ , \tag{2.6a}$$

where K_1 is the coefficient of the next-to-leading-order correction to $\bar{\rho}(Q^2)$ at $\mu^2=Q^2$ in the RS characterized by the coupling constant $\rho(\mu^2)$, i.e., we have (see Sec. III)

$$\bar{\rho}(Q^2)=\rho(\mu^2)\left[1+\left[-\beta_1\ln\frac{Q^2}{\mu^2}+K_1\right]\rho(\mu^2)+\cdots\right]\ . \tag{2.7}$$

Correspondingly,[3] Λ is the scale parameter associated in the standard way to this scheme,

$$\rho(\mu^2)\underset{\mu^2\to\infty}{\sim}\frac{1}{\beta_1\ln\frac{\mu^2}{\Lambda^2}}-\frac{\beta_2}{\beta_1}\frac{\ln\ln\frac{\mu^2}{\Lambda^2}}{\left[\beta_1\ln\frac{\mu^2}{\Lambda^2}\right]^2} \tag{2.8}$$

with no $O(1/\ln^2\mu^2/\Lambda^2)$ corrections.

A useful equivalent form of Eq. (2.6a) is

$$\beta_1\ln\frac{Q^2}{\bar{\Lambda}^2}=\frac{1}{\bar{\rho}}+\frac{\beta_2}{\beta_1}\ln(\beta_1\bar{\rho})+\int_0^{\bar{\rho}}dx\left[\frac{1}{x^2}-\frac{\beta_2}{\beta_1}\frac{1}{x}+\frac{\beta_1}{\bar{\beta}(x)}\right]\ , \tag{2.6b}$$

where we absorbed the constant K_1 into an "effective scale" $\bar{\Lambda}$ defined by the relation

$$\beta_1\ln\frac{\bar{\Lambda}^2}{\Lambda^2}=K_1\ . \tag{2.9}$$

$\bar{\Lambda}$ is a RS-invariant, but process-dependent, quantity, which represents the boundary condition for Eq. (2.5), and is related to $\bar{\rho}$ by the obvious analog of Eq. (2.8). Note that the RS dependence of Λ and K_1 cancels in Eq. (2.9).

Comparison of Eqs. (2.4) and (2.6a) yields immediately the function $F^{-1}(\sigma)$. Since Eq. (2.6a) is exact, and K_1 is known from a next-to-leading-order calculation of σ, any approximation to Eq. (2.6a) can be viewed as an approximation to the $\bar{\beta}$ function. We therefore arrive at the important result that *perturbation theory, in the present approach, is essentially the perturbative expansion of each $\bar{\beta}$ function.* Dropping the integral on the right-hand side of Eq. (2.6a), one obtains a simple approximation, which corresponds to keeping only the first two terms in the expansion of $1/\bar{\beta}$:

$$\frac{1}{\bar{\beta}(\bar{\rho})}=-\frac{1}{\beta_1}\left[\frac{1}{\bar{\rho}^2}-\frac{\beta_2}{\beta_1}\frac{1}{\bar{\rho}}-\left[\frac{\bar{\beta}_3}{\beta_1}-\frac{\beta_2^2}{\beta_1^2}\right]+\cdots\right]\ . \tag{2.10}$$

This approximation gives

$$\beta_1\ln\frac{Q^2}{\Lambda^2}=\frac{1}{\bar{\rho}}+\frac{\beta_2}{\beta_1}\ln(\beta_1\bar{\rho})+K_1+O(\bar{\rho})\ . \tag{2.11a}$$

Alternatively, one may use the standard two-loop approximation to $\bar{\beta}$, i.e., keep only the first two terms in Eq. (2.5), which gives

$$\beta_1\ln\frac{Q^2}{\Lambda^2}=\frac{1}{\bar{\rho}}-\frac{\beta_2}{\beta_1}\ln\left[\frac{1}{\beta_1}\left[\frac{1}{\bar{\rho}}+\frac{\beta_2}{\beta_1}\right]\right]+K_1+O(\bar{\rho})\ . \tag{2.11b}$$

Equations (2.11a) and (2.11b) are, of course, equivalent, up to $O(\bar{\rho})$ terms. To this order, and to this order only (because of the two-loop universality of $\bar{\beta}$), they are also equivalent[8] to the choice

$$\mu^2=Q^2\exp\left[-\frac{K_1}{\beta_1}\right]\ , \tag{2.12}$$

which makes the next-to-leading-order correction in Eq. (2.7) vanish[13] (see Sec. III). The $O(\bar{\rho})$ corrections to Eq. (2.11) depend on the three-loop coefficient $\bar{\beta}_3$. For instance, if the expansion Eq. (2.10) is used, one gets

$$\beta_1\ln\frac{Q^2}{\Lambda^2}=\frac{1}{\bar{\rho}}+\frac{\beta_2}{\beta_1}\ln(\beta_1\bar{\rho})+K_1+\left[\frac{\bar{\beta}_3}{\beta_1}-\frac{\beta_2^2}{\beta_1^2}\right]\bar{\rho}+O(\bar{\rho}^2)\ . \tag{2.13}$$

In such a way, a systematic expansion of Q^2/Λ^2 as a function of $\bar{\rho}$ (or, equivalently, of σ) is achieved. It is therefore clear that the present effective-charge method works in an unambiguous way to all orders, not just to next-to-leading order; beyond two loops, however, it is no more equivalent to a choice of renormalization point. To conclude this section, we stress the following points.

(1) The method of effective charges abandons the use of a universal, fixed RS, and deemphasizes the discussion of convergence properties of ordinary perturbation series in

powers of a fixed renormalized coupling constant $\alpha_s(\mu^2)$ [there are no more such series left with the definition of Eq. (2.4)]. Instead, each physical quantity σ is completely characterized in a RS-invariant manner, by an effective scale $\overline{\Lambda}$ (related to some standard scale parameter Λ of QCD) and a generalized Gell-Mann—Low function $\overline{\beta}$. The various $\overline{\beta}$ functions so obtained are now recognized as the relevant perturbative objects. Such a step is quite natural in a theory like QCD where, by the effect of dimensional transmutation, the only parameter (for zero mass quarks) is a scale Λ: Eq. (2.6) takes into account this basic fact in the most direct manner. We note in this respect that the use of a fixed renormalized coupling constant to parametrize physical quantities in the continuum limit does not seem to be necessary at a more fundamental level, in a nonperturbative approach to renormalization. In fact, such an object does not need to appear explicitly in the exact QCD prediction $\sigma = F(Q^2/\Lambda^2)$, where Λ can be defined to be any "spontaneous" mass scale, such as the proton mass, or the string tension, with no reference to a renormalized coupling constant; for such nonperturbative definitions of Λ, Eq. (2.6a) still makes sense, but K_1 is no more calculable by Feynman diagrams, of course.

(2) Consequently, the convergence of perturbation theory, for a given process σ and energy range Q, depends now essentially on the nontrivial convergence properties of each $\overline{\beta}$ function (i.e., on the behavior of $\overline{\beta}_i$, $i \geq 3$). The fact that Eq. (2.4) looks (by construction) like a leading-order result is completely irrelevant in this respect. No artificial and suspicious "fastest apparent convergence" (FAC)[14] criterion needs to be invoked. In fact, the present method provides an unambiguous criterion[15] for the validity of perturbation theory for each process: one has simply to require that the effective charge $\overline{\rho}$ be small enough for the expansion of $\overline{\beta}(\overline{\rho})$ to converge (in the asymptotic-series sense). Up to two loops, this requirement gives the condition

$$\frac{\beta_2}{\beta_1}\overline{\rho} \ll 1 \ . \qquad (2.14a)$$

The usefulness of this condition with respect to higher orders rests on the essential physical assumption (to be checked case by case by detailed calculations) that higher-order $\overline{\beta}_i$'s are well behaved, in the sense that

$$\left| \frac{\overline{\beta}_{i+1}}{\overline{\beta}_i} \right| \sim O\left[\frac{\beta_2}{\beta_1} \right] . \qquad (2.15)$$

In particular, the magnitude of $\overline{\beta}_3$, the first nonuniversal coefficient, has to be checked. Alternatively, one must require that Q be large enough so that condition (2.14a) is satisfied. In the two-loop approximation, this requirement gives the condition

$$Q^2 \gg \overline{\Lambda}^2 = \Lambda^2 \exp\left[\frac{k_1}{\beta_1} \right] , \qquad (2.14b)$$

which also ensures that a solution to Eq. (2.11), when solved for $\overline{\rho}$, exists. Conditions (2.14) are RS invariant, but the domain of validity of perturbation theory becomes now a process-dependent question, which is physically natural.

We further note that the approximations to Eq. (2.6a) based on the expansions of $\overline{\beta}$ or $1/\overline{\beta}$ [Eqs. (2.5) or (2.10)], are not equivalent, to a given order in $\overline{\rho}$, to a truncated expansion in $1/\ln(Q^2/\Lambda^2)$. In particular, the solutions to the transcendental equation (2.11) contain all powers in $1/\ln(Q^2/\Lambda^2)$. Nevertheless, one can check that, when condition Eq. (2.14b) is satisfied, these solutions are given, to a good approximation, by the analog of Eq. (2.8):

$$\overline{\rho}(Q^2) = \frac{1}{\beta_1 \ln \frac{Q^2}{\overline{\Lambda}^2}} - \frac{\beta_2}{\beta_1} \frac{\ln\ln \frac{Q^2}{\overline{\Lambda}^2}}{\left[\beta_1 \ln \frac{Q^2}{\overline{\Lambda}^2} \right]^2} \ . \qquad (2.16)$$

(3) When the next-to-leading-order correction term in Eq. (2.7) is large, the use of Eq. (2.11) [or equivalently, choosing μ as in Eq. (2.12)] amounts, if both the effective charge $\overline{\rho}$ and the RS are "well behaved" (in a sense to be explained in Sec. III), to a resummation of the most important higher-order corrections to Eq. (2.7): a renormalization-group improvement of perturbation theory is achieved (see Sec. III). Conversely, small next-to-leading-order corrections in a given RS [in particular, zero corrections with μ as given in Eq. (2.12)] imply essential agreement between Eq. (2.11) and ordinary perturbation theory, as we have already mentioned. The extent to which the agreement between the method of effective charges and ordinary perturbation theory [with μ chosen as in Eq. (2.12)] will persist in higher orders is also discussed in Sec. III.

(4) The present method generalizes to all processes and to all orders the well-known Λ_n scheme[2] introduced in the second-order analysis of deep-inelastic scattering [see, in particular, Eq. (2.16)]. The process dependence of the various effective scales $\overline{\Lambda}$ [Eq. (2.9)] is the neatest test of the so-called next-to-leading-order QCD effects. The fact that the Λ_n scheme can be so naturally extended to all orders clarify, we believe, its theoretical meaning.

(5) The results obtained by the present method may depend crucially on the choice of the physical quantity to which it is applied.[8,9] For instance, if R_1 and R_2 are two distinct physical quantities, the prediction for the ratio $R = R_1/R_2$ (or the sum $S = R_1 + R_2$) may depend sensitively on whether the effective-charge method is applied separately to R_1 and R_2, or directly to R (or S). As a preliminary, but not precise enough, general orientation, we shall stick to the rule that the first procedure is the correct one, if R_1 and R_2 are related to Feynman diagrams referring to different processes, since R (or S) is in this case more an artificial construct. This viewpoint also has the advantage to exploit more completely the information contained in the Feynman diagrams for R_1 and R_2. Unfortunately, this prescription is not always sufficient to remove all ambiguities (see Sec. VI C for an example). We conclude that the choice of the "right" physical quantities, for which the associated effective charges and $\overline{\beta}$ functions can be expected to be well behaved [see Eq. (2.15)], becomes now the relevant theoretical question (which replaces the previous question[10] of the choice of a "good" RS).

III. CONNECTION WITH ORDINARY PERTURBATION THEORY: RENORMALIZATION-GROUP-IMPROVED PERTURBATION THEORY

A. Renormalization-group structure of perturbative coefficients

Since the effective charge $\bar{\rho}$ is just a particular example of a renormalized coupling constant, the properties of the perturbation expansion of $\bar{\rho}$ (hence of σ) in powers of another renormalized coupling constant ρ follows from (more or less) well-known general renormalization-group relations which we now recall. Subtracting from Eq. (2.6b) its analog for $\rho(\mu^2)$,

$$\beta_1 \ln\frac{\mu^2}{\Lambda^2} = \frac{1}{\rho} + \frac{\beta_2}{\beta_1}\ln(\beta_1\rho) + \int_0^\rho dx \left[\frac{1}{x^2} - \frac{\beta_2}{\beta_1}\frac{1}{x} + \frac{\beta_1}{\beta(x)}\right], \quad (3.1)$$

one gets the *exact* relation

$$\frac{1}{\beta_1} c_1(Q^2/\mu^2) \equiv -\ln\frac{Q^2}{\mu^2} + \ln\frac{\bar{\Lambda}^2}{\Lambda^2} = \int_0^\rho \frac{dx}{\beta(x)} - \int_0^{\bar{\rho}} \frac{dx}{\bar{\beta}(x)}, \quad (3.2)$$

where a limit process is understood at the lower ends of the integrals in Eq. (3.2), and the two-loop universality of β and $\bar{\beta}$ guarantees a finite result is obtained. Taking the derivative of both sides of Eq. (3.2) at fixed Q^2/μ^2 (and $\bar{\Lambda}^2/\Lambda^2$) gives the relation

$$\left.\frac{\partial\bar{\rho}}{\partial\rho}\right|_{Q^2/\mu^2} = \frac{\bar{\beta}(\bar{\rho})}{\beta(\rho)}. \quad (3.3)$$

Equations (3.2) and (3.3) are completely symmetrical with respect to ρ and $\bar{\rho}$. We are now going to take an unsymmetrical viewpoint, by looking at them as implicit equations for $\bar{\rho}$, considered as a function of the two variables $\ln(Q^2/\mu^2)$ and ρ. Solving for $\bar{\rho}$ in Eq. (3.2) as a power series in ρ, one obtains

$$\bar{\rho} = \rho[1 + c_1(Q^2/\mu^2)\rho + c_2(Q^2/\mu^2)\rho^2 + c_3(Q^2/\mu^2)\rho^3 + c_4(Q^2/\mu^2)\rho^4 + \cdots]$$

with

$$\begin{aligned}
c_1(Q^2/\mu^2) &= -\beta_1\ln\frac{Q^2}{\mu^2} + \beta_1\ln\frac{\bar{\Lambda}^2}{\Lambda^2},\\
c_2(Q^2/\mu^2) &= c_1^2(Q^2/\mu^2) + \frac{\beta_2}{\beta_1}c_1(Q^2/\mu^2) + \frac{\bar{\beta}_3-\beta_3}{\beta_1},\\
c_3(Q^2/\mu^2) &= c_1^3(Q^2/\mu^2) + \frac{5}{2}\frac{\beta_2}{\beta_1}c_1^2(Q^2/\mu^2) + \left[\frac{3\bar{\beta}_3}{\beta_1} - \frac{2\beta_3}{\beta_1}\right]c_1(Q^2/\mu^2) + \frac{1}{2}\frac{\bar{\beta}_4-\beta_4}{\beta_1},\\
c_4(Q^2/\mu^2) &= c_1^4(Q^2/\mu^2) + \frac{13}{3}\frac{\beta_2}{\beta_1}c_1^3(Q^2/\mu^2) + \left[6\frac{\bar{\beta}_3}{\beta_1} - 3\frac{\beta_3}{\beta_1} + \frac{3}{2}\frac{\beta_2^2}{\beta_1^2}\right]c_1^2(Q^2/\mu^2)\\
&\quad + \left[\frac{2\bar{\beta}_4}{\beta_1} - \frac{\beta_4}{\beta_1} + 3\frac{\beta_2}{\beta_1}\frac{\bar{\beta}_3-\beta_3}{\beta_1}\right]c_1(Q^2/\mu^2) + \frac{1}{3}\frac{\bar{\beta}_3-\beta_3}{\beta_1}\left[\frac{5\bar{\beta}_3}{\beta_1} - \frac{6\beta_3}{\beta_1}\right] - \frac{1}{6}\frac{\beta_2}{\beta_1}\frac{\bar{\beta}_4-\beta_4}{\beta_1} + \frac{1}{3}\frac{\bar{\beta}_5-\beta_5}{\beta_1},
\end{aligned} \quad (3.4)$$

where β_i are the coefficients of the β function.

