Radiative-recoil corrections to muonium and positronium hyperfine splitting

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We present a detailed description of the calculation of radiative-recoil corrections to the ground-state hyperfine splitting in muonium and positronium to order $\alpha^2 E_F$, the results of which were previously reported. All these corrections involve two-photon exchanges with one-loop radiative corrections to either a photon or a lepton. The QED vacuum polarization corrections are evaluated completely analytically to the order of interest. The hadronic contributions are estimated and found to be very small. As a preliminary to the lepton line calculation, a compact expression is derived for the radiative correction to such lines. This factor is then applied in a number of different contexts: the recalculation of the old nonrecoil result, which is known analytically; the analytic evaluation of terms of order $\alpha^2(m_e/m_\mu)\ln(m_\mu/m_e)E_F$, which arise from the electron leg; and the nonlogarithmic terms from both lines, which require numerical calculations. The muonium results are $\nu_{\mu^+e^-}=(\alpha/\pi)^2(m_e/m_\mu)\,[-2\ln^2(m_\mu/m_e)+\frac{13}{12}\ln(m_\mu/m_e)+18.18\pm0.58]E_F$ and those for positronium are $\nu_{e^+e^-}=\alpha^2(-1.788\pm0.004)E_F$, where E_F for positronium does not include the annihilation contribution.

I. INTRODUCTION

Recent precision measurements of the ground-state hyperfine splitting (hfs) in muonium[1] have generated considerable activity[2–8] to complete the calculation of contributions of relative order 10^{-6}. Since strong and weak corrections are negligible at this order, the comparison of theory to experiment provides an especially clear test of relativistic two-body bound-state theory in quantum electrodynamics.

The leading contribution to the hfs in muonium is given in terms of the Fermi splitting

$$h\Delta\nu\cong E_F\equiv\tfrac{16}{3}\alpha^2 R_\infty c\left[\frac{m_r}{m_e}\right]^3\frac{\mu_\mu}{\mu_B}\ ,$$

where μ_B is the Bohr magneton, and μ_μ is the muon's magnetic moment. Corrections to the leading order multiply E_F by an expansion in powers of α and (m_e/m_μ), where the terms may also have factors of $\ln\alpha^{-1}$ or $\ln(m_\mu/m_e)$. Present efforts center on terms of order α^3 and $\alpha^2 m_e/m_\mu$; owing to a proper treatment of reduced-mass effects, there are no terms of order $\alpha(m_e/m_\mu)^2$ or $(m_e/m_\mu)^3$. Recoil corrections are characterized by factors of (m_e/m_μ). They occur either as purely dynamical effects associated with exchanged photons (simply, recoil corrections) or as radiative corrections to such contributions (radiative-recoil corrections). Recently Bodwin, Yennie, and Gregorio[6] have completed the calculation of recoil corrections of relative order $\alpha^2 m_e/m_\mu$. The importance of radiative-recoil corrections at this level of accuracy was first pointed out by Caswell and Lepage,[5] who calculated the leading term of order $(\alpha/\pi)^2(m_e/m_\mu)\times\ln^2(m_\mu/m_e)$. The present work, whose results have been previously reported,[7] is an extension of their work to the analytic calculation of singly logarithmic terms and the analytic and numerical calculation of nonlogarithmic terms. The only approximation made is the neglect of external wave-function momenta: their inclusion would give corrections with an additional power of α. Since our results are valid to all orders in m_e/m_μ, results for positronium of relative order α^2 can also be obtained.

This paper is structured as follows: In the remainder of the Introduction, we review the leading one-loop recoil correction of relative order

$$(\alpha/\pi)(m_e/m_\mu)\ln(m_\mu/m_e)$$

and the nonrecoil radiative corrections. Vacuum polarization effects, including an estimate of the hadronic contribution, are treated in Sec. II. We have also calculated these terms for the case of positronium. Section III is devoted to a derivation of various expressions for radiative corrections to the lepton lines. The number of diagrams which must be considered is reduced by the use of the Fried-Yennie gauge, which is discussed further in Appendix A. Some of the forms are based on an extension of the well-known low-energy theorem for Compton scattering.[9] For the present application, this approximation is justified using formal operator techniques in Appendix B. In Sec. IV various applications are presented: a rederivation of the leading nonrecoil correction is given first, then corrections to the electron and muon lines in muonium are described, and finally corrections for positronium are given. Section V summarizes our results and gives a comparison with experiment. Appendixes C and D are a brief presentation of some technical points.

A. General background of the calculation

Various formulations of the bound-state problem in quantum electrodynamics (QED) have been given.[2,3,10,11] While this problem has many subtleties, we can give here a simplified description of the main ideas which will be adequate for our present purposes. The principal ingredients of any formulation are (i) a basic wave function resulting from some simplification of the general formulation and (ii) a set of perturbation kernels incorporating effects not included in the basic wave function. The optimum manner of separating the physics into the wave function and perturbations is neither unique nor obvious. At a minimum, the basic wave function should include the nonrelativistic physics which dominates Coulomb bound states. The "large-large" portion of the wave function is, for our present purposes, well approximated by a wave function derived from a Schrödinger equation with reduced mass and instantaneous Coulomb interaction. For the ground-state hyperfine splitting, which is our present interest, it is an eigenstate of the operator $\vec{\sigma}_e \cdot \vec{\sigma}_\mu$ ($+1$ for total spin 1 and -3 for total spin 0). A more sophisticated wave function incorporates the relativistic behavior of the electron,[2,11] but this refinement is not needed here.

Given the large-large component part of the wave function $\phi_{\rm NR}$, the small components may be constructed approximately using

$$\phi = \left[1+\frac{\vec{\alpha}_e\cdot\vec{p}}{2m_e}\right]\left[1-\frac{\vec{\alpha}_\mu\cdot\vec{p}}{2m_\mu}\right]\phi_{\rm NR} \;. \tag{1.1}$$

Here the electron (muon) Dirac matrices are $\vec{\alpha}_e,\beta_e$ ($\vec{\alpha}_\mu,\beta_\mu$) and the masses are m_e and m_μ. The electron's momentum in the center-of-mass frame is $\vec{p}$.