The general structure of the perturbative coefficients is seen to be

$$c_n(Q^2/\mu^2) = \tilde{c}_n(Q^2/\mu^2) + \frac{1}{n-1}\frac{\bar{\beta}_{n+1}-\beta_{n+1}}{\beta_1} \quad (3.5)$$

with

$$\tilde{c}_n(Q^2/\mu^2) = \sum_{j=0}^{n} a_{j,n} c_1^j(Q^2/\mu^2),$$

where $a_{n,n}=1$, the $a_{j,n}$ depend only on the $\bar{\beta}_i,\beta_i$ coefficients with $i \le n$, and $a_{0,n} \ne 0$ only for $n \ge 4$. In particular, we recover Eqs. (2.7) and (2.9). Noting that $c_1(Q^2/\mu^2)$ can be rewritten as

$$c_1(Q^2/\mu^2) = -\beta_1\ln\frac{Q^2}{\bar{\Lambda}^2} + \beta_1\ln\frac{\mu^2}{\Lambda^2}. \quad (3.6)$$

We deduce from Eq. (3.4) that the RS dependence of the c_i's is entirely controlled[9] by the quantities $\ln(\mu^2/\Lambda^2)$ and the β_i's, which is not surprising, since they determine the expansion parameter ρ by Eq. (3.1). On the other hand, the effective charge $\bar{\rho}$ depends solely on the RS-invariant quantities[17] $\ln(Q^2/\bar{\Lambda}^2)$ and $\bar{\beta}_i$ [cf. Eq. (2.6b)]. The latter are the only physically relevant objects and we see they are mixed in a rather complicated way in the c_i's with irrelevant RS-dependent parameters. We note that the scheme-dependent and scheme-invariant parameters are clearly separated in Eq. (3.2), which represent an explicit solution, in terms of the generalized Stückelberg—Peterman—Gell-Mann—Low function $\bar{\beta}$, to the Stückelberg-Peterman-[18]Callan-Symanzik[19] equations (which express RS independence in differential form):

$$\frac{d\bar{\rho}}{d\lambda_i}=\left[\frac{\partial}{\partial\lambda_i}+\beta_i(\rho)\frac{\partial}{\partial\rho}\right]\bar{\rho}(Q^2/\bar{\Lambda}^2,\bar{\beta}_j,\rho,\lambda_i)=0\ ,\quad i=0,1,\ldots\ , \tag{3.7}$$

where we consider $\bar{\rho}$ as a function of ρ and the RS labeling parameters[9] λ_i (at fixed $Q^2/\bar{\Lambda}^2$ and $\bar{\beta}_j$):

$$\lambda_0=\ln\frac{\mu^2}{\Lambda^2}\ ,\quad \lambda_i=\frac{\beta_{i+2}}{\beta_1}\quad (i\geq 1) \tag{3.8}$$

and[20]

$$\beta_i(\rho)\equiv\left.\frac{\partial\rho}{\partial\lambda_i}\right|_{\substack{\lambda_j\\ i\neq j}}$$

[ρ is itself an implicit function of the λ_i, through Eq. (3.1)].

Reciprocally, one can obtain the coefficients of the β and $\bar{\beta}$ functions from the knowledge of the single $\ln(Q^2/\mu^2)$ and the constant terms in the c_i's. The relevant formulas are essentially contained already in the original article of Gell-Mann and Low.[21] Expanding $\bar{\rho}$ and ρ in powers of $\ln(Q^2/\mu^2)$, one writes

$$\begin{aligned}\bar{\rho}&=\delta_0(\rho)+\delta_1(\rho)\ln\frac{Q^2}{\mu^2}+\delta_2(\rho)\ln^2\frac{Q^2}{\mu^2}+\cdots\ ,\\ \rho&=\gamma_0(\bar{\rho})+\gamma_1(\bar{\rho})\ln\frac{\mu^2}{Q^2}+\gamma_2(\bar{\rho})\ln^2\frac{\mu^2}{Q^2}+\cdots\ .\end{aligned} \tag{3.9}$$

Using Eq. (3.3), one easily derives the relations

$$\begin{aligned}\beta(\rho)&=n\frac{\delta_n(\rho)}{\delta'_{n-1}(\rho)}\ ,\\ \bar{\beta}(\bar{\rho})&=n\frac{\gamma_n(\bar{\rho})}{\gamma'_{n-1}(\bar{\rho})}\ ,\end{aligned} \tag{3.10}$$

where a prime means the derivative with respect to $\rho,\bar{\rho}$ and also

$$\bar{\beta}(\bar{\rho})=\delta_1(\rho)$$

with

$$\bar{\rho}=\delta_0(\rho)\ . \tag{3.11}$$

Equation (3.10) displays the well-known fact that the knowledge of the single $\ln(Q^2/\mu^2)$ and constant terms determine all higher-order logarithms, as well as the β and $\bar{\beta}$ functions. In practice, however, it is easier to get the $n+1$ $\bar{\beta}$ coefficient $\bar{\beta}_{n+1}$ from the knowledge of c_n and $\tilde{c}_n$ [see Eq. (3.5)], i.e., the first n perturbative coefficients, once the first $n+1$ β coefficients of a given base RS [say, the minimal-subtraction (MS) scheme] have been computed in a simpler way by standard means (i.e., from the vertex and wave-function renormalization constants); note also that $\tilde{c}_{n+1}$ can in turn be predicted. For instance, Eq. (3.4) shows that $\bar{\beta}_3$ is known from the knowledge of β_1, β_2, β_3, and c_1,c_2, which in turn predicts $\tilde{c}_3$.[22]

B. Renormalization-group-improved perturbation theory

The use of Eq. (3.2) with $\beta,\bar{\beta}$ (or $1/\beta,1/\bar{\beta}$) truncated to nth order clearly amounts to a resummation to all orders of the first n leading powers of c_1 (plus some parts of the other nonleading powers) [see Eq. (3.4)]. Generalizing standard practice, we call this resummation *renormalization-group-improved perturbation theory.*[25] As an example, we give the one- and two-loop renormalization-group-improved formulas (using the expansion of the inverse β and $\bar{\beta}$ functions):

$$c_1(Q^2/\mu^2)=-\left[\frac{1}{\bar{\rho}}-\frac{1}{\rho}\right]\quad\text{(one loop)}\ , \tag{3.12}$$

$$c_1(Q^2/\mu^2)=-\left[\frac{1}{\bar{\rho}}-\frac{1}{\rho}+\frac{\beta_2}{\beta_1}\ln\frac{\bar{\rho}}{\rho}\right]\quad\text{(two loop)}\ . \tag{3.13}$$

We note that the one-loop improved formula coincides with the ordinary next-to-leading formula for the expansion of the inverse charge $1/\bar{\rho}$. The usefulness of renormalization-group-improved perturbation theory depends essentially on the magnitude of $|c_1|$. We distinguish two cases:

(a) Large $|c_1|$, i.e., $|c_1|\gg\beta_2/\beta_1$. This is the case where the improved formulas are expected to be useful, since the dominant part of each c_i are then resummed, provided the $\bar{\beta}$ and β functions are "well behaved" in the sense of Eq. (2.15), i.e.,

$$\left|\frac{\bar{\beta}_{i+1}}{\bar{\beta}_i}\right|\sim\left|\frac{\beta_{i+1}}{\beta_i}\right|\sim\frac{\beta_2}{\beta_1}\ .$$

We stress that there is no correlation, *a priori*, between the magnitude of $|c_1|$ and that of the β_i and $\bar{\beta}_i$ coefficients: c_1 expresses a relation between two couplings, whereas β_i and $\bar{\beta}_i$ are intrinsic to each coupling. Now, $|c_1|$ may be large either because $|\ln(Q^2/\mu^2)|$ is large, or because $|\ln(\bar{\Lambda}^2/\Lambda^2)|$ (the "constant term") is large: these two kind of logarithms play an entirely symmetrical role. The case of large $|\ln(Q^2/\mu^2)|$ is the most familiar one, but that of large $|\ln(\bar{\Lambda}^2/\Lambda^2)|$ has attracted much attention since the advent of next-to-leading-order QCD calculations, where $|c_1|$ was found to be large in usual schemes even after putting $\mu=Q$. Actually, the distinction between these two cases is essentially a matter of convention: one can always pass from large $|\ln(Q^2/\mu^2)|$ to large $|\ln(\bar{\Lambda}^2/\Lambda^2)|$ merely by redefining Q (or μ), for given values of $\bar{\rho}$ and ρ (hence without changing the value of c_1); in particular, there is always a convention for which $\ln(\bar{\Lambda}^2/\Lambda^2)=0$, for given ρ and $\bar{\rho}$. It should therefore be clear that the present method represents a renormalization-group improvement of perturbation theory in the most classical sense.[26]

Assuming β and $\bar{\beta}$ are well behaved, there is still a limitation on the applicability of improved perturbation theory, which stems from the condition that the solution of Eq. (3.2), considered as an implicit equation for $\bar{\rho}$ with β and $\bar{\beta}$ (or their inverses) truncated at a given order, exists and satisfies the consistency condition Eq. (2.14a) [assuming ρ satisfies the analog condition $(\beta_2/\beta_1)\rho\ll 1$].

For instance, the solution of Eq. (3.12),

$$\bar{\rho}=\frac{\rho}{1-c_1(Q^2/\mu^2)\rho}\ , \tag{3.14}$$

exists only if

$$c_1(Q^2/\mu^2)\rho<1 \tag{3.15a}$$

[the value $\rho=1/c_1(Q^2/\mu^2)$ is the Landau ghost], and furthermore condition Eq. (2.14a) requires

$$\left[\frac{\beta_2}{\beta_1}+c_1(Q^2/\mu^2)\right]\rho<1\ . \tag{3.15b}$$

So, if $(\beta_2/\beta_1)\rho<1$ and $c_1<0$, the conditions Eqs. (3.15) are automatically satisfied, but they impose a restriction on $c_1\rho$ if $c_1>0$; in particular, next-to-leading corrections $c_1\rho$ larger than 100% cannot be handled in this latter case simply because $\bar{\rho}$ becomes too large.

Another way to see that $c_1\rho<1$ when $c_1>0$ is to let the renormalization point $\mu\to\infty$ (for given $\bar{\rho}$ and Q); then

$$c_1(Q^2/\mu^2)\sim\beta_1\ln\frac{\mu^2}{\Lambda^2}\to+\infty\ ,$$

but

$$c_1(Q^2/\mu^2)\rho=-\beta_1\ln\frac{Q^2}{\Lambda^2}\rho+\beta_1\ln\frac{\mu^2}{\Lambda^2}\rho\underset{\mu^2\to\infty}{\sim}1-\frac{\ln(Q^2/\bar{\Lambda}^2)}{\ln(\mu^2/\Lambda^2)}\ ,$$

i.e., the correction tends to 100% from below.

(b) Small $|c_1|$. In such a case, one naturally expects agreement between ordinary and improved perturbation theory. We note that it is always possible to choose μ as in Eq. (2.12), in such a way that $c_1=0$ (Ref. 13). Let us examine the consequences of this choice, in the two-loop, and beyond the two-loop, approximation for β and $\bar{\beta}$.

(i) Up to two loops. We wish to compare the predictions of Eq. (3.13) with those of ordinary perturbation theory in next-to-leading order:

$$\bar{\rho}=\rho[1+c_1(Q^2/\mu^2)\rho]\ .$$

When $c_1=0$, it is clear that they are the same, i.e., they both predict $\bar{\rho}=\rho$ [assuming $(\beta_2/\beta_1)\rho<1$, $(\beta_2/\beta_1)\bar{\rho}<1$]. Therefore, in this approximation, all charges are universal when $c_1=0$. This fact is an immediate consequence of the two-loop universality of the β functions, which implies that all effective charges are related by a mere rescaling of their argument up to two loops.

(ii) Beyond two loops. It is clear from Eq. (3.4) that, if β_i and $\bar{\beta}_i$ are well behaved, the coefficients c_i are under control when $c_1=0$, so that the ordinary perturbation expansion converges and should give an answer close to that obtained from the improved formulas [provided $(\beta_2/\beta_1)\rho\ll 1$]. For instance, going to third order in the ordinary expansion, and putting $c_1=0$, one gets

$$\bar{\rho}=\rho\left[1+\frac{\bar{\beta}_3-\beta_3}{\beta_1}\rho^2\right]\ . \tag{3.16}$$

On the other hand, the improved formula for $\bar{\rho}$, with the β and $\bar{\beta}$ functions truncated to three loops, predicts, when reexpanded in powers of ρ [see Eq. (3.4)]:

$$c_3(Q^2/\mu^2)=\tilde{c}_3(Q^2/\mu^2)=0\ ,$$

which means that improved and nonimproved formulas in fact agree to $O(\rho^4)$. Actually, this exact agreement is an accident, peculiar to this order, and to the use of truncated expansions for β and $\bar{\beta}$; if instead the inverse β and $\bar{\beta}$ functions had been truncated to the same order, the reader can easily check that the improved and nonimproved results would differ to $O(\rho^4)$ by the amount

$$\frac{\bar{\beta}_3-\beta_3}{\beta_1}\times\frac{\beta_2}{\beta_1}\times\rho^4\ ,$$

i.e., the relative correction is $(\beta_2/\beta_1)\rho$, which is still satisfactory when $(\beta_2/\beta_1)\rho\ll 1$.

C. Well-behaved RS: Definition of good universal conventions

The previous results suggest that we define a well-behaved RS as one whose β function is well behaved, i.e., satisfy the condition

$$\left|\frac{\beta_{i+1}}{\beta_i}\right|\sim O\left[\frac{\beta_2}{\beta_1}\right]\ . \tag{3.17}$$

Ordinary perturbation theory in such RS will then give reliable results for those physical quantities whose associated $\bar{\beta}$ function is also well behaved [in the sense of Eq. (2.14a)], provided the renormalization point μ is chosen as in Eq. (2.12), so that $c_1=0$. For these well-behaved RS, and well-behaved effective charges $\bar{\rho}$, the fastest apparent convergence choice of Eq. (2.12) is therefore *a posteriori* justified. We conclude that the method of effective charges is compatible with the existence of a well-behaved RS, i.e., of a "good universal convention" in the sense of Ref. 10, and actually, if successful, implies its existence, although such RS are by no means unique. Examples of well-behaved RS are easy to find, and in fact all familiar RS's seem to share this property. The most convenient one is probably the MS scheme,[27] its β_3 coefficient has been computed[28] and it is well behaved,

$$\frac{\beta_3}{\beta_2}=7.92\ ,\quad \frac{\beta_2}{\beta_1}=6.16\ \text{(for 4 flavors)}\ . \tag{3.18}$$

We also note that the 't Hooft RS, defined[29] by

$$\beta(\rho)\equiv-\beta_1\rho^2-\beta_2\,\rho^3, \tag{3.19}$$

is well behaved by construction. Momentum-subtraction RS's[3] also appear to be well behaved (see below). Anyway, there is no particular interest to use these RS's to make predictions, since the latter can be given in a simpler and entirely RS-invariant way using the $\bar{\beta}$ functions only.

What about well-behaved effective charges? Once a well-behaved RS has been adopted, and μ chosen as in Eq. (2.12), the appearance of anomalously large higher-order coefficients in the expansion of $\bar{\rho}$ is a signal that the $\bar{\beta}$ function is not well behaved. There is then little hope to make perturbative predictions for $\bar{\rho}$. To the author's knowledge, calculations allowing to extract the values of three-loop $\bar{\beta}_3$ coefficients for effective charges of physical interest have not yet been completed. A plausible model for such effective charges, however, is provided by the (gauge-dependent) coupling constants defined by various three-point vertices. For the latter, the two-loop calculations needed to extract $\bar{\beta}_3$ [once β_3 is known (Ref. 28)] have been performed,[30] and one finds that their $\bar{\beta}_3$ are also well behaved (which also shows that momentum-subtraction schemes are examples of "good universal conventions"[10]).

IV. RENORMALIZATION-GROUP-INVARIANT OPERATORS, GREEN'S FUNCTIONS, AND WILSON-COEFFICIENT FUNCTIONS

In this section, we show that not only physical quantities, but also Green's functions (and Wilson coefficients) can be related to renormalization-group-invariant effective charges.

A. Renormalization-group-invariant operators

Consider the simplest case of a multiplicatively renormalizable bare operator O_b which does not mix under renormalization. The basic statement of multiplicative renormalizability, in the framework of cutoff regularization, is that

$$O_R(\mu^2)=\lim_{M^2\to\infty} O_b(M^2)Z^{-1}(\mu^2,M^2)\ , \tag{4.1}$$

where M is the cutoff, μ is the renormalization point, $Z(\mu^2,M^2)$ the operator wave-function renormalization constant, and $O_R(\mu^2)$ the cutoff-independent renormalized operator. Consequently, the anomalous dimension $\gamma_R(\mu^2)$, defined by the relation

$$\gamma_R(\mu^2)=\lim_{M^2\to\infty} \mu^2\frac{\partial Z(\mu^2,M^2)}{\partial\mu^2}Z^{-1}(\mu^2,M^2)\ , \tag{4.2}$$

is cutoff independent. Let us from now on assume that terms which vanish as $M^2\to\infty$ have been dropped from $Z(\mu^2,M^2)$, order by order in perturbation theory, and that only logarithmic-divergent and constant terms have been retained. Then the limit operation in Eqs. (4.1) and (4.2) can be omitted, i.e., we have

$$\gamma_R(\mu^2)=\mu^2\frac{\partial Z(\mu^2,M^2)}{\partial\mu^2}Z^{-1}(\mu^2,M^2) \tag{4.3a}$$

identically in M^2. We now observe that Eq. (4.3a) implies that the μ and the M dependences of $Z(\mu^2,M^2)$ factorize, i.e., that we can write

$$Z(\mu^2,M^2)=Z_R(\mu^2)Z_b{}^{-1}(M^2)$$

with (4.4)

$$\gamma_R(\mu^2)=\mu^2\frac{\partial Z_R}{\partial\mu^2}Z_R{}^{-1}(\mu^2)\ ,$$

where the functions Z_R and Z_b depend, respectively, on the operator renormalization scheme and the regularization method considered. From Eq. (4.1) (with the limit operation omitted), and Eq. (4.4), we deduce

$$O_R(\mu^2)\times Z_R(\mu^2)=O_b(M^2)\times Z_b(M^2)\equiv O_{\text{inv}}\ , \tag{4.5}$$

where we introduced O_{inv}, the renormalization-group-invariant operator[31] associated to O_b. O_{inv} depends neither on the cutoff M, nor on the renormalization point μ. Its overall normalization is, however, arbitrary, since one can always "renormalize":

$$Z_b\to\lambda Z_b\ ,\quad Z_R\to\lambda Z_R\ \text{(where } \lambda \text{ is a constant)}$$

without changing $Z(\mu^2,M^2)$ in Eq. (4.4).

We note also that Eq. (4.4) implies the relation

$$Z^{-1}(\mu^2,M^2)M^2\frac{\partial Z(\mu^2,M^2)}{\partial M^2}=-\gamma_b(M^2)=-M^2\frac{\partial Z_b}{\partial M^2}Z_b{}^{-1}(M^2)\ , \tag{4.3b}$$

where $\gamma_b(M^2)$ is the anomalous dimension describing the cutoff dependence of $O_b(M^2)$ in the continuum limit. Equations (4.3) are equivalent to the Callan-Symanzik equations

$$\left[\mu^2\frac{d}{d\mu^2}+\gamma_R(\mu^2)\right]O_R(\mu^2)=0$$

and (4.6)

$$\left[M^2\frac{d}{dM^2}+\gamma_b(M^2)\right]O_b(M^2)=0\ ,$$

which show the analogy between cutoff and the renormalization point. The cutoff dependence of a bare operator has the same form as the renormalization-point dependence of a renormalized operator. In fact, the bare operator can be identified, in the continuum limit, with an operator renormalized by "minimum (cutoff) subtraction." The functions γ_b and γ_R (and Z_b and Z_R) can be computed in perturbation theory (see next section). Generalization of the previous considerations to the case of operator mixing is easy: Z_b and Z_R (and γ_b and γ_R) are simply replaced by $N\times N$ matrices if O_b is an N-component operator. Equation (4.5) then defines an N-component renormalization-group-invariant operator (O_b and O_R are row vectors) defined up to an arbitrary constant matrix multiplication.

B. Renormalization-group-invariant Green's functions

Next, we consider Green's functions. Take as a simple example the bare propagator (chosen to be dimensionless):

$$\langle\phi_b\phi_b\rangle=\int d^4x\, e^{ip\cdot x}\langle 0\,|\,T\phi_b(x)\phi_b(0)\,|\,0\rangle=G_b(p^2/M^2,\rho_b)\ , \tag{4.7}$$

where ϕ_b is some bare elementary local field operator, and ρ_b the bare coupling constant. We assume G_b is multiplicatively renormalizable. Then, we can extract the cutoff dependence of G_b using the relation

$$\phi_b(x)=\phi_{\rm inv}(x)\frac{1}{[Z_b(M^2)]^{1/2}} .$$

Hence

$$G_b(p^2/M^2,\rho_b)=G_{\rm inv}(p^2)Z_b^{-1}(M^2) \tag{4.8}$$

with

$$G_{\rm inv}(p^2)=\int d^4x\, e^{ip\cdot x}\langle 0\,|\,T\phi_{\rm inv}(x)\phi_{\rm inv}(0)\,|\,0\rangle . \tag{4.9}$$

Equation (4.8) shows that the p and M dependences of G_b factorize. We call $G_{\rm inv}$ the renormalization-group-invariant Green's function associated to G_b. Similarly, if $G_R(p^2/\mu^2,\rho)$ is the corresponding renormalized Green's function, the p and μ dependences factorize, and we have[32]

$$G_R(p^2/\mu^2,\rho)=G_{\rm inv}(p^2)Z_R^{-1}(\mu^2) . \tag{4.10}$$

Equation (4.8) suggests a very simple way to compute $G_{\rm inv}(p^2)$. Taking the logarithmic derivative of Eq. (4.8) with respect to p^2, we get

$$p^2\frac{\partial G_b(p^2/M^2,\rho_b)}{\partial p^2}[G_b(p^2/M^2,\rho_b)]^{-1} =p^2\frac{\partial G_{\rm inv}(p^2)}{\partial p^2}[G_{\rm inv}(p^2)]^{-1} \equiv H(p^2) . \tag{4.11}$$

We see that $H(p^2)$ is cutoff independent, and computable in perturbation theory. In general, no $O(\rho_b^0)$ term is present, and $H(p^2)$ is given by the expansion,[34] in powers of the bare coupling constant ρ_b:

$$H(p^2)=\gamma_1[\rho_b+c_1(p^2/M^2)\rho_b{}^2+\cdots]\equiv\gamma_1\rho_H(p^2) , \tag{4.12}$$

where γ_1 is the one-loop anomalous dimension of ϕ_b, and ρ_H the effective charge associated to H. We note that H is simply the anomalous dimension of ϕ_b in the familiar renormalization convention

$$G_R(p^2/\mu^2,\rho)\,|_{p^2=\mu^2}\equiv 1 , \tag{4.13}$$

since, in this case, $G_{\rm inv}(p^2)\equiv Z_R(p^2)$ [see Eq. (4.10)]. $H(p^2)$ can be calculated directly from the Feynman diagrams for G_b [see Eq. (4.11)]. Then, a simple integration yields $G_{\rm inv}$ (up to an arbitrary multiplicative factor). Let us integrate Eq. (4.11) in terms of the effective charge ρ_H. Using

$$\int \frac{dp^2}{p^2}\rho_H(p^2)=\int dx\frac{x}{\beta_H(x)} ,$$

where β_H is the β function associated to ρ_H, we get

$$\ln G_{\rm inv}(p^2)=-\frac{\gamma_1}{\beta_1}\ln\rho_G(p^2)+{\rm const} ,$$

i.e.,

$$G_{\rm inv}(p^2)={\rm const}\times[\rho_G(p^2)]^{-\gamma_1/\beta_1} , \tag{4.14}$$

where

$$\rho_G=\rho_H\exp\left\{-\int_0^{\rho_H}dx\left[\frac{1}{x}+\frac{\beta_1 x}{\beta_H(x)}\right]\right\} \tag{4.15}$$

is the "integrated effective charge" defined by

$$p^2\frac{d\ln\rho_G}{dp^2}=-\beta_1\rho_H . \tag{4.16}$$

In summary, we have shown that the invariant Green's function $G_{\rm inv}$ can be described in terms of an effective charge ρ_H (or its integral ρ_G). For completeness, we mention that $G_{\rm inv}$ can also be derived easily from the familiar solution of the renormalization-group equation for G_R (or G_b),

$$G_R[p^2/\mu^2,\rho]=G_R[1,\rho(p^2)]\exp\left[\int_\rho^{\rho(p^2)}dx\frac{\gamma_R(x)}{\beta(x)}\right] , \tag{4.17}$$

where we considered γ_R as a function of the coupling ρ, rather than of the scale μ^2. Isolating the singularity at $x=0$ of the integrand in Eq. (4.17), one recovers Eq. (4.10) with

$$Z_R^{-1}(\mu^2)={\rm const}\times[\rho(\mu^2)]^{\gamma_1/\beta_1} \times\exp\left\{-\int_0^{\rho(\mu^2)}dx\left[\frac{\gamma_R(x)}{\beta(x)}+\frac{\gamma_1}{\beta_1}\frac{1}{x}\right]\right\} \tag{4.18}$$

and

$$G_{\rm inv}(p^2)=G_R[1,\rho(p^2)]Z_R(p^2)$$

(similar equations hold for G_b with the substitutions $G_R\leftrightarrow G_b$, $Z_R\leftrightarrow Z_b$, $\gamma_R\leftrightarrow\gamma_b$, $\rho\leftrightarrow\rho_b$, $\mu\leftrightarrow M$). We stress that $G_{\rm inv}(p^2)$ is operator and coupling-constant RS invariant: In particular, the overall normalization constant in Eqs. (4.14) and (4.18) is completely independent of γ_R and of the definition of ρ. We disagree with Ref. 33 on this question.

Generalization to other Green's functions is straightforward. For instance, if one considers the Green's function $\langle\phi_b O_b\phi_b\rangle$ obtained by insertion of the multiplicatively renormalizable local operator O_b, carrying zero momentum, all the previous considerations are still valid: One simply has to replace the factor $Z_b^{-1}(M^2)$ in Eq. (4.8) by $Z_{\phi_b}{}^{-1}(M^2)Z_{O_b}{}^{-1}(M^2)$ (where Z_{ϕ_b} and Z_{O_b} are the wave-function renormalization constants relating ϕ_b and O_b to $\phi_{\rm inv}$ and $O_{\rm inv}$). To treat the case of operator mixing, one introduces the $N\times N$ matrix G_b whose elements are

$$G_b^{ij}=\frac{\langle\phi_b^i O_b^j\phi_b^i\rangle}{\langle\phi_b^i\phi_b^i\rangle}\quad (i,j=1,\ldots,N) .$$

Then we have

$$G_b^{ij}(p^2/M^2,\rho_b)=G_{\rm inv}^{ik}(p^2)[Z_{O_b}{}^{-1}(M^2)]^{kj} , \tag{4.19}$$

where $Z_{O_b}{}^{-1}(M^2)$ is the inverse of the wave-function renormalization matrix of O_b, which satisfies

$$O_b^j(M^2)[Z_{O_b}(M^2)]^{jk}=O_{\rm inv}^k . \tag{4.20}$$

The following matrix equation then replaces Eq. (4.11):

$$\left[p^2\frac{\partial G_b}{\partial p^2}\right](G_b)^{-1}=\left[p^2\frac{\partial G_{\rm inv}}{\partial p^2}\right](G_{\rm inv})^{-1}$$

$$\equiv H(p^2)\ , \qquad (4.21)$$

where $H(p^2)$ is an $N\times N$ matrix,[35] calculable in perturbation theory, and whose elements can be related to effective charges. Given $H(p^2)$, integration of Eq. (4.21) yields $G_{\rm inv}$.

Subtractively renormalizable Green's functions may also be related to renormalization-group-invariant Green's functions and effective charges. For dimensionless Green's functions, such as the hadronic vacuum polarization $\Pi_b(Q^2,M^2)$ in QCD, only the corresponding "Adler function" $Q^2 d\Pi_b/dQ^2$ has physical significance, and is related to an effective charge $\rho_\pi(Q^2)$ defined by

$$Q^2\frac{d\Pi_b}{dQ^2}=A+B\rho_\pi(Q^2)\ , \qquad (4.22)$$

where A and B are calculable constants. On the other hand, dimensional Green's functions appear to have physical meaning: their values at zero momentum are often related to physical quantities. A well-known example is the correlation function

$$\phi_b(Q^2)=i\int d^4x\, e^{iQ\cdot x}\langle 0\,|\,T(\,\alpha_s F\widetilde{F}(x)\ ,$$

$$\alpha_s F\widetilde{F}(0))\,|\,0\rangle\,|_{\rm no\ quarks}\ ,$$

where $F\widetilde{F}(x)$ is the topological charge density. $\phi_b(Q^2=0)$ gives the small-θ dependence of the vacuum energy and, in the large-N limit, is related to the η' mass.[36] However, in perturbation theory $\phi_b(Q^2)$ requires a subtractive renormalization. The corresponding renormalization-group-invariant Green's function $\phi_{\rm inv}(Q^2)$ is defined by the relation

$$\phi_b(Q^2,M^2)=\phi_{\rm inv}(Q^2)-CQ^4\rho_b[1+O(\rho_b)]\ . \qquad (4.23a)$$

The second term on the right-hand side of Eq. (4.23a) represents the unphysical, M-dependent subtraction term, and

$$\phi_{\rm inv}(Q^2)=CQ^4\rho_\phi(Q^2)\ , \qquad (4.23b)$$

where C is a calculable constant, and ρ_ϕ an effective charge. Since the subtraction term depends on Q^2 only through the overall factor Q^4, we have $\phi_b(Q^2=0,M^2)=\phi_{\rm inv}(Q^2=0)$. Note also that $\phi_b(Q^2,M^2)$ is $O(\rho_b{}^2)$, whereas $\phi_{\rm inv}(Q^2)$ is $O(\rho_b)$.

C. Renormalization-group-invariant Wilson-coefficient functions

We consider as the simplest example the operator-product expansion of the nonsinglet part of the product of two electromagnetic currents $J(x)J(0)$, which we write symbolically as

$$J(x)J(0)\underset{x\to 0}{\sim}\sum_n O^n(\mu^2)C^n(x^2,\mu^2)\ , \qquad (4.24)$$

where only the leading twist-two contribution has been written, the sum runs over spin-n, twist-two nonsinglet operators O^n, renormalized at point μ and $C^n(x^2,\mu^2)$ are the corresponding coefficient functions. Since the electromagnetic current operator $J(x)$ does not need to be renormalized, the product $J(x)J(0)$ is μ independent, and so is each term $O^n(\mu^2)C^n(x^2,\mu^2)$ in the expansion. We can therefore write

$$O^n(\mu^2)C^n(x^2,\mu^2)=O^n_{\rm inv}C^n_{\rm inv}(x^2)\ , \qquad (4.25)$$

where

$$O^n_{\rm inv}=O^n(\mu^2)Z^n(\mu^2)$$

and

$$C^n_{\rm inv}(x^2)=[Z^n(\mu^2)]^{-1}C^n(x^2,\mu^2)\ . \qquad (4.26)$$

Equation (4.26) defines the μ-independent, renormalization-group-invariant coefficient function $C^n_{\rm inv}(x^2)$ corresponding to $C^n(x^2,\mu^2)$. Note that $C^n_{\rm inv}$ is defined only up to an arbitrary normalization.[37] Generalization to the singlet case, where operator mixing occurs, is easily done by considering Z^n in Eq. (4.26) as a matrix, and C^n as a column vector. $C^n_{\rm inv}(x^2)$ is then defined up to an arbitrary constant matrix multiplication. We note that with the use of $O^n_{\rm inv}$ and $C^n_{\rm inv}$, the explicit μ dependence of the right-hand side of Eq. (4.24) has disappeared, which is useful in many applications: one does not have to worry about the choice of μ (the "factorization scale").