Some perturbation kernels without radiative corrections are shown in Fig. 1. Since their interpretation is reasonably clear intuitively, a detailed derivation will not be given here. A complete discussion may be found in Ref. 11. The heavy fermion lines in the figure represent the wave function in the center-of-mass frame. The light fermion lines represent Feynman propagators. The exchanged photon in Fig. 1(a) is labeled T to indicate that it is a transverse photon. It yields the leading contribution to the hyperfine splitting,

$$E_F = \tfrac{8}{3}\alpha^4 \frac{m_r^{\,3}}{m_e m_\mu}(1+a_\mu) \tag{1.2a}$$

which is now expressed in natural units ($\hbar = c = 1$), with

$$m_r \equiv m_e m_\mu/(m_e+m_\mu) \;.$$

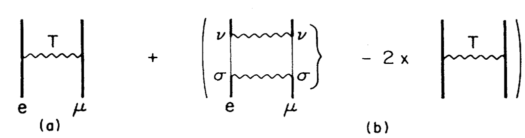

FIG. 1. (a) The leading contribution to the hfs. T indicates a transverse photon interaction. The heavy fermion lines represent the two-particle wave function. (b) The leading recoil correction to the hfs. The two photons in the first diagram are treated covariantly. The light fermion lines represent Feynman propagators. The subtracted term removes the leading contribution, which occurs twice in the first diagram. The brackets on the muon line indicate that the crossed diagram is to be added.

If the relativistic properties of the electron are taken into account, (1.2a) should be multiplied by a function of α, which for the ground state is $(1+\frac{3}{2}\alpha^2+\cdots)$. Also, it is conventional to use the total magnetic moment of the nucleus in calculating the leading term. This accounts for the factor $(1+a_\mu)$, where a_μ corresponds to the anomalous moment of the muon. The electron's anomalous moment is conventionally treated along with other radiative corrections. The distinction between the electron and muon is made because the electron's electromagnetic structure is probed by momenta of order of m_e while that of the muon requires momenta of order m_μ. Thus, for the largest corrections, which occur at lower momenta, the total muon magnetic moment does factor out. Not surprisingly, this factorization breaks down in connection with recoil corrections, as will be seen in Sec. IV. In what follows, recoil contributions are therefore expressed in terms of

$$\widetilde{E}_F \equiv \tfrac{8}{3}\alpha^4 \frac{m_r^{\,3}}{m_e m_\mu} \;. \tag{1.2b}$$

Figure 1(b) includes effects not already accounted for in Fig. 1(a). In the first term of the parentheses, the photons are in the covariant Feynman gauge. This may seem strange since the problem has been (effectively) formulated in the Coulomb gauge. However, we have made a gauge transformation *within the kernels* to the covariant gauge. The leftover contributions are too small by a factor of α to be of interest in the present work. The method of making this rearrangement is described in Ref. 11. It is easy to see that this term contains the leading contribution exactly twice: if one photon has space indices and the other time indices and the intermediate muon is put on the mass shell, the integral equation satisfied by the wave equation gives back the leading-order result. This can happen two ways, and the subtraction removes these contributions.

While it may not be completely self-evident, the removal of the contribution where the muon is on the mass shell in Fig. 1(b) means that the result depends dynamically on the muon, i.e., it is a recoil correction. Also, since the electron becomes more relativistic, higher powers of α are produced. The resulting order of magnitude is $\alpha(m_e/m_\mu)E_F$. Incidentally, the hfs contribution where both photons have time indices is negligible in this order. The contribution where both indices are spatial has no leading-order part, i.e., it is purely a recoil correction.

1. Description of the one-loop recoil correction [Fig. 1(b)]

Let us study the one-loop kernel in Fig. 1(b). The electron factor of this kernel is

$$\frac{\gamma_\nu(\not{p}+\gamma_0 E'+m_e)\gamma_\sigma}{p^2-\gamma^2+2E'p_0+i\epsilon} \;, \tag{1.3a}$$

where p^μ is the loop four-momentum carried by the electron relative to $(E',\vec{0})$. The origin of the loop four-momentum can be chosen arbitrarily. A convenient choice is $E'^2-m_e^{\,2}=E''^2-m_\mu^{\,2}=-\gamma^2$, where $E'+E''=E$ is the total energy. We use this choice here, but this is more for consistency with the general treatment of recoil than because it affects our calculation. To a good approximation, the relationship between E and E' is

$$E = m_\mu + E' + \frac{E'^2 - m_e^2}{2m_\mu} \cong m_\mu + m_e - \frac{\gamma^2}{2m_R} \quad (1.3b)$$

and $\gamma \approx m_R \alpha$.[11] The γ matrices in (1.3a) refer, of course, to the electron, while those in (1.4) below refer to the muon.

The muon factor for the same kernel is

$$M^{\nu\sigma} \equiv \frac{\gamma^\nu(\gamma_0 E'' - \not{p} + m_\mu)\gamma^\sigma}{p^2 - \gamma^2 - 2E'' p_0 + i\epsilon} + \frac{\gamma^\sigma(\gamma_0 E'' + \not{p} + \not{p}_{\rm ext} + m_\mu)\gamma^\nu}{(p + p_{\rm ext})^2 - \gamma^2 + 2E'' p_0 + i\epsilon} . \quad (1.4)$$

In the second term of (1.4), which comes from the crossed graph, $p^\mu_{\rm ext} = (0, \vec{p}_{\rm ext})$ is a combination of wave-function momenta. Now we may make some approximations in (1.4). In the small-momentum region ($|p| \ll m_e$), the p_0 poles of (1.3a) and the second term of (1.4) are on the same side of the axis. Consequently, the product of those two expressions cannot yield an important contribution from the low-momentum region. The consequence is that we may neglect $\vec{p}_{\rm ext}$ and the small components of the wave functions; corrections have more powers of α than we need to include here.

We can get an idea of how the nonrelativistic region works by studying the first term of (1.4). For small $|p_\lambda|$ ($\ll m_\mu$), and neglecting γ, the muon propagator can be written

$$\frac{1}{p^2 - 2m_\mu p_0 + i\epsilon} \xrightarrow{\sim} \frac{1}{-2m_\mu p_0 + i\epsilon} \cong \frac{-2\pi i \delta(p_0)}{2m_\mu} - \frac{1}{2m_\mu p_0 + i\epsilon} .$$

The second term in the propagator emphasizes large momenta, as in the discussion of the previous paragraph; actually, it approximately cancels the nonrecoil part of the second term of (1.4). The $\delta(p_0)$ term corresponds to the leading nonrecoil contribution which we want to subtract. This must be done carefully in order to keep the correct reduced-mass dependence. Consequently, we use the more elaborate identity

$$\frac{1}{p^2 - \gamma^2 - 2E'' p_0 + i\epsilon} = \frac{-2\pi i \delta(p_0)}{2E} - \frac{1}{2E(p_0 + i\epsilon)} - \frac{p^2 - \gamma^2 + 2E' p_0}{2E(-p_0 + i\epsilon)(p^2 - \gamma^2 - 2E'' p_0 + i\epsilon)} . \quad (1.5)$$

The $\delta(p_0)$ term now precisely reproduces the lowest-order contribution which is to be subtracted, as described earlier. The (approximated) second denominator of (1.4) may be similarly rearranged:

$$\frac{1}{p^2 - \gamma^2 + 2E'' p_0 + i\epsilon} = \frac{1}{2E(p_0 + i\epsilon)} + \frac{p^2 - \gamma^2 + 2E' p_0}{2E(p_0 + i\epsilon)(p^2 - \gamma^2 + 2E'' p_0 + i\epsilon)} - \frac{2(p^2 - \gamma^2)}{2E(p_0 + i\epsilon)(p^2 - \gamma^2 + 2E'' p_0 + i\epsilon)} . \quad (1.6)$$

At this stage, only relativistic (for the electron) loop momenta can contribute. That means that we may neglect small components of the wave function and external momenta in the photon propagators, with errors of higher order in α than is presently needed. Also, the $(\gamma_0 E'' + m_\mu)$ terms in the numerators of (1.4) cannot contribute to the order of interest. After all these steps, (1.4) becomes

$$M^{\nu\sigma} - \frac{2\pi i \delta(p_0)}{2E} \gamma^\nu \not{p} \gamma^\sigma = \frac{-\gamma^\nu \not{p} \gamma^\sigma}{2E} \left\{ (p^2 - \gamma^2 + 2E' p_0) \left[\frac{-1}{(-p_0 + i\epsilon)(p^2 - \gamma^2 - 2E'' p_0 + i\epsilon)} + \frac{1}{(p_0 + i\epsilon)(p^2 - \gamma^2 + 2E'' p_0 + i\epsilon)} \right] - \frac{2(p^2 - \gamma^2)}{(p_0 + i\epsilon)(p^2 - \gamma^2 + 2E'' p_0 + i\epsilon)} \right\} . \quad (1.7)$$

Taken together with the electron factor (whose numerator is now $\gamma_\nu \not{p} \gamma_\sigma$), the terms in the square brackets cancel each other (reverse the sign of p_0 in one of them, taking into account the symmetry of the rest of the integrand). In the remaining term of (1.7) we may neglect γ^2 everywhere (corrections have extra powers of α). As a calculational device, it is convenient to give one of the photons a small mass^2, which we label s. As a consequence of subtracting the nonrecoil contribution, there is no infrared divergence; and ultimately we set s equal to zero. This device allows us to separate integrals into parts which would otherwise be divergent. The use of s as a regulator also permits us to apply the result immediately to the vacuum polarization, where s becomes the mass^2 of the state contributing to the photon propagator.

In order to obtain a hyperfine-splitting contribution from the product of (1.3a) and (1.7), two of the three γ matrices in each factor must be spatial and one must have a time index. Then with spherical averaging, as appropriate, we find for this product

$$\frac{\frac{2}{3}\vec{\sigma}_e\cdot\vec{\sigma}_\mu(3p_0{}^2-2\vec{p}\,^2)}{(m_\mu{}^2-m_e{}^2)(p^2+i\epsilon)} \times\left[\left[\frac{2m_\mu{}^2}{p^2+2m_\mu p_0+i\epsilon}-\frac{2m_\mu{}^2}{2m_\mu p_0+i\epsilon}\right] -\left[\frac{2m_e{}^2}{p^2+2m_e p_0+i\epsilon}-\frac{2m_e{}^2}{2m_e p_0+i\epsilon}\right]\right]. \quad (1.8)$$

The advantage of this final rearrangement is that it separates the electron and muon denominators, which simplifies the subsequent integrals. The integrals over wave-function momenta yield the square of the nonrelativistic wave function at the origin of coordinate space. Consequently, the contribution of Fig. 1(b) to the hyperfine splitting becomes [see Fig. 1(b)]

$$\Delta E=\widetilde{E}_F\frac{m_e m_\mu}{m_\mu{}^2-m_e{}^2}16\pi\alpha \times\int\frac{d^4p}{-(2\pi)^4 i}\frac{3p_0{}^2-2\vec{p}\,^2}{(p^2+i\epsilon)(p^2-s+i\epsilon)} \times[P(m_\mu,p)-P(m_e,p)]\,, \quad (1.9)$$

where

$$P(m,p)=\frac{2m^2}{p^2+i\epsilon}\left[\frac{1}{p^2+2mp_0+i\epsilon}-\frac{1}{2mp_0+i\epsilon}\right] =\frac{-2m^2}{(p^2+2mp_0+i\epsilon)(2mp_0+i\epsilon)}\,.$$

The first form of P is somewhat more convenient for calculation of the integral.

The integral in (1.9) may be carried out by the usual Feynman parameter techniques, and the result expressed as [see Fig. 1(b)]

$$\Delta E=\widetilde{E}_F\frac{m_e m_\mu}{m_\mu{}^2-m_e{}^2}\frac{\alpha}{2\pi}\int_0^1 d\lambda[S(m_\mu,\lambda)-S(m_e,\lambda)], \quad (1.10)$$

where

$$S(m,\lambda)=m^2\left[\frac{16-6\lambda-\lambda^2}{m^2\lambda^2+s(1-\lambda)}-\frac{16}{m^2\lambda^2+s(1-\lambda)^2}\right].$$

The two terms of S correspond to the two terms of the first form of P. Roughly speaking, the first term represents the one-loop part of Fig. 1(b) and the second the subtraction. The subtraction piece is easily integrated with the result $-8\pi m/\sqrt{s}$; however, it is better to note that this cancels the leading singularity at $\lambda=0$ when $s\to 0$. It is easy to see that for $s \ll m^2$

$$\int_0^1 d\lambda\, S(m,\lambda)=-9-3\ln\frac{m^2}{s}+O\left[\frac{\sqrt{s}}{m}\right]. \quad (1.11)$$

Thus [see Fig. 1(b)],

$$\Delta E=\widetilde{E}_F\left[-\frac{3\alpha}{\pi}\frac{m_e m_\mu}{m_\mu{}^2-m_e{}^2}\ln\frac{m_\mu}{m_e}\right], \quad (1.12)$$

the constant and the $\ln s$ having canceled upon subtraction. Of the total coefficient -3, the $3p_0{}^2$ in (1.9) contributes $+3$ and the $-2\vec{p}\,^2$ contributes -6.

2. Review of the nonrecoil radiative corrections

The precise separation of the radiative corrections into nonrecoil and recoil pieces is somewhat a matter of definition. This is because the reduced-mass dependence already takes into account some aspects of the finite mass of the nucleus. We want as much as possible to have the "natural" reduced-mass dependence appear in the leading terms rather than treat the difference between m_r and m_e as a recoil correction. For example, the m_r dependence in (1.2) simply reflects the fact that the dominant contribution depends on the nonrelativistic wave function at the origin. The m_e occurs in the denominator because it arises from the electron's magnetic moment, which is not a reduced-mass effect. Equation (1.2) was not written down by guesswork, of course; careful analysis shows that it is the true leading term.[11] When we proceed beyond lowest order, however, different methods of organizing the calculation can lead to differences in the division between reduced mass and dynamical recoil corrections.