V. EXTENSION TO THE CASE OF NONZERO FERMION MASSES

We now consider QCD (or QED) with a single massive fermion (for simplicity). Extension of the method of effective charges to this case is rather straightforward. Actually, there is now a further difficulty due to the additional freedom in the definition of the renormalized fermion mass. The treatment we shall suggest gets rid only of the RS ambiguity related to the definition of the renormalized coupling constant when a fixed mass definition (such as the pole of the fermion propagator) is used. However, the use of the "scale-invariant" renormalized mass parameter $\hat{m}$[29,38] (associated to running masses in mass-independent RS) leads to an essentially RS-invariant treatment. The reason is that the parameter $\hat{m}$ is universal, i.e., is the same for all mass-independent RS, and parametrize the cutoff dependence of the bare mass in the continuum limit. Since this question, to the author's knowledge, has not been treated in much detail in the literature, we sketch the basic arguments which lead to the previous statements.

Let $m_R(\mu^2)$ be the running mass in a mass-independent RS (such as the MS scheme). We recall that its μ dependence is given by the equation

$$\mu^2\frac{dx}{d\mu^2}=-x[1+\gamma^m(\rho)]\ , \qquad (5.1a)$$

where $x\equiv[m_R(\mu^2)/\mu]^2$, ρ is the RS coupling constant, and $\gamma^m(\rho)$ the mass-anomalous-dimension function. Integration of Eq. (5.1a) gives[39]

$$m_R{}^2(\mu^2)=\hat{m}\,{}^2[\beta_1\rho(\mu^2)]^{\gamma_1^m/\beta_1}\times\left[1+\left[\frac{\gamma_2^m}{\beta_1}-\frac{\beta_2}{\beta_1}\frac{\gamma_1^m}{\beta_1}\right]\rho(\mu)+\cdots\right], \tag{5.2a}$$

where we put $\gamma^m(\rho)=\gamma_1^m\rho+\gamma_2^m\rho^2+\cdots$.

Similarly, the cutoff dependence of the bare mass in the continuum limit is given by the analog of Eqs. (5.1a) and (5.2a):

$$M^2\frac{dx_b}{dM^2}=-x_b[1+\gamma_b^m(\rho_b)]\,, \tag{5.1b}$$

where $x_b\equiv[m_b(M^2)/M]^2$, m_b is the bare mass, ρ_b the bare coupling constant, M the cutoff, and

$$m_b{}^2(M^2)=\hat{m}_b{}^2[\beta_1\rho_b(M^2)]^{\gamma_1^m/\beta_1}\times\left[1+\left[\frac{\gamma_{2,b}^m}{\beta_1}-\frac{\beta_2}{\beta_1}\frac{\gamma_{1,b}^m}{\beta_1}\right]\rho_b(M^2)+\cdots\right] \tag{5.2b}$$

with

$$\gamma_b^m(\rho_b)=\gamma_{1,b}^m\rho_b+\gamma_{2,b}^m\rho_b{}^2+\cdots\,.$$

We note that we must have

$$\hat{m}=\hat{m}_b\,,\qquad \gamma_1^m=\gamma_{1,b}^m \tag{5.3}$$

in order that the mass-renormalization constant $m_b{}^2(M^2)/m_R{}^2(\mu^2)$ be a power series in ρ (or ρ_b) with mass-independent coefficients, with the $O(\rho^0)$ term normalized to unity; we then get the familiar looking result

$$\frac{m_b{}^2(M^2)}{m_R{}^2(\mu^2)}=1+\rho(\mu^2)\left[-\gamma_1^m\ln\frac{M^2}{\mu^2}+\text{const}\right]\cdots\,. \tag{5.4}$$

Equation (5.3) states the universality of $\hat{m}$, as well as that of the one-loop mass-anomalous dimension $\gamma_1^m=6\times\frac{4}{3}$ in QCD (two-loop anomalous dimension γ_2^m are not universal, however, as is well known). Coming back now to the method of effective charges, it is convenient to distinguish the cases $Q\neq0$ and $Q=0$, where Q is the external momentum.

A. Nonzero external momentum

1. Fixed renormalized mass

We assume mass renormalization has been performed, using a fixed mass definition (such as $m=$pole of fermion propagator). The obvious generalizations of Eqs. (2.4) and (2.5) are

$$\sigma(Q^2,m^2)=A(m^2/Q^2)+B(m^2/Q^2)[4\pi\bar{\rho}(Q^2,m^2)]^d \tag{5.5}$$

(we assume d is a constant), and

$$Q^2\frac{\partial\bar{\rho}}{\partial Q^2}=\bar{\beta}(\bar{\rho},m^2/Q^2)\,. \tag{5.6a}$$

Actually, for $\bar{\rho}=\bar{\rho}(Q^2,m^2,M^2,\rho_b)$ one can define two other β functions:

$$\text{(i)}\ \ \bar{\beta}_m(\bar{\rho},m^2/Q^2)=m^2\left.\frac{\partial\bar{\rho}}{\partial m^2}\right|_{Q^2,M^2,\rho_b}, \tag{5.6b}$$

$$\text{(ii)}\ \ \bar{\beta}_M(\bar{\rho},m^2/Q^2)=M^2\left.\frac{\partial\bar{\rho}}{\partial M^2}\right|_{Q^2,m^2,\rho_b}=-Q^2\left.\frac{\partial\bar{\rho}}{\partial Q^2}\right|_{m^2/Q^2,M^2,\rho_b}. \tag{5.6c}$$

Homogeneity of $\bar{\rho}$ implies the relation

$$\bar{\beta}+\bar{\beta}_m+\bar{\beta}_M=0\,. \tag{5.6d}$$

The $\bar{\beta}_M$ function is specially useful for physical quantities $\sigma(Q^2,m^2,M^2,\rho_b)$ defined with on-shell external fermions ($Q^2=m^2$), such as the electron $g-2$ in QED, or the quarkonium gluonic width in QCD[8] (see Sec. V B I). We also note that Eq. (5.6c) is no more difficult to integrate than the zero-mass equation, since m^2/Q^2 is fixed.

Predictions for the effective charge $\bar{\rho}$ can be done in terms of $\bar{\beta}$ and a boundary condition $\bar{\Lambda}$ [related to the large-Q behavior of $\bar{\rho}$ by Eq. (2.16): we assume that $\bar{\beta}(\bar{\rho},m^2/Q^2)$ has a zero-mass limit, and that $\bar{\rho}$ coincide with an effective charge of the zero-mass theory at large Q; these assumptions are certainly true in QCD[40]]. However, one can easily check that the functional form of $\bar{\beta}(\bar{\rho},m^2/Q^2)$, as well as the definition of the effective charge $\bar{\rho}$, depend in fact on the definition chosen for m; given one definition, all other possible choices are generated by relations of the form

$$m=\overline{m}\{1+c\rho(\overline{m}^2)[1+O(\rho)]\}\,. \tag{5.7}$$

It is clear that once Eq. (5.7) is substituted into Eqs. (5.5) and (5.6), a new effective charge $\bar{\rho}$ and (or) $\bar{\beta}(\bar{\rho},m^2/Q^2)$ function will result.

2. Running masses in mass-independent RS: Use of the scale-invariant mass $\hat{m}$

As we have argued above, $\hat{m}$ can be considered as a RS-invariant parameter. However, if one merely substitutes into the unrenormalized (or renormalized in any mass-independent RS) Feynman diagrams for σ the expression Eq. (5.2b) for $m_b{}^2$ [or the expression Eq. (5.2a) for $m_R{}^2$], the result will have a very complicated structure with respect to ρ_b (or ρ_{MS}) and no clear effective charge picture will emerge. This difficulty can be circumvented if one uses a small-mass expansion (for $\hat{m}\,{}^2\ll Q^2$) or a heavy-mass expansion (for $\hat{m}\,{}^2\gg Q^2$), depending on the kinematical region of interest. Consider for instance the small-mass expansion of σ:

$$\sigma=\sigma_0(Q^2/M^2,\rho_b)+\frac{m_b{}^2}{Q^2}\sigma_1(Q^2/M^2,\rho_b)+\frac{m_b{}^4}{Q^4}\left[\sigma_2(Q^2/M^2,\rho_b)+\tilde{\sigma}_2(Q^2/M^2,\rho_b)\ln\frac{m_b{}^2}{Q^2}\right]+\cdots\,, \tag{5.8}$$

where we used the fact that the strongest singularity at zero *bare* mass is presumably of the form $m_b{}^3\ln m_b{}^2$ (by analogy with a similar result of Ref. 40, for the singularity at zero renormalized mass in a mass-independent RS). Next, substituting Eq. (5.2b) for m_b, we obtain an expansion of the form

$$\sigma=\sigma_0(Q^2/M^2,\rho_b)+\frac{\hat{m}^2}{Q^2}\hat{\sigma}_1(Q^2/M^2,\rho_b)+\frac{\hat{m}^4}{Q^4}\left[\hat{\sigma}_2(Q^2/M^2,\rho_b)+\hat{\tilde{\sigma}}_2(Q^2/M^2,\rho_b)\ln\frac{\hat{m}^2}{Q^2}\right]+\cdots\,, \tag{5.9}$$

where each term is a function of the zero-mass theory, and can be associated to an effective charge. Of particular interest is the case where $\sigma_0\equiv 0$: σ is then an observable appropriate to the study of light-quark masses. For instance, the leading term in the small-mass expansion of

$$F_5(Q^2)=\frac{d^2\psi_5}{(dQ^2)^2}\,,$$

where $\psi_5(Q^2)$ is the correlation function:

$$\psi_5(Q^2)=\langle\partial_\mu A^\mu \partial_\nu A^{\nu +}\rangle$$

(where A_μ is the charged strangeness-nonchanging axial-vector current) has the form[41] in the $\overline{\text{MS}}$ scheme in QCD (for two quark flavors of equal mass):

$$F_5(Q^2)=\frac{3}{8\pi^2}\frac{4m_R{}^2(Q^2)}{Q^2}\left[1+\frac{4\times 11}{3}\rho(Q^2)+\cdots\right]+O\left[\frac{m_R{}^4}{Q^4}\ln\frac{m_R{}^2}{Q^2}\right]. \tag{5.10}$$

Upon substitution of Eq. (5.2a) (at $\mu^2=Q^2$), one obtains

$$F_5(Q^2)=\frac{3}{8\pi^2}\frac{4\hat{m}^2}{Q^2}[\beta_1\rho(Q^2)]^{\gamma_1^m/\beta_1}\left[1+\left[\frac{\gamma_2^m}{\beta_1}-\frac{\beta_2}{\beta_1}\frac{\gamma_1^m}{\beta_1}+\frac{4\times 11}{3}\right]\rho(Q^2)+\cdots\right]+O\left[\frac{\hat{m}^4}{Q^4}\ln\frac{\hat{m}^2}{Q^2}\right]$$
$$\equiv\frac{3}{8\pi^2}\frac{4\hat{m}^2}{Q^2}[\beta_1\bar{\rho}(Q^2)]^{\gamma_1^m/\beta_1}+O\left[\frac{\hat{m}^4}{Q^4}\ln\frac{\hat{m}^2}{Q^2}\right]. \tag{5.11}$$

Hence, through the effective charge $\bar{\rho}(Q^2)$, we get a RS-invariant prediction for $F_5(Q^2)$, in leading order in the chiral-symmetry-breaking parameter $\hat{m}$.

Finally, we mention that for effective charges $\bar{\rho}(Q^2,m^2)$, defined with on-mass-shell external fermions ($Q^2=m^2$), one can give RS-invariant predictions in terms of the new β function

$$\hat{\beta}(\bar{\rho})=\hat{m}^2\frac{\partial\bar{\rho}}{\partial\hat{m}^2}\bigg|_{Q^2=m^2}$$

[see Sec. V B 2].

B. Zero-momentum observables

Typical examples are the electric charge in QED or the derivatives of the hadronic vacuum polarization at $Q=0$ in QCD.

1. Fixed renormalized mass

When $Q=0$, the derivative with respect to Q in Eq. (5.6) is no more available, but it can be replaced by a derivative with respect to the renormalized mass m. If the zero-momentum observable σ has the expansion (where mass renormalization has been performed)

$$\sigma(m^2)=A+B(\rho_b)^d[1+\sigma_1(m^2/M^2)\rho_b+\cdots]\,,$$

we define again an effective charge $\bar{\rho}$ by the identity

$$\sigma(m^2)\equiv A+B[\bar{\rho}(m^2)]^d\,,$$

and we introduce the generalized Callan-Symanzik function

$$m^2\frac{\partial\bar{\rho}}{\partial m^2}\bigg|_{\rho_b,M^2}=\bar{\beta}(\bar{\rho})\,. \tag{5.12}$$

Equation (5.12) is the analog of the Callan-Symanzik function associated to the electric charge in QED.

We note that the mass variation can equally be performed at fixed (ρ,μ^2) in any mass-independent RS, leading to the same $\bar{\beta}$ function. Also, the structure of the perturbation expansion of $\bar{\rho}(m^2)$ in any mass-independent RS is given by the analog of Eq. (3.4), with Q replaced by m. However, the resulting $\bar{\beta}$ function now depends upon the definition of m beyond two loops. If m_1 and m_2 are two different definitions, and $\bar{\beta}_1(\bar{\rho})$, $\bar{\beta}_2(\bar{\rho})$ the corresponding Callan-Symanzik functions, we have, putting

$$\varphi(\bar{\rho})\equiv\left[\frac{m_2}{m_1}\right]^2,$$

$$\frac{d\ln\varphi}{d\bar{\rho}}=\frac{1}{m_2{}^2}\frac{dm_2{}^2}{d\bar{\rho}}\bigg|_{\rho_b,M^2}-\frac{1}{m_1{}^2}\frac{dm_1{}^2}{d\bar{\rho}}\bigg|_{\rho_b,M^2}=\frac{1}{\bar{\beta}_2(\bar{\rho})}-\frac{1}{\bar{\beta}_1(\bar{\rho})}\,. \tag{5.13}$$

Hence

$$\ln\varphi=\int_0^{\bar\rho}dx\left[\frac{1}{\bar\beta_2(x)}-\frac{1}{\bar\beta_1(x)}\right]. \tag{5.14}$$

A similar formalism can be applied to those effective charges $\bar\rho(Q^2,m^2,M^2,\rho_b)$ defined with on-mass-shell external fermions ($Q^2=m^2$), although they are not genuine zero-momentum observables. According to a remark made earlier (Sec. V A 1), they can be determined by the $\bar\beta_M$ function [see Eq. (5.6c)], i.e., equivalently, the mass derivative

$$m^2\frac{\partial\bar\rho}{\partial m^2}\bigg|_{m^2/Q^2=1,\rho_b,M^2}=-\bar\beta_M\left[\bar\rho,\frac{m^2}{Q^2}=1\right]. \tag{5.15}$$

2. *Scale-invariant mass* $\hat m$

A completely RS-invariant solution (within the class of mass-independent RS) can be given if one considers the following modification of the Callan-Symanzik method. Instead of taking the variation of the zero-momentum effective charge ρ (we drop the bar for ease of notation) with respect to the renormalized mass m at fixed ρ_b and M, consider the variation of ρ with respect to the bare mass m_b at fixed ρ_b and M:

$$m_b{}^2\frac{\partial\rho}{\partial m_b{}^2}\bigg|_{\rho_b,M^2}=\hat\beta(\rho), \tag{5.16a}$$

where we introduced the modified Callan-Symanzik function $\hat\beta(\rho)$. The fact that the bare-mass derivative in Eq. (5.16a) is finite has been known long ago in QED,[42] where the particular zero-momentum observable considered was the electric charge. The physical content of this property becomes clear when one realizes that Eq. (5.16a) is equivalent to the relation

$$\hat m^2\frac{\partial\rho}{\partial\hat m^2}\bigg|_{\rho_b,M^2}=\hat\beta(\rho), \tag{5.17}$$

i.e., the bare-mass derivative, at fixed ρ_b and M, is simply the derivative with respect to the scale-invariant mass $\hat m$. The latter statement is an obvious consequence of the fact that m_b is proportional to $\hat m$ at fixed ρ_b and M [see Eq. (5.2b)]. We also note that the same $\hat\beta$ function is obtained if one takes the derivative of ρ with respect to the running mass m_R, at fixed ρ_R and μ, in any mass-independent RS (such as the MS scheme)

$$m_R{}^2\frac{\partial\rho}{\partial m_R{}^2}\bigg|_{\rho_R,\mu^2}=\hat\beta(\rho) \tag{5.16b}$$

(where ρ_R is the coupling constant associated to the considered RS).