We use (1.5) as the definition of the separation between reduced-mass and dynamical recoil corrections. With pure Coulomb exchanges in the ladder approximation, the first term leads to the Dirac equation for the electron in an external Coulomb field. However, there is a modification which produces an effective reduced-mass dependence in the nonrelativistic domain. The treatment of the muon factor multiplies each free Dirac propagator by an additional factor of approximately $m_\mu/(m_\mu+m_e)$ (with relative error $\sim\alpha^2 m_e/m_\mu$). In the nonrelativistic domain, where the electron numerator is approximately $2m_e$, the combination produces a numerator $2m_r$, leading to the correct reduced-mass dependence in the wave function. The remaining terms in (1.5) are regarded as dynamical corrections. The second term, while not having a recoil dependence, is canceled by a term from the crossed Coulomb graph in the same manner as in the discussion of (1.7). The final term has a numerator which cancels the electron denominator yielding a contribution which comes from the very high momentum region and has an extra factor of $1/m_\mu$. It seems entirely plausible to regard this residue as a dynamical recoil term. By the way, suppose we had tried to use the crude approximation preceding (1.5). Then the $\delta(p_0)$ term would not have had the extra factor $m_\mu/(m_\mu+m_e)$, and the result would have led to a wave function without reduced-mass dependence. The other terms would then have been a complicated combination of terms needed to fix up reduced-mass dependence together with true recoil corrections. Thus

(1.5), which was discovered after much trial and error, seems to be a very useful choice for the separation.

Although it is not so important for present purposes, we can indicate briefly how the reduced-mass dependence gets into Dirac wave functions. The factor $m_\mu/(m_\mu + m_e)$ multiplying the free Dirac propagator can equally well be regarded as multiplying the Coulomb interaction, changing its strength according to

$$\alpha \to \alpha m_\mu/(m_\mu+m_e) \equiv \alpha' \ .$$

Now, as is well known, the Dirac wave functions depend on two arguments: $\alpha' m_e \vec{r}$ and α'. The first argument is the same as $\alpha m_r \vec{r}$. With one exception, the second argument is not important for our present discussion.[12] The exception is that the size of the small component of the Dirac wave function relative to the large component is proportional to $\alpha' = \alpha m_r/m_e$. This is the origin of $1/m_e$ in (1.2).

Now we may turn to a discussion of radiative corrections. One of these (a_μ) is already contained in (1.2a), and justification for the factorization of $(1+a_\mu)$ in leading corrections was given there. When radiative corrections probe large momenta $(\sim m_\mu)$, as in recoil corrections, there need not be such factorization.

a. Vacuum polarization.

Other than the anomalous moment effects, the easiest radiative correction to discuss is vacuum polarization.[13] The various nonrecoil contributions are illustrated in Fig. 2. In Fig. 2(a) the vacuum polarization occurs in the exchanged transverse photon, while in Figs. 2(b) and 2(c) it occurs in adjacent Coulomb interactions. Contributions from nonadjacent Coulomb interactions are of higher order in α. As mentioned earlier, vacuum polarization effectively replaces a massless photon propagator by one with mass$^2 = s$, where s is a variable to be integrated with a certain spectral weight function.

To see how these calculations work, we note that the spin analysis associated with the transverse photon exchange produces a factor

$$\frac{\vec{\sigma}_e \times (\vec{p}\,'-\vec{p})}{2m_e} \cdot \frac{\vec{\sigma}_\mu \times (\vec{p}\,'-\vec{p})}{2m_\mu} \doteq \frac{2}{3} \frac{\vec{\sigma}_e \cdot \vec{\sigma}_\mu (\vec{p}\,'-\vec{p})^2}{4 m_e m_\mu} , \tag{1.13}$$

where $\vec{p}$ $(\vec{p}\,')$ is the electron's momentum before (after) the transverse photon interaction. With vacuum polarization, some of the integration momenta are characterized by $\sqrt{s}$ while others are characterized by γ. To see how this works with each diagram of Fig. 2, suppose we select

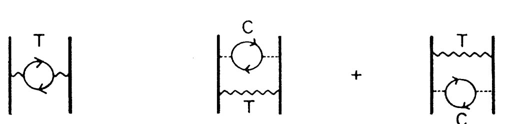

FIG. 2. The nonrecoil vacuum polarization contributions. (a) The vacuum polarization occurs in the transverse photon interaction of Fig. 1(a). (b) and (c) Vacuum polarization in an adjacent Coulomb interaction of Fig. 1(a). The intermediate heavy lines correspond to the nonrecoil first term of (1.5).

the term $\vec{p}\,'^2$ from (1.13). In Fig. 2(b), $\vec{p}\,'$ is the momentum in the loop between the transverse photon and the Coulomb interaction with the vacuum polarization. Examination of the integrand shows that this momentum is large while that in the wave functions is small. In Fig. 2(c), the momenta in the upper wave function and the loop below T both tend to be large. The consequence is that the resulting contribution has an additional power of α. In Fig. 2(a) the upper wave function has large momenta and the lower one has small momenta. The result is identical to the contribution from Fig. 2(b): then Figs. 2(a) and 2(b) are the same except for a change in the position of the vacuum polarization. The loop integrations are identical. The $\vec{p}^{\,2}$ from (1.13) works in a similar way. The $-2\vec{p}\,'\cdot\vec{p}$ requires two momenta to be large together and it produces an additional factor of α.

In summary, the result is extremely simple. It is exactly the subtraction term discussed with the recoil correction, but doubled to take into account the two possible insertions of the vacuum polarization and integrated with respect to s. This gives

$$\Delta E(\text{nonrecoil vac pol}) = E_F m_r \frac{4\alpha^2}{\pi} \int_{s_{\text{th}}} \frac{\rho(s)ds}{\sqrt{s}} \ . \tag{1.14}$$

For the electron contribution to the vacuum polarization

$$s = 4{m_e}^2/(1-x^2) \tag{1.15a}$$

and

$$\rho(s)ds = \frac{x^2(1-\frac{1}{3}x^2)}{1-x^2}dx, \quad 0<x<1 \ . \tag{1.15b}$$

The electron contribution alone would give $E_F(3\alpha^2/4)(m_r/m_e)$. This differs from the usual result by a reduced-mass factor (m_r/m_e). However, by adding the muon contribution to the vacuum polarization (with m_r/m_μ), we find

$$\Delta E(\text{leading nonrecoil vac pol}) = E_F \frac{3\alpha^2}{4} \tag{1.16}$$

which is symmetric in the masses. For other contributions to the vacuum polarization (i.e., hadrons and heavy leptons) there is no point in defining separate nonrecoil contributions. Such effects will be estimated in the following section.

b. Electron line radiative corrections.

The nonrecoil radiative corrections to the electron are more complicated than the vacuum polarization,[8,13–15] so our description of them will be very sketchy. The simplest part is given by the electron's anomalous magnetic moment (represented by a_e); this simply adds a_e to the coefficient multiplying E_F. The more difficult part is the binding correction which depends on the strength (α) of the external potential. Although the old calculations are done in a very different way, the leading binding corrections should be contained in Fig. 3. They are of relative order α^2 [they amount to $\alpha^2(\ln 2 - \frac{13}{4})$ in the coefficient multiplying E_F], so according to our prescription they should be multiplied by

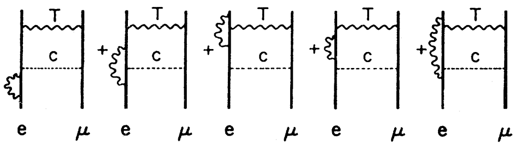

FIG. 3. Various nonrecoil radiative corrections to the hfs. The heavy lines have the same meaning as in Figs. 1 and 2. Another set is obtained by interchanging T and C.

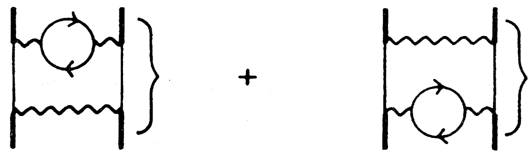

FIG. 4. Radiative-recoil corrections arising from vacuum polarization. It is understood that the contributions of Fig. 2 are to be subtracted from the original expressions to obtain recoil corrections.

$$m_r/m_e = 1 - m_e/(m_e+m_\mu) .$$

The deviation of this factor from 1 would be of importance at the present level of interest. As shown in Sec. IV, there are other contributions of this same order of magnitude arising from the terms in (1.7) and the radiative correction in the muon line. This contribution has no significance separately from the other ones, but it does indicate the size of radiative-recoil corrections from the lepton lines.

Beyond the corrections of order α^2 are ones of order α^3 [or $\alpha(Z\alpha)^2$] (Refs. 8 and 13–15). These are too small to require recoil corrections.

II. VACUUM POLARIZATION RADIATIVE-RECOIL CORRECTIONS

The effect of vacuum polarization corrections to the one-loop recoil contribution is given by replacing either photon propagator by

$$\frac{1}{q^2+i\epsilon} \to \left[\frac{\alpha}{\pi}\right] \int_{s_{\rm th}}^{\infty} \frac{\rho(s)ds}{q^2-s+i\epsilon} . \tag{2.1}$$

Accordingly, (1.9) should be multiplied by $2\alpha/\pi$ and integrated with respect to s. Equation (1.10) then becomes

$$\Delta E = \left[\frac{\alpha}{\pi}\right]^2 \frac{m_e m_\mu}{m_\mu{}^2 - m_e{}^2}\widetilde{E}_F \times \int_{s_{\rm th}}^{\infty} ds\, \rho(s) \int_0^1 d\lambda [S(m_\mu,\lambda) - S(m_e,\lambda)] , \tag{2.2}$$

where $S(m,\lambda)$ is given in (1.10). Equation (2.2) serves as a starting point for further specialization. Note that this expression is symmetric in the masses, and $S(m,\lambda)$ already incorporates the subtraction of nonrecoil contributions.

(i) Muonium

Here the vacuum loop is the usual $O(\alpha)$ radiative correction with weight

$$\rho(s) = \left[1 - \frac{4m^2}{s}\right]^{1/2} \left[1 + \frac{2m^2}{s}\right] \Big/ 3s , \tag{2.3}$$

where m is the mass of the lepton in the vacuum polarization loop in Fig. 4, and the contributions of both electron and muon are to be included. Thus, each S has two terms. Very conveniently, $S(m_\mu,\lambda)$ with muon vacuum polarization and $S(m_e,\lambda)$ with electron vacuum polarization cancel trivially. The remainder is still symmetric in m_μ and m_e, but we may use the large mass ratio to drop the contribution of $S(m_e,\lambda)$ with muon vacuum polarization. Thus, we retain only $S(m_\mu,\lambda)$ with the electron vacuum polarization. Making the change of variable $s = 4m_e{}^2/(1-x^2)$ and denoting $\eta = m_e/m_\mu$ we arrive at

$$\Delta E = \left[\frac{\alpha}{\pi}\right]^2 \frac{m_e m_\mu}{(m_\mu{}^2 - m_e{}^2)}\widetilde{E}_F \times \int_0^1 dx\, x^2(1-\tfrac{1}{3}x^2) \int_0^1 d\lambda\, Q(\lambda,x,\eta) , \tag{2.4}$$

where

$$Q(\lambda,x,\eta) = \frac{16-6\lambda-\lambda^2}{\lambda^2(1-x^2)+4\eta^2(1-\lambda)} - \frac{16}{\lambda^2(1-x^2)+4\eta^2(1-\lambda)^2} .$$

The leading recoil contribution from (2.4) (with $\eta \ll 1$) is

$$\Delta E^M(VP) = \left[\frac{\alpha}{\pi}\right]^2 \frac{m_e}{m_\mu}\widetilde{E}_F \left[-2\ln^2\frac{m_\mu}{m_e} - \frac{8}{3}\ln\frac{m_\mu}{m_e} - \frac{28}{9} - \frac{\pi^2}{3} + O(\eta)\right]$$

$$= -6.61(1.0+0.250+0.113)\ \text{kHz} . \tag{2.5}$$

The double logarithm agrees with the result found previously by Caswell and Lepage.[5] The remaining terms are exact for $\eta \ll 1$ and they agree to within 1% with a numerical computation by Lepage.[16]

A convenient method of obtaining this approximation will be described briefly. One notes that the sensitive region of the two-dimensional integrand for small η is $\lambda \sim 0$, $x \sim 1$. If the λ integration is done first, the exact form must be used rather than the asymptotically leading result (1.11). A simpler way to proceed is to rearrange the x dependence as follows:

$$x^2(1-\tfrac{1}{3}x^2) = \tfrac{2}{3}x - \tfrac{1}{3}x(1-x)^2(2+x) . \tag{2.6}$$

The $\frac{2}{3}x$ term retains the $x \sim 1$ singular behavior but permits the x integration to be done simply. The subsequent λ integration with $\eta \ll 1$ is also simple (although a little lengthy). The remaining term in (2.6) has the singularity at $x = 1$ suppressed. Then (1.11) may be used and the resulting x integration presents no problem.