Let us now discuss, in terms of the $\hat\beta(\rho)$ function, the ultraviolet structure of the expansion of the zero-momentum observable ρ in powers of the bare charge ρ_b, when no mass renormalization has been performed (i.e., we are interested in ρ as a function of ρ_b, $m_b{}^2$, and M^2). We shall show that $\hat\beta$ can be computed without performing any explicit mass renormalization. We start from the Callan-Symanzik equation which expresses the cutoff independence of ρ:

$$\left[M^2\frac{\partial}{\partial M^2}+\beta_b(\rho_b)-\gamma_b^m(\rho_b)m_b{}^2\frac{\partial}{\partial m_b{}^2}\right]\rho\left[\frac{m_b{}^2}{M^2},\rho_b\right]=0, \tag{5.18}$$

where β_b is the bare coupling β function

$$M^2\frac{\partial\rho_b}{\partial M^2}=\beta_b(\rho_b). \tag{5.19}$$

Homogeneity gives the relation

$$m_b{}^2\frac{\partial\rho}{\partial m_b{}^2}=-M^2\frac{\partial\rho}{\partial M^2}, \tag{5.20}$$

which allows us to put Eq. (5.18) in a modified form, where m_b plays a purely passive role (like an external kinematical scale variable),

$$\left[M^2\frac{\partial}{\partial M^2}+\hat\beta_b(\rho_b)\frac{\partial}{\partial\rho_b}\right]\rho\left[\frac{m_b{}^2}{M^2},\rho_b\right]=0 \tag{5.21}$$

and we defined

$$\hat\beta_b(\rho_b)=\frac{\beta_b(\rho_b)}{1+\gamma_b^m(\rho_b)}=-\beta_1\rho_b{}^2-(\beta_2-\beta_1\gamma_{1,b}^m)\rho_b{}^3+\cdots. \tag{5.22}$$

Equation (5.21) is of a standard type, and to solve it we note that together with Eqs. (5.16)a) and (5.20), it implies the relation

$$\frac{d\rho}{\hat\beta(\rho)}=\frac{d\rho_b}{\hat\beta_b(\rho_b)}. \tag{5.23}$$

Hence, by integration,

$$\ln\frac{m_b{}^2}{M^2}-\frac{K_1}{\beta_1}=\int_0^{\rho}\frac{dx}{\hat\beta(x)}-\int_0^{\rho_b}\frac{dx}{\hat\beta_b(x)}, \tag{5.24}$$

where the $\ln m_b{}^2/M^2$ term arises from taking into account Eq. (5.16a), and K_1 is an integration constant. The procedure used here is simply the reverse of the one followed in Sec. III, Eq. (5.24) being the direct analog of Eq. (3.2) with the substitutions

$$\bar\rho\leftrightarrow\rho,\quad \rho\leftrightarrow\rho_b,$$
$$Q^2\leftrightarrow m_b{}^2,\quad \mu^2\leftrightarrow M^2,$$
$$\bar\beta\leftrightarrow\hat\beta,\quad \beta\leftrightarrow\hat\beta_b,$$

and

$$\ln\frac{\bar\Lambda^2}{\Lambda^2}\leftrightarrow\frac{K_1}{\beta_1}.$$

Actually, all the results of Sec. III are valid with these substitutions; in particular, the $\hat\beta$ function can be obtained from the single $\ln(m_b{}^2/M^2)$ and the constant terms in the bare expansion of ρ. In short, at zero momentum, if we do not perform explicit mass renormalization, we have a new effective renormalization group characterized by the various $\hat{\beta}$ functions. The latter are two-loop universal, as follows from Eq. (5.23), but differ, beyond one loop, from the corresponding β functions [see Eq. (5.22)]. Furthermore, the "new renormalization-group-improved" relation between two zero-momentum effective charges ρ and $\bar{\rho}$ now reads, with obvious notations,

$$-\left[\frac{\bar{K}_1}{\beta_1}-\frac{K_1}{\beta_1}\right]=\int_0^{\bar{\rho}}\frac{dx}{\hat{\bar{\beta}}(x)}-\int_0^{\rho}\frac{dx}{\hat{\beta}(x)} . \qquad (5.25)$$

Equation (5.25) provides an alternative to the similar renormalization-group improvement obtained using ordinary β functions associated to fixed-mass definitions.

We also note that the solution of Eq. (5.17), including the proper boundary conditions, may be derived from Eq. (5.24). For this purpose, we eliminate the "unphysical" parameters m_b and M^2 from Eq. (5.24), using the relations which express the bare coupling dependence of the cutoff and of the bare mass. These are

$$\ln\frac{M^2}{\Lambda_b{}^2}=\int_0^{\rho_b}\frac{dx}{\beta_b(x)} , \qquad (5.26)$$

where $\Lambda_b{}^2$ is the regularization-dependent physical scale occurring as a boundary condition to Eq. (5.19), and

$$\ln\frac{m_b{}^2}{\hat{m}^2}=\int_0^{\rho_b}dx\left[\frac{1}{\beta_b(x)}-\frac{1}{\hat{\beta}_b(x)}\right] \qquad (5.27)$$

[Eq. (5.27) follows from integrating Eq. (5.1b), and using Eqs. (5.22) and (5.26)]. We thus get

$$\ln\frac{\hat{m}^2}{\Lambda_b{}^2}=\frac{K_1}{\beta_1}+\int_0^{\rho}\frac{dx}{\hat{\beta}(x)} , \qquad (5.28)$$

where integrals like

$$\int_0^{\rho}\frac{dx}{\beta(x)}$$

have to be interpreted as

$$\frac{1}{\beta_1\rho}+\frac{\beta_2}{\beta_1{}^2}\ln(\beta_1\rho)+\int_0^{\rho}dx\left[\frac{1}{\beta_1x^2}-\frac{\beta_2}{\beta_1{}^2}\frac{1}{x}+\frac{1}{\beta(x)}\right] .$$

In accordance with similar previous remarks, we stress that all the above results also hold if one considers the expression of ρ in any mass-independent RS, m_b being replaced by the running mass, ρ_b by the running coupling constant associated to this RS, and M by the renormalization point.

Finally, we mention that effective charges $\bar{\rho}(Q^2,m^2)$ defined with on-mass-shell external fermions ($Q^2=m^2$), although not strictly zero-momentum objects, can also be described in terms of the $\hat{\beta}$ function defined by

$$\hat{m}^2\frac{\partial\bar{\rho}}{\partial\hat{m}^2}\bigg|_{Q^2=m^2}=\hat{\beta}(\bar{\rho}) . \qquad (5.29)$$

Typical examples where this remark applies are the g-2 of the electron in QED or the quarkonium gluonic width in QCD.

VI. APPLICATIONS

This section is a brief survey, intended to illustrate the main features of the method of effective charges and, also, to point out some problems it encounters. Since most of the physical quantities σ depend upon more than one kinematical momentum scale, we have first to mention the straightforward extension required when several scales are present: one simply considers σ as a function of one overall momentum scale, keeping all the external momenta in fixed ratios to each other (as is usually done when one deals with renormalization-group equations). We are then back in effect to a one-scale problem, but it is clear that useful results (i.e., well-behaved effective charges) will be obtained only if one stays in the "deep Euclidean region," where the ratios of kinematical invariants take on finite values, away from any phase-space boundary. When two or more distinct scales are involved in an essential way (in the sense that their ratio becomes large), deeper physical understanding is required: one has to find a way to go back to a one-scale problem before defining the relevant effective charges (see, however, Sec. VI E). This is what happens for those deep-inelastic processes where a factorization between long and short distances has been established. Using renormalization-group-invariant operators and coefficient functions (see Sec. IV), one then recovers and generalizes some very simple formulas, originally due to Bardeen and Buras[12] in the case of structure-function moments, which have the following features.

(i) No "factorization scale" need be introduced; hence, no factorization scale ambiguity arises.

(ii) Factorization between long and short distances is implemented in the neatest way: all the Q dependences, calculable in perturbation theory, are contained in the renormalization-group-invariant coefficient functions and directly related to effective charges; whereas the nonperturbatively calculable part is isolated into some constant, scale-independent, RS-invariant normalization factors, which represent the hadron matrix element of renormalization-group-invariant operators. We now turn to some specific processes.

A. Moments of deep-inelastic structure functions (nonsinglet case)

We shall use in general in the following the notation of Buras.[12] Starting from the standard operator-product expansion result for the nonsinglet moments $M_k^{\rm NS}(n,Q^2)$ $(k=1,2)$,[43] and introducing renormalization-group-invariant operator and coefficient functions, we get, keeping only twist two operators,

$$\begin{aligned}M_k^{\rm NS}(n,Q^2)&=A_n^{\rm NS}(\mu^2)C_{k,n}^{\rm NS}(Q^2/\mu^2,\rho)\\&=A_n^{\rm NS}(\mu^2)Z_n^{\rm NS}(\mu^2)[Z_n^{\rm NS}(\mu^2)]^{-1}C_{k,n}^{\rm NS}(Q^2/\mu^2,\rho)\\&\equiv\bar{A}_n^{\rm NS}C_k^{\rm NS}(n,Q^2) ,\end{aligned} \qquad (6.1)$$

where $A_n^{\rm NS}(\mu^2)$ is the hadron matrix element of the nonsinglet, twist-two, spin-n operator $O_n^{\rm NS}(\mu^2)$ renormalized at point μ, $C_{k,n}^{\rm NS}(Q^2/\mu^2,\rho)$ is the Fourier transform of the associated Wilson coefficient function and $\bar{A}^{\rm NS}$ and $C_k^{\rm NS}(n,Q^2)$ are the corresponding renormalization-group-invariant quantities. The long-distance physics is contained in $\bar{A}_n^{\rm NS}$, which is a constant[44] (at fixed n), RS-independent, parameter-free prediction of QCD (defined up to an arbitrary overall normalization), but noncalculable perturbatively. All the Q dependence is contained in $C_k^{\rm NS}(n,Q^2)$, which is given, up to next-to-leading order, by the expansion[12]

$$C_k^{\rm NS}(n,Q^2)=[\rho(Q^2)]^{d_{\rm NS}^n}[1+R_{k,n}^{\rm NS}\rho(Q^2)+\cdots]$$
$$\equiv[\bar{\rho}_k^{\rm NS}(n,Q^2)]^{d_{\rm NS}^n}, \qquad (6.2)$$

where $d_{\rm NS}^n=\gamma_{\rm NS}^{1,n}/\beta_1$ ($\gamma_{\rm NS}^{1,n}$ is the one-loop anomalous dimension of $O_n^{\rm NS}$), and $R_{k,n}^{\rm NS}$ can be found (or easily deduced) from Ref. 12. The use of the effective charge $\bar{\rho}_k^{\rm NS}(n,Q^2)$ is equivalent, up to next-to-leading order, to the Λ_n scheme,[2] as mentioned in Sec. II.

We note incidentally that the fact that $\bar{A}_n^{\rm NS}$ is not calculable perturbatively does not prevent the method of effective charges to be applied to the perturbatively calculable part $C_k^{\rm NS}(n,Q^2)$, which is well defined. The latter may be obtained from the standard solution of the renormalization-group equation for $C_{k,n}^{\rm NS}(Q^2/\mu^2,\rho)$,

$$C_{k,n}^{\rm NS}(Q^2/\mu^2,\rho)=C_{k,n}^{\rm NS}[1,\rho(Q^2)]\exp\left[-\int_\rho^{\rho(Q^2)}dx\frac{\gamma_{\rm NS}^n(x)}{\beta(x)}\right] \qquad (6.3)$$

by the analogs of Eqs. (4.10) and (4.18):

$$C_{k,n}^{\rm NS}(Q^2/\mu^2,\rho)=Z_n^{\rm NS}(\mu^2)C_k^{\rm NS}(n,Q^2)$$

with

$$Z_n^{\rm NS}(\mu^2)=[\rho(\mu^2)]^{-\gamma_{\rm NS}^{1,n}/\beta_1}\times\exp\left\{\int_0^{\rho(\mu^2)}dx\left[\frac{\gamma_{\rm NS}^n(x)}{\beta(x)}+\frac{\gamma_{\rm NS}^{1,n}}{\beta_1}\frac{1}{x}\right]\right\} \qquad (6.4)$$

and

$$C_k^{\rm NS}(n,Q^2)=[Z_n^{\rm NS}(Q^2)]^{-1}C_{k,n}^{\rm NS}[1,\rho(Q^2)]\ .$$

Alternatively, one can make a prediction in terms of the effective charge $\rho_k^{\rm NS}(n,Q^2)$ associated[5,8] to the derivative

$$[M_k^{\rm NS}]^{-1}Q^2\frac{dM_k^{\rm NS}}{dQ^2}=[C_k^{\rm NS}]^{-1}Q^2\frac{dC_k^{\rm NS}}{dQ^2}$$
$$\equiv-\gamma_{\rm NS}^{1,n}\rho_k^{\rm NS}(n,Q^2)\ . \qquad (6.5)$$

Equation (6.5) is a physical all-orders generalization[45] of the Altarelli-Parisi equation[46] in moment space.

The procedure suggested here is similar to the method used in Sec. IV A to compute renormalization-group-invariant Green's functions; in particular, the relation between $\bar{\rho}_k^{\rm NS}(n,Q^2)$ and $\rho_k^{\rm NS}(n,Q^2)$ is given by the analogs of Eqs. (4.15) and (4.16).

B. Moments of structure functions (singlet case)

The operator-product expansion gives in this case

$$M_k^s(n,Q^2)=\sum_{a=\psi,G}A_n^a(\mu^2)C_{k,n}^a(Q^2/\mu^2,\rho)$$
$$=\sum_{\substack{a,b=\psi,G\\ i=+,-}}A_n^a(\mu^2)[Z_n(\mu^2)]^{ai}[Z_n^{-1}]^{ib}\times C_{k,n}^b(Q^2/\mu^2,\rho)$$
$$\equiv\sum_{i=+,-}\bar{A}_n^iC_k^i(n,Q^2)\ , \qquad (6.6)$$

where $Z_n(\mu^2)$ is the 2×2 wave-function renormalization matrix of singlet quark and gluon twist-2 operators $O_n^a(\mu^2)$, defined as in Eq. (4.20) [with M replaced by μ, and $O_b^i(M^2)$ by $O_n^a(\mu^2)$]. Similar to $\bar{A}_n^{\rm NS}$, $\bar{A}_n^i$ is a RS-independent constant two-dimensional vector (defined up to an arbitrary constant matrix multiplication), which contains the long-distance physics. In perturbation theory, one finds,[12] with a particular choice for the arbitrary constant matrix,

$$C_k^i(n,Q^2)=[\rho(Q^2)]^{d_i^n}[1+R_{k,n}^i\rho(Q^2)+\cdots]$$
$$\equiv[\bar{\rho}_k^i(n,Q^2)]^{d_i^n}\quad(i=+,-)\ , \qquad (6.7)$$

where $d_i^n=\gamma_i^n/\beta_1$ (γ_n^n are the eigenvalues of the one-loop anomalous dimension matrix) and $R_{k,n}^i$ can be easily deduced from results in Ref. 12. Each moment of the singlet structure function can therefore be described in terms of *two* effective charges $\bar{\rho}_k^i(n,Q^2)$ $(i=+,-)$. We note that the associated effective scales $\bar{\Lambda}_k^i(n)$ are different, which should be taken into account in a phenomenological analysis of leptoproduction data. The invariant coefficient functions $C_k^i(n,Q^2)$ may be obtained from the solution of the renormalization-group equation for $C_{k,n}^a(Q^2/\mu^2,\rho)$. One finds,[12]

$$C_{k,n}^a(Q^2/\mu^2,\rho)=\sum_{b=\psi,G}\left[T\exp\left[-\int_\rho^{\rho(Q^2)}dx\frac{\hat{\gamma}^n(x)}{\beta(x)}\right]\right]_{ab}\times C_{k,n}^b[1,\rho(Q^2)]$$
$$=\sum_{i=+,-}[Z_n(\mu^2)]^{ai}C_k^i(n,Q^2)\ , \qquad (6.8)$$

where $\hat{\gamma}^n(x)$ is the anomalous dimension matrix

$$Z_n(\mu^2)=\hat{V}_n(\rho)\hat{U}_n\begin{bmatrix}(\rho)^{-d_+^n} & 0\\ 0 & (\rho)^{-d_-^n}\end{bmatrix}, \qquad (6.9a)$$

where the 2×2 matrices $\hat{V}_n$ and $\hat{U}_n$ can be found in Ref. 12, and

$$C_k^i(n,Q^2)=\sum_{a=\psi,G}[Z_n^{-1}(Q^2)]^{ia}C_{k,n}^a[1,\rho(Q^2)]\ . \qquad (6.9b)$$

Calculation of $C_k^i(n,Q^2)$ may also be done using Altarelli-Parisi matrix-type equations for the singlet quark and gluon structure-function 2×2 matrix[47]

$$\hat{M}^{s}(n,Q^2,p^2)=\begin{bmatrix} M^{s}_{2,\psi}(n,Q^2,p^2) & M^{s}_{1,\psi}(n,Q^2,p^2) \\ M^{s}_{2,G}(n,Q^2,p^2) & M^{s}_{1,G}(n,Q^2,p^2) \end{bmatrix} \quad (6.10a)$$

(p^2 is the parton momentum square) related to the 2×2 invariant coefficient function matrix

$$\hat{C}(n,Q^2)=\begin{bmatrix} C_2^{+}(n,Q^2) & C_1^{+}(n,Q^2) \\ C_2^{-}(n,Q^2) & C_1^{-}(n,Q^2) \end{bmatrix} \quad (6.10b)$$

by the physical Altarelli-Parisi equation[47]

$$[\hat{M}^{s}]^{-1}Q^2\frac{d\hat{M}^{s}}{dQ^2}=[\hat{C}]^{-1}Q^2\frac{d\hat{C}}{dQ^2} \equiv -\hat{\Gamma}(n,Q^2)\ , \quad (6.11)$$

where the elements of the matrix $\hat{\Gamma}(n,Q^2)$ are given by effective charges, and we used the fact that

$$\hat{M}^{s}(n,Q^2,p^2)=\hat{\hat{A}}_n(p^2)\hat{C}(n,Q^2)\ , \quad (6.12)$$

where $\hat{\hat{A}}_n(p^2)$ is a Q-independent 2×2 matrix.