(ii) Positronium

For positronium the recoil and nonrecoil effects have the same order since all loop momenta are characterized by m_e. Thus, we do not separate the nonrecoil contribution, and the complete result is contained in the first term of S. Taking the limit of (2.2) as $m_\mu \to m_e$, we find

$$\Delta E = \left[\frac{\alpha}{\pi}\right]^2 E_F^{\text{Ps}} \times \int_{4m_e^2}^{\infty} ds\, \rho(s) s \int_0^1 d\lambda \frac{m_e^2(1-\lambda)(16-6\lambda-\lambda^2)}{[m_e^2\lambda^2+s(1-\lambda)]^2} , \tag{2.7}$$

where E_F^{Ps} is the positronium Fermi energy (~ 116.8 GHz). We retain only the electron contribution to the vacuum polarization since higher masses give a contribution of relative order m_e^2/s. Following the steps used in the previous section which lead to (2.4) we find

$$\Delta E = \left[\frac{\alpha}{\pi}\right]^2 E_F^{\text{Ps}} \int_0^1 dx\, x^2(1-\tfrac{1}{3}x^2) \times \int_0^1 d\lambda \frac{4(1-\lambda)(2-\lambda)(8+\lambda)}{[(1-x^2)\lambda^2+4(1-\lambda)]^2} . \tag{2.8}$$

These integrals can be evaluated explicitly to give

$$\Delta E^{\text{Ps}}(\text{VP}) = \frac{5}{3}\left[\frac{\alpha}{\pi}\right]^2 E_F^{\text{Ps}} = 1.05 \text{ MHz} . \tag{2.9}$$

This result has also been checked numerically by Lepage who obtained agreement to within a few percent.[16]

(iii) Hadronic contribution

As in the positronium case, there is no purpose in evaluating the nonrecoil contribution separately. Returning to Eq. (2.2), we then express the hadronic contribution by

$$\Delta E = \left[\frac{\alpha}{\pi}\right]^2 m_e m_\mu \widetilde{E}_F \int_{4m_\pi^2}^{\infty} ds\, \rho(s) \times \int_0^1 d\lambda \frac{16-6\lambda-\lambda^2}{m_\mu^2\lambda^2+s(1-\lambda)} , \tag{2.10}$$

where we have expanded the mass factor to first order in m_e/m_μ.

(a) Pion component

Assuming that the spectrum is dominated by the ρ resonance plus 2-π background we can express the spectral function in terms of the pion form factor as

$$\rho(s) = \frac{(s-4m_\pi^2)^{3/2}}{12 s^{5/2}} |F_\pi(s)|^2 . \tag{2.11}$$

We use the Gounaris-Sakurai representation of the pion form factor[17]

$$|F_\pi(s)|^2 = \frac{m_\rho^4\left[1+\dfrac{\Gamma_\rho^d}{m_\rho}\right]^2}{(s-m_\rho^2)^2+m_\rho^2\Gamma_\rho^2\left[\dfrac{s-4m_\pi^2}{m_\rho^2-4m_\pi^2}\right]^{3/2}\dfrac{m_\rho^2}{s}} . \tag{2.12}$$

The fitting parameters are uncertain by a few percent. As representative values we have chosen $m_\rho = 775$ MeV, $\Gamma_\rho = 150$ MeV, and $d = \frac{1}{2}$.[18]

The integrals over λ in (2.10) can be evaluated analytically while the remaining integration over s is handled numerically. The result is

$$\Delta E^M(\text{VP;pion}) = 63.8\left[\frac{\alpha}{\pi}\right]^2 \frac{m_e m_\mu}{m_\rho^2}\widetilde{E}_F = 0.14(1) \text{ kHz} . \tag{2.13}$$

The uncertainty of 0.01 kHz reflects the range of choice of fitting parameters.

It is interesting to compare this result with a calculation which retains only the ρ-pole contribution:

$$\rho(s) = \frac{4\pi^2}{f_\rho^2}\delta(s-m_\rho^2) . \tag{2.14}$$

The resulting parameter integral can be expanded in m_μ/m_ρ to give

$$\Delta E^M(\text{VP};\rho) = \left[\frac{\alpha}{\pi}\right]^2 \frac{m_e m_\mu}{m_\rho^2}\widetilde{E}_F \frac{4\pi^2}{f_\rho^2}\left[9\ln\left[\frac{m_\rho}{m_\mu}\right]^2 + \frac{15}{2} + \cdots\right] \tag{2.15}$$

Estimating the ρ coupling as $f_\rho^2/4\pi = 2.2$, we find

$$\Delta E^M(\text{VP};\rho^0) = 61.3\left[\frac{\alpha}{\pi}\right]^2 \frac{m_e m_\mu}{m_\rho^2}\widetilde{E}_F , \tag{2.16}$$

which agrees to 4% with the complete form-factor result in (2.13).

(b) Higher-mass vector mesons

Contributions from the ω and ϕ can be estimated from Eq. (2.15). For the couplings we take[18]

$$\frac{f_\omega^2}{4\pi} = 18.4(1.8) \text{ and } \frac{f_\phi^2}{4\pi} = 11.0(1.7) .$$

Then

$$\Delta E^M(\text{VP};\omega) = 0.016 \text{ kHz}$$

and

$$\Delta E^M(\text{VP};\phi) = 0.017 \text{ kHz} . \tag{2.17}$$

(c) Higher-mass background and heavy leptons

For s above the threshold mass $M^2 \sim 1$ GeV^2 we can parametrize the spectral function by

$$\rho(s) = \frac{R(s)\rho(e^+e^- \to \mu^+\mu^-)}{4\pi^2\alpha} \simeq \left[\frac{\alpha}{\pi}\right]\frac{R}{3s} . \tag{2.18}$$

Proceeding as in our treatment of the ρ pole we expand the λ integration in m_μ^2/s. The result is just (2.15) with the replacement $m_\rho^2 \to s$. The remaining integration over s is straightforward and yields the result

$$\Delta E^M(\mathrm{VP};M^2)=\left[\frac{\alpha}{\pi}\right]^2 R\frac{m_e m_\mu}{M^2}\widetilde{E}_F\left[3\ln\left[\frac{M}{m_\mu}\right]^2+\frac{11}{2}+\cdots\right], \tag{2.19}$$

where we have assumed R is constant above threshold. Taking $R\sim 2$ we find

$$\Delta E^M(\mathrm{VP};M^2)\sim 0.05\ \mathrm{kHz}\ . \tag{2.20}$$

A heavy lepton contribution can also be obtained from (2.19) by using $R=1$ and $M=2m_{\mathrm{lepton}}$; the τ contribution, for example, is quite negligible compared with the uncertainty of other terms.

Assembling the various contributions we find a net hadronic contribution of

$$\Delta E^M(\mathrm{VP;hadronic})=0.22(3)\mathrm{kHz}\ . \tag{2.21}$$

For positronium, the hadronic vacuum polarization contribution is $O(10^{-9}E_F)$ and can be neglected.

III. LEPTON LINE RADIATIVE CORRECTIONS—FORMULATION

In this section we set up the radiative corrections for the lepton lines. Our goal is to derive a general, and very useful, expression for a lepton factor with lowest-order radiative corrections taken into account. This part of the analysis is somewhat simplified by the requirement that we are interested only in terms of relative order α^2. In the next section, this expression is applied to a number of different contributions. For convenience, we describe radiative corrections to the electron line. However, in this section no approximations are made which are specific to the electron, and the results are equally applicable to the muon line.

A simplification in our analysis arises because for all our intended applications the momentum in the lines containing the radiative corrections is large compared to the characteristic momentum in the wave function. This permits us to restrict our attention to the set of diagrams shown in Fig. 5. As mentioned in Sec. I and elaborated in Appendix A, the use of these diagrams alone for the order of interest is not self-evident. When a large number of Coulomb interactions are spanned by the virtual photon, the low-momentum region of the integral may contribute inverse powers of α so that all numbers of interactions may contribute to a given order. To circumvent this complication, we advocate the use of a special gauge (sometimes referred to as the Fried-Yennie gauge[19]) for the radiative correction photons. In this gauge, there is a compensation which assures us that the diagrams of Fig. 5 are adequate for the present calculation. This is explained in detail in Appendix A, but the main features are described here briefly. The occurrence of inverse powers of α is associated with the infrared behavior. In bound states there is no actual infrared divergence because the electron is slightly off-mass-shell. However, in the Feynman gauge the various functions become singular as their external four-momenta approach the mass shell. In the Fried-Yennie gauge, the behavior near the mass shell is much smoother and causes less difficulty.[20,21]

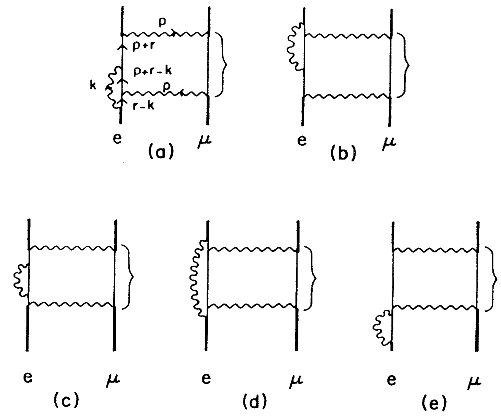

FIG. 5. Radiative-recoil corrections arising from the electron line, with a subtraction of Fig. 3 contributions understood.

We should make a warning about one point which is possibly confusing. The contributions of Fig. 5 contain the nonrecoil anomalous moment contributions redundantly. These effects are ordinarily accounted for at a previous level, so they are simply discarded as they arise in the analysis. Since they have a well-defined order, this may be done without distorting the terms of interest. Because of the prescriptions to be used in evaluating the contributions from Fig. 5, these lowest-order contributions would not agree with the actual anomalous moment contributions; but that is of no significance.

In addition to the infrared region, the ultraviolet region requires a few remarks. Because of the Ward identity, the divergences in the internal and external self-energies cancel those of the vertices. As usual, the correct renormalization is obtained by including only one of the external self-energies. As the leading logarithms of order $\alpha^2\ln^2(m_\mu/m_e)$ are associated with divergent renormalizations, it is not surprising that they cancel, as found by Caswell and Lepage.[5] However, it is awkward to have to deal with them since they complicate the calculation of nonleading logarithms and constants. In the approach to be presented here, we have found a way to avoid calculating these spurious quantities. Briefly, it is to use numerator factors to cancel denominators in such a way that cancellations between various diagrams become manifest.

After obtaining the cancellations mentioned in the preceding paragraph, the integration over photon momentum is carried out with the help of Feynman parameter techniques. The result is quite lengthy but is in a convenient form for corrections to the electron line. For corrections to the muon line, it is not convenient because it does not manifest the low-energy Compton scattering theorem. Therefore, it is rearranged with the help of extensive and judicious integrations by parts with respect to the Feynman parameters x and y; techniques for doing this are described in Appendix C. A more direct method of demonstrating the low-energy theorem by formal techniques is explained in Appendix B; but the results there are not actually applied to a calculation.

A. Momentum-space analysis

Certain denominators occur in more than one of the diagrams of Fig. 5. They are

$$D_0=p^2-\gamma^2+2E'p_0+i\epsilon \tag{3.1a}$$

which occurs in (1.3) and Figs. 5(a)—5(c) and 5(e);

$$D_i=k^2-2k\cdot p-2k_0E'+p^2-\gamma^2+2E'p_0+i\epsilon \tag{3.1b}$$

which occurs in Fig. 5(a)—5(d); and

$$D_e=k^2-2k_0E'-\gamma^2+i\epsilon \tag{3.1c}$$

which occurs in Figs. 5(a), 5(b), 5(d), and 5(e). The trick to canceling ultraviolet divergences and double logarithms is to arrange numerator terms which cancel various denominators and permit pieces from different graphs to compensate.

1. Electron self-energy diagrams [Figs. 5(c) and 5(e)]

Let q be the four-momentum in the electron line $[q=r=(E',\vec{0})$ for an external line or $q=r+p$ for the internal line]. Then the electron self-energy is

$$\Sigma_2(q)=4\pi\alpha\int\frac{d^4k}{(2\pi)^4i}\frac{1}{k^2+i\epsilon}\frac{N_\Sigma}{(q-k)^2-m^2+i\epsilon}-\delta m_2\ , \tag{3.2a}$$

where

$$N_\Sigma=\gamma_\sigma(\not q-\not k+m)\gamma^\sigma+2\not k(\not q-\not k+m)\not k/(k^2+i\epsilon) \tag{3.2b}$$

$$=-3(\not q-m)+3m+\frac{4k\cdot q\not k-k^2\not q}{k^2+i\epsilon}\ . \tag{3.2c}$$

Rather than renormalize on mass shell, we follow a somewhat unconventional procedure. Because of Ward's identity, we simply allow the divergences in the vertices and self-energies to cancel each other and make no effort to compensate the finite parts. After integration and mass renormalization, Σ_2 takes the form

$$\Sigma_2=[B(q^2)+F(q)](\not q-m)\ , \tag{3.3a}$$

where

$$B(q^2)\equiv4\pi\alpha\int\frac{d^4k}{(2\pi)^4i}\frac{-3}{(k^2+i\epsilon)[(q-k)^2-m^2+i\epsilon]}\ . \tag{3.3b}$$

For the special cases $q=r$ $(q=p+r)$ we use the notation $B_e(B_i)$. As will be seen shortly, the constants B are simply and conveniently canceled by terms from the vertex functions. F, defined by (3.3a), is worked out explicitly in (3.8) below. In spite of its appearance, the last term of (3.2c) is convergent in the ultraviolet because of symmetry. The second term is rendered finite by the mass renormalization. These terms yield only single powers of $\ln(m_e/m_\mu)$ and they are analyzed further in Sec. III B.