C. Violation of the Callan-Gross relation

As an application of the previous results, let us discuss the QCD prediction for the moments

$$M_L^{\rm NS}(n,Q^2)=\int_0^1 dx\, x^{n-2}F_L^{\rm NS}(x,Q^2)$$

of the longitudinal nonsinglet structure function

$$F_L^{\rm NS}(x,Q^2)=F_2^{\rm NS}(x,Q^2)-2xF_1^{\rm NS}(x,Q^2)\ ,$$

which vanish in the parton model (Callan and Gross, Ref. 48). This will give us an example of a typical problem the method of effective charges has to face. The problem is that the prediction for $M_L^{\rm NS}(n,Q^2)$ can be done *a priori* in at least two different ways, which are not obviously compatible.

(i) One can apply the method of effective charges separately to $M_1^{\rm NS}(n,Q^2)$ and $M_2^{\rm NS}(n,Q^2)$, then deduce

$$M_L^{\rm NS}(n,Q^2)=M_2^{\rm NS}(n,Q^2)-M_1^{\rm NS}(n,Q^2)\ .$$

(ii) Or one can attempt to apply it directly to $M_L^{\rm NS}(n,Q^2)$, since the next-to-leading-order corrections to $M_L^{\rm NS}(n,Q^2)$ have recently been computed.[49,50]

TABLE I. The values of $\bar{\Lambda}_K^{\rm NS}(n)/\Lambda_{\overline{\rm MS}}$ for four flavors.

n	$\bar{\Lambda}_1^{\rm NS}(n)/\Lambda_{\overline{\rm MS}}$	$\bar{\Lambda}_2^{\rm NS}(n)/\Lambda_{\overline{\rm MS}}$
2	1.05	1.34
4	1.66	1.79
6	2.01	2.10
8	2.28	2.34
10	2.49	2.54

Let us consider in turn these two approaches

(i) Using Eqs. (6.1) and (6.2), we get

$$M_L^{\rm NS}(n,Q^2)=M_2^{\rm NS}(n,Q^2)-M_1^{\rm NS}(n,Q^2) =\bar{A}_n^{\rm NS}\{[\bar{\rho}_2^{\rm NS}(n,Q^2)]^{d^n_{\rm NS}}-[\bar{\rho}_1^{\rm NS}(n,Q^2)]^{d^n_{\rm NS}}\}\ . \quad (6.13)$$

Hence

$$\frac{M_L^{\rm NS}(n,Q^2)}{M_2^{\rm NS}(n,Q^2)}=1-\left[\frac{\bar{\rho}_1^{\rm NS}(n,Q^2)}{\bar{\rho}_2^{\rm NS}(n,Q^2)}\right]^{d^n_{\rm NS}}\ . \quad (6.14)$$

The $\bar{\rho}_k^{\rm NS}(n,Q^2)$ are computed from Eq. (2.16) in terms of the scale Λ of a standard RS, using the relation (Para and Sachrajda, Ref. 2)

$$\frac{\bar{\Lambda}_k^{\rm NS}(n)}{\Lambda}=\exp\left[\frac{R_{k,n}^{\rm NS}}{2\beta_1 d^n_{\rm NS}}\right]\ . \quad (6.15)$$

Values of $\bar{\Lambda}_k^{\rm NS}(n)/\Lambda$ are given in Table I for $\Lambda=\Lambda_{\overline{\rm MS}}$ (the values for $k=2$ had already been obtained by Para and Sachrajda, Ref. 2).

(ii) Alternatively, we have, in any RS,

$$M_L^{\rm NS}(n,Q^2)=\bar{A}_n^{\rm NS}(R_{2,n}^{\rm NS}-R_{1,n}^{\rm NS})[\rho(Q^2)]^{1+d^n_{\rm NS}}[1+R_{L,n}^{\rm NS}\rho(Q^2)+\cdots] \equiv\bar{A}_n^{\rm NS}(R_{2,n}^{\rm NS}-R_{1,n}^{\rm NS})[\bar{\rho}_L^{\rm NS}(n,Q^2)]^{1+d^n_{\rm NS}}\ . \quad (6.16)$$

Hence

$$\frac{M_L^{\rm NS}(n,Q^2)}{M_2^{\rm NS}(n,Q^2)}=(R_{2,n}^{\rm NS}-R_{1,n}^{\rm NS})\frac{[\bar{\rho}_L^{\rm NS}(n,Q^2)]^{1+d^n_{\rm NS}}}{[\bar{\rho}_2^{\rm NS}(n,Q^2)]^{d^n_{\rm NS}}}\ . \quad (6.17)$$

(iii) Actually, there is a third way to compute the ratio $M_L^{\rm NS}(n,Q^2)/M_2^{\rm NS}(n,Q^2)$ from the relation [we used Eq. (6.4)]

$$\frac{M_L^{\rm NS}(n,Q^2)}{M_2^{\rm NS}(n,Q^2)}=\frac{\bar{A}_n^{\rm NS}[Z_n^{\rm NS}(Q^2)]^{-1}C_{L,n}^{\rm NS}[1,\rho(Q^2)]}{\bar{A}_n^{\rm NS}[Z_n^{\rm NS}(Q^2)]^{-1}C_{2,n}^{\rm NS}[1,\rho(Q^2)]} =\frac{C_{L,n}^{\rm NS}[1,\rho(Q^2)]}{C_{2,n}^{\rm NS}[1,\rho(Q^2)]}=(R_{2,n}^{\rm NS}-R_{1,n}^{\rm NS})\rho(Q^2)[1+(R_{L,n}^{\rm NS}-R_{2,n}^{\rm NS})\rho(Q^2)+\cdots] \equiv(R_{2,n}^{\rm NS}-R_{1,n}^{\rm NS})\bar{\rho}_n(Q^2)\ . \quad (6.18)$$

We shall discard the latter alternative, since the effective charge $\bar{\rho}_n(Q^2)$ is a more artificial construct, less directly related to Feynman diagrams. Furthermore, in case of discrepancy between approaches (i) and (ii), we tend to favor approach (i), which treats more symmetrically $M_1^{\rm NS}(n,Q^2)$ and $M_2^{\rm NS}(n,Q^2)$, which both start in order $O(\rho^{d_{\rm NS}^n})$. This means we assume both $\bar{\rho}_1^{\rm NS}(n,Q^2)$ and $\bar{\rho}_2^{\rm NS}(n,Q^2)$ have well-behaved $\bar{\beta}$ functions (for not too large n). If one takes this point of view, it is not necessary to compute $R_{L,n}^{\rm NS}$ to make a well-defined prediction for $M_L^{\rm NS}(n,Q^2)$. What is then the new information contained in $R_{L,n}^{\rm NS}$? It appears that the knowledge of $R_{L,n}^{\rm NS}$ is equivalent to that of

$$\Delta\bar{\beta}_3^{\rm NS}(n)=\bar{\beta}_{3,2}^{\rm NS}(n)-\bar{\beta}_{3,1}^{\rm NS}(n)\ ,$$

where $\bar{\beta}_{3,k}^{\rm NS}(n)$ $(k=1,2)$ are the three-loop coefficients of the $\bar{\beta}$ functions associated to $\bar{\rho}_k^{\rm NS}(n,Q^2)$, i.e., for the first time we have an indication of the deviation from universality for three-loop physical $\bar{\beta}$ functions. Indeed, it is easy to derive the relation

$$R_{L,n}^{\rm NS}=(1+d_{\rm NS}^n)\frac{R_{1,n}^{\rm NS}+R_{2,n}^{\rm NS}}{2d_{\rm NS}^n}+\frac{\beta_2}{\beta_1}\left[1+\frac{\Delta\bar{\beta}_3^{\rm NS}(n)}{\beta_2}\frac{d_{\rm NS}^n}{R_{2,n}^{\rm NS}-R_{1,n}^{\rm NS}}\right]. \tag{6.19}$$

At the present time, two calculations of $R_{2,n}^{\rm NS}$ have been performed.[49,50] In view of the rather large discrepancies between them, we avoid giving detailed numerical results. For illustrative purposes, however, we shall use the result $R_{L,n=4}^{\rm NS}=33.05$ (in the $\overline{\rm MS}$ scheme for four flavors) of Ref. 49 to compare the predictions for the $n=4$ value of the ratio $M_L^{\rm NS}(n,Q^2)/M_2^{\rm NS}(n,Q^2)$ obtained using the three methods described above. We choose $Q=50\Lambda_{\overline{\rm MS}}$. The values obtained for $M_L^{\rm NS}(n=4,Q^2)/M_2^{\rm NS}(n=4,Q^2)$ using methods (i), (ii), and (iii) above are then, respectively, 0.018, 0.054, and 0.020. We see that (i) and (ii) differ by a factor of 3, whereas (i) and (iii) are consistent with each other. As explained previously, we tend to favor the result of method (i). We also mention that the results of Ref. 49 give smaller values of $|\Delta\bar{\beta}_3^{\rm NS}(n)|$ than those of Ref. 50. For instance, for $n=10$ we get [using Eq. (6.19)]

$$\frac{\Delta\bar{\beta}_3^{\rm NS}(n=10)}{\beta_2}=0.5$$

with the result of Ref. 49, whereas

$$\frac{\Delta\bar{\beta}_3^{\rm NS}(n=10)}{\beta_2}=-1.7$$

with the result of Ref. 50. In any case, both results are compatible with the assumption that $\bar{\beta}_{3,2}^{\rm NS}(n)$ and $\bar{\beta}_{3,1}^{\rm NS}(n)$ are well behaved (we recall $\beta_2/\beta_1=6.16$ in QCD for four flavors), and both show that $\Delta\bar{\beta}_3^{\rm NS}(n)$ decrease with increasing n.

D. Effective charges for structure functions (nonsinglet case)

Effective charges may also be related directly to structure functions, without having to introduce the moments. The simplest example is probably afforded by the kernel of the physical[45] Altarelli-Parisi equation in the nonsinglet case [which is the inverse Mellin transform of Eq. (6.5)]:

$$Q^2\frac{dF_k^{\rm NS}(x,Q^2)}{dQ^2}=F_k^{\rm NS}(x,Q^2)\otimes P_k^{\rm NS}[x,\rho(Q^2)]\ , \tag{6.20}$$

where $\otimes$ is the convolution product, and

$$\begin{aligned}P_k^{\rm NS}[x,\rho(Q^2)]&=[p_1^{\rm NS}(x)\rho(Q^2)+p_{2,k}^{\rm NS}(x)\rho^2(Q^2)+\cdots]_+\\&=\left[p_1^{\rm NS}(x)\left[\rho(Q^2)+\frac{p_{2,k}^{\rm NS}(x)}{p_1^{\rm NS}(x)}\rho^2(Q^2)+\cdots\right]\right]_+\\&\equiv[p_1^{\rm NS}(x)\rho_k^{\rm NS}(x,Q^2)]_+\ .\end{aligned} \tag{6.21}$$

In Eq. (6.21),

$$p_1^{\rm NS}(x)=2\times\frac{4}{3}\left[\frac{1-x^2}{1-x}\right]$$

is the one-loop nonsinglet Altarelli-Parisi kernel, and $p_{2,k}^{\rm NS}(x)/p_1^{\rm NS}(x)$ as well as the whole effective charge $\rho_k^{\rm NS}(x,Q^2)$, are ordinary functions of x, not distributions [note that $\rho_k^{\rm NS}(x,Q^2)$ is not the inverse Mellin transform of $\rho_k^{\rm NS}(n,Q^2)$]. Equation (6.20) can be formally integrated, with the result

$$\begin{aligned}\ln_\otimes\{F_k^{\rm NS}(x,Q^2)\}&=A_k^{\rm NS}(x)-\left[\frac{p_1^{\rm NS}(x)}{\beta_1}\ln\rho(Q^2)+\frac{1}{\beta_1}\left[p_{2,k}^{\rm NS}(x)-\frac{\beta_2}{\beta_1}p_1^{\rm NS}(x)\right]\rho(Q^2)+\cdots\right]_+\\&=A_k^{\rm NS}(x)-\left[\frac{p_1^{\rm NS}(x)}{\beta_1}\ln\bar{\rho}_k^{\rm NS}(x,Q^2)\right]_+\ ,\end{aligned} \tag{6.22}$$

where $\bar{\rho}_k^{\rm NS}(x,Q^2)$ is the "integrated effective charge" [Eq. (4.15)] corresponding to $\rho_k^{\rm NS}(x,Q^2)$, and $A_k^{\rm NS}(x)$ is an integration constant. Furthermore, $\ln_\otimes\{F_k^{\rm NS}(x,Q^2)\}$ is the inverse Mellin of $\ln M_k^{\rm NS}(n,Q^2)$. All the complexities of integrating the Altarelli-Parisi equation in x space are now reduced to the mathematical problem of constructing the quantity $\ln_\otimes\{F_k^{\rm NS}(x,Q^2\}$, given $F_k^{\rm NS}(x,Q^2)$. It has been shown[51] that this result may be obtained by very simple analytic formulas, assuming some simple analytic fit for $F_k^{\rm NS}(x,Q^2)$. We stress that the x dependence of the effective scales $\bar{\Lambda}_k^{\rm NS}(x)$ [or $\Lambda_k^{\rm NS}(x)$] associated to $\bar{\rho}_k^{\rm NS}(x,Q^2)$ [or $\rho_k^{\rm NS}(x,Q^2)$] constitutes the cleanest test of the next-to-leading-order QCD effects in x space.

E. A comment on the $x\to 1$ behavior of the Altarelli-Parisi kernel

From the explicit calculation[45,47] of $p_{2,k}^{\rm NS}(x)$, it is known that

$$c_1(x)\equiv\frac{p_{2,k}^{\rm NS}(x)}{p_1^{\rm NS}(x)}\underset{x\to 1}{\sim}\beta_1\ln\frac{1}{1-x}\ . \qquad (6.23)$$

Equation (6.23) implies that perturbation theory (both improved and nonimproved) for the effective charge

$$\rho_k^{\rm NS}(x,Q^2)=\rho(Q^2)[1+c_1(x)\rho(Q^2)+c_2(x)\rho^2(Q^2)+\cdots]$$

[or $\bar{\rho}_k^{\rm NS}(x,Q^2)$] breaks down as $x\to 1$ at fixed Q^2, since it implies

$$[Q/\Lambda_k^{\rm NS}(x)]^2\underset{x\to 1}{\sim}\frac{Q^2(1-x)}{\Lambda^2}\sim\frac{W^2}{\Lambda^2}\ . \qquad (6.24)$$

On the other hand, no singularity appears as $x\to 1$ at fixed $W^2=Q^2(1-x)/x$. Kinematical considerations[52] suggest that W^2 is the natural variable in the $x\to 1$ region. In fact, it has been argued[53] that the leading singularities as $x\to 1$ in each order of perturbation theory are reproduced by performing the rescaling $Q^2\to Q^2(1-x)$ in the argument of the running coupling constant in the leading-order Altarelli-Parisi kernel

$$p_1^{\rm NS}(x)\rho(Q^2)\to p_1^{\rm NS}(x)\rho[Q^2(1-x)]\ . \qquad (6.25)$$

Concerning nonleading singularities, the simplest possibility would be that the effective charge $\rho_k^{\rm NS}(x,Q^2)$ has a finite limit as $x\to 1$ at fixed W^2 order by order in perturbation theory in ρ, i.e., that the rescaling $Q^2\to Q^2(1-x)$, performed in each order, absorbs all $x=1$ singularities. In such a case, $\rho_k^{\rm NS}(x,Q^2)$ would be a well-behaved effective charge as $x\to 1$ provided the limit is taken at fixed W^2. The simple change of variable $Q^2\to W^2$ would then have reduced the problem to an essentially single-scale problem, and renormalization-group-improved perturbation theory would be applicable in the region $W^2\gg\Lambda^2$ at $x\simeq 1$.

F. Sudakov form factor of the electron in QED

This is a well-known case of a so-called "double-logarithm" resummation problem, which provides us with another typical example of the possibility, mentioned in the introduction to Sec. VI, of decomposing, in some kinematical limits, a two-scale amplitude into several functions of a single scale, to each of which the method of effective charges may in turn be applied. Let $F_1(Q/m,\alpha,\lambda/m)$ be the Dirac on-shell electron form factor, where $Q^2\equiv -q^2>0$, q^2 is the momentum transfer, α the fine-structure constant, m the electron mass, and λ a photon mass used as infrared regulator. It is well known that the infrared divergences exponentiate:[54]

$$F_1(Q/m,\alpha,\lambda/m)=\exp\left[\frac{\alpha}{\pi}B(Q/m)\ln\frac{\lambda}{m}\right]F(Q/m,\alpha)\ , \qquad (6.26)$$

where B is a known function, and the "reduced form factor" $F(Q/m,\alpha)$ is free of infrared divergences. We are interested in the large-momentum transfer behavior $Q/m\to\infty$, and shall therefore deal with the asymptotic reduced form factor $F_\infty(Q/m,\alpha)$, defined by dropping all terms which vanish as $m/Q\to 0$, order by order in perturbation theory. $F_\infty(Q/m,\alpha)$ is known[54] up to $O(\alpha^2)$:

$$F_\infty(Q/m,\alpha)=1+\frac{\alpha}{\pi}\left[-t^2+\tfrac{3}{2}t-1+\frac{\pi^2}{12}\right]+\left[\frac{\alpha}{\pi}\right]^2\left[\tfrac{1}{2}t^4-\tfrac{31}{18}t^3+\left[\frac{229}{72}-\frac{\pi^2}{3}\right]t^2+\text{const}\right]+\cdots$$

$$(t\equiv\ln Q/m)\ . \qquad (6.27)$$

We shall now make use of the new information that the structure of the "Sudakov logarithms" t^n is governed by the following equation:[54–56]

$$\left[m\frac{\partial}{\partial m}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}-\left[\gamma_0(\alpha)+\gamma_1(\alpha)\ln Q/m\right]\right]F_\infty\left(\frac{Q}{m},\alpha\right)=0\ , \qquad (6.28)$$

where, coming back to standard conventions, we define

$$m\frac{\partial\alpha}{\partial m}=\alpha\beta(\alpha)\ ,$$
$$\beta(\alpha)=\beta_1\frac{\alpha}{\pi}+\beta_2\left[\frac{\alpha}{\pi}\right]^2+\cdots\ , \qquad (6.29)$$

and $\gamma_0(\alpha),\gamma_1(\alpha)$ are anomalous dimension-type functions.

To solve Eq. (6.28), it is instructive to proceed as follows. We first rewrite it as

$$m\frac{d\ln F_\infty}{dm}=\gamma_0(\alpha)+\gamma_1(\alpha)\ln\frac{Q}{m}\ , \qquad (6.30)$$

where

$$m\frac{d}{dm}\equiv m\frac{\partial}{\partial m}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}\ ,$$

and differentiate both sides of Eq. (6.30) with respect to $\ln Q$:

$$\frac{d^2 \ln F_\infty}{d\ln m\, d\ln Q}=\gamma_1(\alpha)=\gamma_{11}\frac{\alpha}{\pi}+\gamma_{12}\left[\frac{\alpha}{\pi}\right]^2+\cdots . \tag{6.31}$$

Then, integrating Eq. (6.31) with respect to $\ln m$, we get

$$Q\frac{d\ln F_\infty}{dQ}=\frac{\gamma_{11}}{\beta_1}\ln\left[\frac{\bar{\alpha}(m)}{\bar{\rho}(Q)}\right], \tag{6.32}$$

where

$$\bar{\alpha}(m)=\alpha\left[1+\frac{\alpha}{\pi}\delta_1+\cdots\right]$$

is an m-dependent "integrated" effective charge solution of the equation

$$m\frac{d}{dm}(\ln\bar{\alpha})=\frac{\beta_1}{\gamma_{11}}\gamma_1(\alpha) \tag{6.33}$$

and

$$-\frac{\gamma_{11}}{\beta_1}\ln\bar{\rho}(Q)$$

is a (Q-dependent) integration constant, expressed in terms of an effective charge

$$\bar{\rho}(Q)=\alpha\left[1+\frac{\alpha}{\pi}\left[\beta_1\ln\frac{Q}{m}+e_1\right]+\cdots\right].$$

The latter satisfies the homogeneous Callan-Symanzik equation

$$\left[m\frac{\partial}{\partial m}+\alpha\beta(\alpha)\right]\bar{\rho}(Q/m,\alpha)=0 \tag{6.34}$$

and is therefore a function of the zero-mass theory (no m dependence remains if $\bar{\rho}$ is expanded in a mass-independent RS, with an arbitrary renormalization point μ).

The last step is to integrate Eq. (6.32) with respect to $\ln Q$, thus getting

$$\ln F_\infty(Q/m,\alpha)=-\frac{\gamma_{11}}{\beta_1}\int_m^Q\frac{dk}{k}\ln\left[\frac{\bar{\rho}(k)}{\bar{\alpha}(m)}\right]+D(\alpha), \tag{6.35}$$

where

$$D(\alpha)=d\frac{\alpha}{\pi}+O\left[\left[\frac{\alpha}{\pi}\right]^2\right]$$

is an m-dependent integration constant related to $\gamma_0(\alpha)$ by the relation

$$m\frac{\partial D}{\partial m}=\gamma_0(\alpha)+\frac{\gamma_{11}}{\beta_1}\ln\left[\frac{\bar{\alpha}(m)}{\bar{\rho}(m)}\right]. \tag{6.36}$$

Equation (6.35) shows that $F_\infty(Q/m,\alpha)$ can be expressed in terms of three effective charges $\bar{\rho}(Q)$, $\bar{\alpha}(m)$, and $(\pi/d)D(\alpha)$, each depending upon one scale variable only (Q or m), the mass singularities being isolated into $\bar{\alpha}(m)$ and $D(\alpha)$.

The coefficients e_1 and δ_1, which determine the effective scales associated to $\bar{\rho}(Q)$ and $\bar{\alpha}(m)$, are easily obtained, expanding Eq. (6.35) in powers of α, and comparing with Eq. (6.27). One gets (using also $\beta_1=\frac{2}{3}$, $\beta_2/\beta_1=\frac{3}{4}$)

$$e_1=-\frac{65}{36}+\frac{\pi^2}{4}, \qquad \delta_1=-\frac{47}{36}+\frac{\pi^2}{4} \tag{6.37}$$

(together with $\gamma_{11}/\beta_1=3$ and $d=-1+\pi^2/12$).

We note that Eq. (6.35) can be cast into the alternative, perhaps more familiar looking form

$$\ln F_\infty(Q/m,\alpha)=-\frac{\gamma_{11}}{\pi}\int_m^Q\frac{dk}{k}\ln\frac{Q}{k}\tilde{\rho}(k)-\frac{\gamma_{11}}{\beta_1}\ln\left[\frac{\bar{\rho}(m)}{\bar{\alpha}(m)}\right]+D(\alpha), \tag{6.38}$$

where the effective charge $\tilde{\rho}(Q)$ is the "derivative" of $\bar{\rho}(Q)$

$$Q\frac{d}{dQ}(\ln\bar{\rho})=\beta_1\frac{\tilde{\rho}(Q)}{\pi} \tag{6.39}$$

and is given by the expansion

$$\tilde{\rho}(Q)=\alpha\left[1+\frac{\alpha}{\pi}\left[\beta_1\ln\frac{Q}{m}+e_1+\frac{\beta_2}{\beta_1}\right]+\cdots\right].$$

$\tilde{\rho}(Q)$ is very simply related to F_∞ by the equation

$$\frac{d^2\ln F_\infty}{(d\ln Q)^2}=-\frac{\gamma_{11}}{\pi}\tilde{\rho}(Q), \tag{6.40}$$

which is the RS-invariant version of Eq. (6.28).

VII. CONCLUSIONS

Let us now summarize the main points of the present method.

(1) The standard perturbative QCD prediction for a physical quantity σ goes through two steps.

(i) One expands σ in powers of some renormalized coupling constant $\alpha_s(Q^2)$.

(ii) One solves the renormalization-group equation for $\alpha_s(Q^2)$ and expands the solution in powers of $1/\ln(Q^2/\Lambda^2)$.

Thus, in the standard approach, perturbation theory is defined in terms of two different series expansions, one for σ and the other for the RS β function. Instead, in the method of effective charges, these two series are viewed only as intermediate computational steps, devoid of fundamental significance, which exist only to be merged in the single series for the $\bar{\beta}(\bar{\rho})$ function associated to the corresponding effective charge. Integration of the resulting renormalization-group-equation then yields the Q^2 dependence of $\bar{\rho}$ in the form $Q^2/\Lambda^2=\bar{F}^{-1}(\bar{\rho})$, free of any scale ambiguity. In practice, the simplest way to obtain the function $\bar{F}^{-1}(\bar{\rho})$ is to proceed in the reverse order to the standard approach. First, one solves the renormalization-group equation for the RS coupling α_s

used in the calculation, getting a function $Q^2/\Lambda^2 = F^{-1}(\alpha_s)$, and second, one inverts the expansion of $\bar{\rho}$ in powers of α_s and substitute for α_s as a function of $\bar{\rho}$ in $F^{-1}(\alpha_s)$. Consideration of the "inverse function" $\bar{F}^{-1}(\bar{\rho})$ is also extremely natural from the following viewpoint. The QCD prediction for any physical quantity having dimension of mass takes the form $m_i/\Lambda = C_i$, where C_i is a pure number, depending, however, on the definition of the quantity m_i. Now, the scale Q can be looked upon as such a physical mass scale, defined by the value of $\bar{\rho}$ (or σ); the function $\bar{F}^{-1}(\bar{\rho})$ is then simply the number C_i corresponding to Q, the $\bar{\rho}$ dependence of $\bar{F}^{-1}$ reflecting the definition dependence of Q.

This approach, which takes from the start into account the dimensional transmutation of the renormalized theory, represents a straightforward extension of the Gell-Mann—Low theory to all physical quantities. In fact, it is closer in spirit to the method of Gell-Mann—Low than to that of Stückelberg-Peterman-Callan-Symanzik: it solves the renormalization-group equations of the latter in terms of the various physical RS-invariant $\bar{\beta}$ function.[57] Consequently, perturbation theory is recognized as being essentially the perturbative expansion of each $\bar{\beta}(\bar{\rho})$ function. From this point of view, the importance of the coming generation of higher-order perturbative QCD calculations stems from the fact that they will allow us to extract the values of the physical three-loop $\bar{\beta}$-function coefficients, these being the first nonuniversal coefficients which could give a true indication of the convergence of perturbation theory for physical quantities.

(2) The method of effective charges gives RS-invariant predictions, in the sense that these predictions depend on no arbitrary quantity foreign to the physical quantity σ of interest. In this respect, it can be viewed as giving its full meaning to one of the main advantages of the minimal-subtraction scheme, to wit, that it is possible to give renormalized predictions without having to compute other Feynman diagrams than those belonging to the quantity under consideration. In fact, no other coupling constant, apart from those which automatically appear, i.e., the effective charge $\bar{\rho}$ and the bare coupling ρ_b (or, equivalently, the MS coupling $\rho_{\rm MS}$) need be introduced. Furthermore, the structure of the bare expansion of σ (with cutoff regularization), or, equivalently, of its expansion in the MS scheme, can be disentangled in terms of the β_b (or $\beta_{\rm MS}$) function and the $\bar{\beta}$ function, which appear in a quite symmetrical way, although only the latter contains the invariant physical information.

Nevertheless, we have shown that, if the $\bar{\beta}$ function is well behaved and its expansion in powers of $\bar{\rho}$ converge, the fastest apparent convergence choice of the renormalization point [Eq. (2.12)] will give reliable results, in essential agreement with the method of effective charges, in any well-behaved RS (like, presumably, the MS scheme). We find it, however, both simpler and nicer to give predictions invariant under the general Stückelberg-Peterman renormalization group in terms of $\bar{\beta}$.

(3) The method of effective charges makes use only of the "kinematics" of renormalizability, it has no dynamical content. It applies in a straightforward way only to those problems where no more than a single energy scale is involved: The one scale which must necessarily accompany the cutoff to make a dimensionless argument in the ultraviolet logarithms. When more than one scale is involved, the present method can *a priori* offer no guidance: specific study is required to learn how to reduce the problem to a single-scale one, two typical examples being offered by the factorization theorem, and the equation governing the Sudakov form factor of the electron in QED.

(4) The present method solves the ambiguity related to the choice of RS. Indeed, for a given effective charge, the prediction obtained is essentially unique, up to the ordinary ambiguity related to the use of perturbation theory for the $\bar{\beta}(\bar{\rho})$ function (e.g., use of truncated expansion of $\bar{\beta}$, or of $1/\bar{\beta}$, or any other reasonable form of approximation). The latter ambiguity poses no problem in practice, if perturbation theory is applicable to $\bar{\beta}(\bar{\rho})$. However, one has to identify *a priori* the "good," well-behaved effective charges, to which (renormalization-group-improved) perturbation theory may safely be applied; this identification may in some cases be the source of a new kind of ambiguity.

(5) When corrections in usual RS are large (for instance, in quarkonium decay)[8], quantitatively different, renormalization-group-improved results are obtained.

(6) In deep-inelastic processes, where a factorization between long and short distances has been established, we have shown that effective charges appear in a natural way in a formalism free of any factorization scale ambiguity, since factorization can be performed without introducing a factorization scale. The difference between the procedure here proposed and the standard one is easy to see explicitly in lowest order. If

$$M(p^2,Q^2) = 1 + \rho\left[-\gamma_1 \ln\frac{Q^2}{p^2} + \text{const}\right] + \cdots$$

is a parton structure-function moment, the standard factorization reads

$$M(p^2,Q^2) = \left[1 + \rho\left[-\gamma_1 \ln\frac{Q^2}{M^2} + \text{const}\right] + \cdots\right] \times \left[1 + \rho\left[-\gamma_1 \ln\frac{M^2}{p^2} + \text{const}\right] + \cdots\right],$$

where M is the factorization scale, whereas we suggest to write instead

$$M(p^2,Q^2) = \left\{\rho\left[1 + \rho\left[-\beta_1 \ln\frac{Q^2}{\mu^2} + \text{const}\right] + \cdots\right]\right\}^{\gamma_1/\beta_1} \times \left\{\rho\left[1 + \rho\left[-\beta_1 \ln\frac{p^2}{\mu^2} + \text{const}\right] + \cdots\right]\right\}^{-\gamma_1/\beta_1},$$

where μ is the renormalization point of the coupling ρ, and each factor is separately renormalization-group invariant. These remarks apply not only to the classic light-cone-dominated processes, but also timelike processes, or Drell-Yan[58] (assuming it factorizes[59]), as is clear from the formal similarity between operator-product expansion and the cut-vertices formalism.[60] We also note that the possibility of eliminating completely the factorization scale may simplify the study of other processes beyond deep-inelastic scattering, such as that of nonleptonic decays.[61]

(7) Finally, we mention that a straightforward extension of the present method to the case where the theory depends upon n bare coupling constants leads to a system of n coupled differential equations, relating a given effective charge to its first $n-1$ derivatives with respect to Q^2.

APPENDIX A: COMMENTS ON STEVENSON'S OPTIMIZED PERTURBATION THEORY

Stevenson has proposed[9] to solve the RS ambiguity by requiring properly defined finite-order approximations to be stationary with respect to the RS-labeling parameters λ_i [see Eq. (3.8)]. We would like (i) to present a general argument, and an explicit example, to show that his "principle of minimum sensitivity" (PMS) method yields results equivalent to the method of effective charges, at least as long as renormalization-group-improved perturbation theory is applicable (this equivalence has been demonstrated in low orders in a different way in Ref. 62); and (ii) to point out some obvious advantages of the method of effective charges over PMS. Let us review these points in turn.

(i) First of all, one should note the close connection between Stevenson's invariants ρ_i and the $\bar{\beta}$-function coefficients $\bar{\beta}_i$ associated to a given physical quantity σ. They are essentially the same objects, being related by simple algebraic relations. For instance, assuming $\sigma \equiv \bar{\rho} \equiv \bar{\alpha}_s/\pi$, and putting

$$\bar{\beta}(\bar{\rho}) = -\sum_{n=1} \bar{\beta}_n (\bar{\rho})^n ,$$

we have

$$\rho_2 = \frac{\bar{\beta}_3}{\beta_1} - \frac{1}{4}\left[\frac{\beta_2}{\beta_1}\right]^2 , \qquad \rho_3 = \frac{1}{2}\frac{\bar{\beta}_4}{\beta_1} , \tag{A1}$$

etc. The easiest way to obtain these relations is to compare the "improvements formulas" and Eq. (5.26) of the first paper in Ref. 9 with our general formulas Eq. (3.4). Also, the invariant ρ_1 is essentially $\beta_1 \ln(Q^2/\bar{\Lambda}^2)$. Now, any approximated prediction $\bar{\rho} = F_{\text{app}}(Q^2/\mu^2, \rho)$, invariant under the renormalization group, and having a well-defined perturbative expansion around $\rho = 0$, can be characterized by an effective $\bar{\beta}_{\text{app}}(\bar{\rho})$ function, with a well-defined expansion around $\bar{\rho} = 0$ (these conditions are met by the PMS ansatz). Consequently, any two such predictions can only differ by the values assigned in $\bar{\beta}_{\text{app}}(\bar{\rho})$ to the unknown (i.e., not yet calculated at a given order of perturbation theory) higher-order coefficients of the exact $\bar{\beta}(\bar{\rho})$ function (the known, calculated $\bar{\beta}_i$'s must necessarily coincide). In particular, the PMS choice for these higher order $\bar{\beta}_i$'s represent one such arbitrary ansatz. This arbitrariness is a reflection of the arbitrariness of the truncation of the β function pointed out in Ref. 33, which comes in the definition of the finite-order approximations considered by Stevenson; note, in particular, that optimization can be performed using different definitions of the original finite-order noninvariant approximations; for instance, one can use approximations based on the truncation of $1/\beta$ instead of β itself (see below). A similar arbitrariness exists in the method of effective charges, which makes no commitment concerning the unknown $\bar{\beta}_i$'s, apart from the assumption they are well behaved [Eq. (2.15)], and can accommodate any guess for them, as long as this guess is consistent with the well-behaved behavior assumed to be checked in the known, calculated $\bar{\beta}_i$'s. In Sec. II, we suggested the two simplest, obvious guesses (truncation of $\bar{\beta}$ or $1/\bar{\beta}$). But the PMS guess for $\bar{\beta}$ is also admissible (although more complicated), since the values suggested by PMS for the unknown $\bar{\beta}_i$'s can only be algebraic combinations of the known ρ_i's, i.e., the known $\bar{\beta}_i$'s and presumably (at least in low orders) are consistent with condition Eq. (2.15) (assuming the known $\bar{\beta}_i$'s satisfy it). Consequently, any PMS prediction for $\bar{\rho}$ can also be considered as a prediction of the method of effective charges. Let us make the previous arguments more concrete by the following example. Consider the simplest approximation of the method of effective charges, Eq. (2.11a), based on the truncation of $1/\bar{\beta}$ at second order. One can solve Eq. (2.11a) as an expansion in powers of $1/\ln Q^2/\Lambda^2$, and obtain the standard result

$$\bar{\rho}(Q^2) = \frac{1}{\beta_1 \ln(Q^2/\Lambda^2)}\left[1 - \frac{\beta_2}{\beta_1}\frac{\ln\ln(Q^2/\Lambda^2)}{\beta_1 \ln(Q^2/\Lambda^2)} + \frac{\ln\lambda^2}{\ln(Q^2/\Lambda^2)} + \cdots\right] , \tag{A2}$$

where we made the further approximation of truncating the expansion beyond third order, and we put

$$\ln\lambda^2 = \frac{K_1}{\beta_1} = \ln\frac{\bar{\Lambda}^2}{\Lambda^2} . \tag{A3}$$

Equation (A2) is ambiguous, since it depends on the arbitrary choice of the parameter Λ (or λ). One can then try to solve this ambiguity by applying the optimization idea of Stevenson directly to Eq. (A2), i.e., find the "optimum" values of λ [hence, using Eq. (A3) of Λ] and of $\bar{\rho}(Q^2)$, by requiring approximation Eq. (A2) to be stationary with respect to variations of Λ (at fixed $\bar{\Lambda}$: note that λ becomes a function of Λ). The result of this exercise is

$$\beta_1 \ln\lambda_{\text{opt}}{}^2 = \frac{\beta_2}{\beta_1}\ln\ln\frac{Q^2}{\Lambda_{\text{opt}}{}^2} - \frac{\beta_2}{2\beta_1} . \tag{A4}$$

Hence

$$\bar{\rho}_{\text{opt}}(Q^2) = \rho\left[1 - \frac{\beta_2}{2\beta_1}\rho\right]$$

with (A5)

$$\beta_1 \ln\frac{Q^2}{\bar{\Lambda}^2} = \frac{1}{\rho} + \frac{\beta_2}{\beta_1}\ln(\beta_1\rho) + \frac{\beta_2}{2\beta_1} .$$

As the reader can easily check, following the method of Ref. 9, Eq. (A5) is precisely the PMS ansatz for $\bar{\rho}$ in second-order optimized perturbation theory, where the perturbative approximations are defined with the inverse

β function truncated at second order (instead of the β function itself as used in Ref. 9). Therefore, optimization in second order can be viewed as an approximate solution of the two-loop renormalization-group equation (2.11a) for the effective charge $\bar{\rho}$. It is clear, however, that there is no particular advantage to substitute Eq. (A5) to the simpler relation (2.11a). This remark leads us to our second point.

(ii) Let us list a few advantages of the method of effective charges.

(a) It is simpler than PMS. One does not need to go through a complicated optimization procedure, especially in higher orders.

(b) Its use allows discussion of the convergence of perturbation theory in terms of the convergence of successive approximations to a single well-defined function $\bar{\beta}(\bar{\rho})$, directly related to Feynman diagrams (Sec. III). The optimization method does not introduce such a function. To deal with a well-defined function, rather than a sequence of optimized approximants, is also advantageous from the point of view of the question of the resummation properties of perturbation theory to all orders. Indeed, the method of effective charges makes no *a priori* commitment on the resummation properties of $\bar{\beta}(\bar{\rho})$ (as seems natural), leaving open the possibility of applying any specific resummation procedure which will emerge as an outcome of future progress in this field. On the other hand, Stevenson's framework appears to be artificially restrictive, since the sequence of optimized approximants automatically either converge[63] or diverge, leaving no freedom for further improvement. The renormalization group cannot by itself provide a "built-in" resummation procedure, as the present method, which satisfies renormalization-group invariance (see Sec. III), explicitly demonstrates. Furthermore, even if a particular sequence of optimized approximants do converge, the answer may well depend on the arbitrary way the original nonoptimized finite-order approximations have been defined.

(c) If one tries to optimize the second-order result for the expansion of the inverse effective charge $1/\bar{\rho}$, the PMS method runs into spurious difficulties,[9] which are not present in the method of effective charges. The latter recognize the origin of these difficulties to stand in the fact that the one-loop renormalization-group-improved perturbation theory formula for $\bar{\rho}$ coincides with the ordinary second-order perturbation theory expression for $1/\bar{\rho}$ [see Eq. (3.12)]. Hence, in second order, the optimization is in some sense already performed, and there is nothing to improve upon.

(d) Stevenson's approach uses the renormalization group in a twofold, and from the present viewpoint, unnecessarily awkward manner: a first time, in a standard way, at the level of the definition of the approximants to which the optimization is intended to be applied (since one uses the μ dependence of the running coupling constant as given by the renormalization group); a second time, in a more unorthodox manner, in the process of optimization itself. Note that the μ dependence of the running coupling constant is obtained, even in Stevenson's approach, by integration of the renormalization-group equation, and not by applying a PMS procedure. In contrast, the method of effective charges uses the renormalization group in a uniform and most standard manner, at the level of the renormalization-group equation for $\bar{\rho}$. Clearly, once it is recognized that $\bar{\rho}(Q)$ defines a particular RS, there is no need to treat the Q dependence of $\bar{\rho}$ in a different way from the μ dependence of the other couplings. Furthermore, the resulting solution for $\bar{\rho}$ is invariant under the general Stückelberg-Peterman renormalization group, as pointed out in Sec. III A. In fact, with a given approximation for β and $\bar{\beta}$, ρ as given by Eq. (3.2) satisfies Eq. (3.7) identically: there is no need to look for a stationary point.

As a last point, we note that the PMS was originally inspired[9] from examples taken outside field theory, where no analog of the $\bar{\beta}(\bar{\rho})$ functions exists. Whereas PMS seems a perfectly sensible procedure in these examples, the existence of the $\bar{\beta}$ functions in field theory (reflecting the dimensional transmutation) renders the whole PMS procedure unnecessary here.

APPENDIX B: COMMENT ON THE BRODSKY, LEPAGE, AND MACKENZIE CRITERION

Inspired by intuitive considerations based on QED, Brodsky, Lepage, and Mackenzie (BLM) have proposed[7] to fix the renormalization point μ of the running coupling constant by requiring all flavor-dependent terms in the next-to-leading-order coefficient in the expansion of a given physical quantity to be absorbed into the coupling, by means of a flavor-independent change of μ.

To explain their procedure, we note that the expansion of an effective charge $\bar{\rho}(Q)$ in an arbitrary RS can be written, up to next-to-leading order, as

$$\bar{\rho}(Q)=\rho(1+c_1\rho+\cdots) \tag{B1}$$

with

$$c_1=-\beta_1\left[\ln\frac{Q^2}{\mu^2}+d_1^*\right]+c_1^* ,$$

where $\beta_1=11-2f/3$ in QCD, and d_1^* and c_1^* are f-independent constants. BLM suggest to fix μ in such a way that $\ln(Q^2/\mu^2)+d_1^*=0$, leaving no explicit f dependence in c_1. Although the value $c_1=c_1^*$ thus obtained depends on the initial choice of RS,[64] the difference between the c_1^*'s associated to two effective charges is RS invariant.[7] This property allows one to give a RS-invariant classification of effective charges, since one can group in the same class those charges which have the same (or close) values of c_1^*. In fact, BLM found that most effective charges have rather similar values of c_1^*, except in the case of Υ decay. This result substantiates their claim that perturbation theory is not reliable for Υ decay.

On the other hand, as explained in this paper, all the RS-invariant information concerning a given physical quantity is contained in its associated effective scale $\bar{\Lambda}$, and the $\bar{\beta}$-function coefficients. Furthermore, we saw that the value of $\bar{\Lambda}$ is the only invariant information available in the next-to-leading-order coefficient, which seems to be contradicted by the BLM finding that some additional information is in fact contained in c_1^*. The question thus naturally arises as to what is the nature of this new information, and of its eventual bearing on the issue of the reliability of perturbation theory.

It turns out that c_1^* has a very simple interpretation in the framework of the method of effective charges, based on the observation that it satisfies the relation

$$c_1^* = -\frac{\bar{\beta}_3^* - \beta_3^*}{\beta_2^*} , \tag{B2}$$

where $\bar{\beta}_3$ and β_3 are, respectively, the effective charge and RS three-loop β-function coefficients, and the asterisk indicates that the various quantities are evaluated at $f=33/2$. To prove Eq. (B2), we note that c_1^* is also the value of c_1 in a world where $\beta_1=0$ [see Eq. (B1)], which happens for $f=\frac{33}{2}$. On the other hand, the second relation in Eq. (3.4) tells us that

$$(c_2 - c_1{}^2)\beta_1 = \beta_2 c_1 + \bar{\beta}_3 - \beta_3 . \tag{B3}$$

Since the c_i's are polynomial in f, the left-hand side of Eq. (B3) vanishes as $f\to\frac{33}{2}$, hence Eq. (B2) follows.

Equation (B2) gives a nice proof and interpretation of the above-mentioned properties of c_1^*: whereas c_1^* is RS dependent (through the β_3^* factor), it contains invariant information (through the $\bar{\beta}_3^*$ factor). However, this information is only relevant to the world of $f=\frac{33}{2}$ flavors, which solves the previously mentioned contradiction. The BLM observation can thus be restated as the fact that "most" processes have similar values of $\bar{\beta}_3^*$, except Υ decay. Using $\beta_2=102-\frac{38}{3}f$ and[28] $\beta_{3,\mathrm{MS}}=\frac{2857}{2}-\frac{5033}{18}f+\frac{325}{54}f^2$, one finds

$$\beta_2^* = -107 ,$$
$$\frac{\beta_{3,\mathrm{MS}}^*}{\beta_2^*} \simeq 14.25 . \tag{B4}$$

On the other hand, for the ratio

$$R \equiv \frac{81\pi e_b{}^2}{10(\pi^2-g)}\alpha^2\frac{\Gamma_g(\Upsilon\to\text{hadrons})}{\Gamma(\Upsilon\to\mu^+\mu^-)}$$
$$\equiv (4\pi\bar{\rho}_\Upsilon)^3 ,$$

where Γ_g is the gluonic width of the Υ, α the electromagnetic coupling constant, and $e_b=-\frac{1}{3}$ is the charge of the b quark, one gets,[7] in the MS scheme,

$$c_{1,\Upsilon}^* \simeq -18.67 .$$

Hence

$$\frac{\bar{\beta}_{3,\Upsilon}^*}{\beta_2^*} = 33.12 \simeq (2.3)\frac{\beta_{3,\mathrm{MS}}^*}{\beta_2^*} . \tag{B5}$$

We conclude that, although the three-loop effective-charge $\bar{\beta}_\Upsilon$-function coefficient turns out to be about twice as large in the case of Υ decay, compared to the β_{MS} three-loop coefficient (which is perhaps not such a large variation), this result is only relevant to the world of $f=\frac{33}{2}$ flavors, very far from the QCD case (at $f=\frac{33}{2}$ the theory is not even asymptotically free). In particular, Eq. (B5) says nothing about the behavior of $\bar{\beta}_{3,\Upsilon}$ around $f=4$, which is the real issue, and therefore we cannot, on the basis of Eq. (B5) alone, endorse the statement of Ref. 7 that perturbation theory is endangered for Υ decay.

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[17]We hope the reader does not get confused by the fact that these quantities are said to be RS invariant, despite the fact that they themselves define a particular RS. The point, of course, is that they do not depend on *another* RS than the one defined by $\bar{\rho}$.

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$$M_1^{\rm NS}(n,Q^2)=2\int_0^1 dx\, x^{n-1}F_1^{\rm NS}(x,Q^2)\,,$$
$$M_2^{\rm NS}(n,Q^2)=\int_0^1 dx\, x^{n-2}F_2^{\rm NS}(x,Q^2)\,.$$

[44]Since $\bar{A}^{\rm NS}_n$ is a dimensionless renormalization-group-invariant quantity depending on no kinematical variable (at fixed n). For moments of structure functions of off-shell partons, $\bar{A}^{\rm NS}_n=\bar{A}^{\rm NS}_n(p^2)$, a renormalization-group-invariant function of the parton momentum square p^2.

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