2. Vertex diagrams [Figs. 5(a) and 5(b)]

Including the effects of the vertices, the electron factor is given by

$$\mathrm{EF(vert)}=4\pi\alpha\int\frac{d^4k}{(2\pi)^4i}\frac{N_v}{(k^2+i\epsilon)D_0D_iD_e}\ , \tag{3.4a}$$

where

$$N_v=(3D_i+3D_e)\gamma_\nu\not p\gamma_\sigma+D_0M_v+N_v'\ ,$$

$$\begin{aligned}N_v'=&-2(\not k\gamma_\nu\not k\not p\gamma_\sigma+\gamma_\nu\not p\not k\gamma_\sigma\not k+k^2\gamma_\nu\not p\gamma_\sigma)-6k\cdot p\gamma_\nu\not p\gamma_\sigma\\&+4(p^2\gamma_\nu\not k\gamma_\sigma-p\cdot k\gamma_\nu\not p\gamma_\sigma)\\&+\gamma_\nu\not p\gamma_\sigma[-12k\cdot r+8p\cdot r+8m^2\\&\qquad+16k\cdot r\,k\cdot(r+p)/(k^2+i\epsilon)]\\&-2m[\not k\gamma_\nu\not p\gamma_\sigma+\gamma_\nu\not p\gamma_\sigma\not k+2k\cdot p\gamma_\nu\gamma_\sigma]\ ,\end{aligned} \tag{3.4b}$$

$$\begin{aligned}M_v=&4\gamma_\nu\not k\gamma_\sigma-3\gamma_\nu\not p\gamma_\sigma+4m\gamma_\nu\gamma_\sigma\\&-8k\cdot r\gamma_\nu\not k\gamma_\sigma/(k^2+i\epsilon)\ ,\end{aligned}$$

where r is the external line four-momentum. The derivation of (3.4b) involves approximations [such as neglecting higher powers of α^2 which arise from factors of $(\not r-m)$ next to the wave function] and the dropping of terms which do not contribute to the hfs.

Not all the terms in (3.4b) need be calculated since several of them are compensated by terms from other diagrams. For example, in the first term, one of the denominators may be canceled and the resulting integrals identified as the same ones appearing in the electron self-energy, but with the opposite sign. Thus, we find

$$(3D_i+3D_e)\gamma_\nu\not p\gamma_\sigma\to-(B_i+B_e)\frac{\gamma_\nu\not p\gamma_\sigma}{D_0} \tag{3.5}$$

which is exactly canceled by the divergent terms from the electron self-energies, as anticipated. The term containing the factor of M_v would lead to $\ln^2(m_\mu/m_e)$ if it were evaluated. However, when the factor of D_0 is canceled out, it turns out that the spanning photon contribution has terms of similar structure which cancel these contributions before integration. With the set of denominators D_eD_i, only nonlogarithmic terms remain.

Although our numerical calculation will give us the complete result for the radiative-recoil correction, it is useful to identify the terms which contain $\ln(m_\mu/m_e)$ and evaluate them analytically. They may later be subtracted from the complete result in order to determine the additive constant. The results for these terms were presented in Ref. 7, but the details have not been given before. To obtain the logarithmic terms, many terms may be dropped from N_v'. Any terms containing an explicit factor of m_e lead to integrals converging at momenta of order of the electron mass. This eliminates the last two terms completely. The third term is only slightly more subtle: when the integral over k is completed, the $\not k$ is to be replaced by $\not r$ or $\not p$. In either case, it cannot yield a logarithm for the hfs. At this point, only the first two terms remain; we may simplify them by neglecting r and examining their behavior in the intermediate momentum range. The integral is easily worked out and found to yield a result which is $-\frac{5}{4}(\alpha/\pi)$ times the uncorrected electron line factor. Instead of making these simplifications here, we convert the noncanceling terms to a Feynman parameter integral in Sec. III B.

3. Spanning photon diagram [Fig. 5(d)]

Following our previous discussion, we find that the spanning photon diagram gives a contribution

$$\mathrm{EF(span)}=4\pi\alpha\int\frac{d^4k}{(2\pi)^4i}\frac{N_{\rm sp}}{(k^2+i\epsilon)D_iD_e^2}\,, \qquad (3.6a)$$

where

$$N_{\rm sp}=D_eM_{\rm sp}+N'_{\rm sp}\,,$$

$$N'_{\rm sp}=\gamma_\nu \not p\gamma_\sigma\left[4r^2+\frac{8(r\cdot k)^2}{k^2}\right]$$

$$-2(k\cdot r\gamma_\nu\not p\gamma_\sigma+m\not k\gamma_\nu\not p\gamma_\sigma+m\gamma_\nu\not p\gamma_\sigma\not k)$$

$$-(2\not k\gamma_\nu\not p\gamma_\sigma\not k+k^2\gamma_\nu\not p\gamma_\sigma)\,, \qquad (3.6b)$$

$$M_{\rm sp}=3\gamma_\nu\not p\gamma_\sigma-4\gamma_\nu\not k\gamma_\sigma-2m\gamma_\nu\gamma_\sigma+\frac{4k\cdot r}{k^2}\gamma_\nu\not k\gamma_\sigma\,.$$

Notice how pieces of the $M_{\rm sp}$ term cancel pieces of (3.4b); this is the cancellation of the spurious $\ln^2(m_\mu/m_e)$ terms which occur in separate diagrams.

The first term of $N'_{\rm sp}$ also deserves some comment; its form arises from the use of the Fried-Yennie gauge. In the Feynman gauge, the separate pieces of this term would have been infrared divergent. Although the combination is infrared finite, it is important to note that it is evaluated with the external lines slightly off the mass shell. The result would be different if one were to introduce a photon mass which is allowed to tend to zero after integration. However, this difference would not affect the final answer as long as the external electron self-energies are treated in a consistent manner. Later on when this term is analyzed with Feynman parameters, we shall employ an integration by parts to manifest this cancellation in a simple way.

As in the discussion of the vertex contributions, we describe how the $\ln(m_\mu/m_e)$ terms may be identified. Terms which may be dropped for this purpose include the noncanceling pieces of the $M_{\rm sp}$ term which are proportional to m_e and hence converge in the low-momentum region. One might think that the same argument could be applied to the first two terms of $N'_{\rm sp}$. However, the situation there is slightly more subtle. Because of the two factors of D_e in the denominator, the k integration produces inverse powers of m_e; and such terms do contribute to the logarithm. Actually, it is possible to use them to compensate against pieces of the external electron self-energy; that was done in the earlier publication.[7] In any case, the $\ln(m_\mu/m_e)$ contribution from (3.6) [after cancellation of $\ln^2(m_\mu/m_e)$ terms] is simple to determine.

4. Summary of the momentum-space expression

The sum of all the noncanceling terms from the previous analysis is

$$\mathrm{EF}=\frac{\gamma_\nu\not p\gamma_\sigma}{D_0}F(r)+\frac{\gamma_\nu(\not p+\not r+m)F(p+r)\gamma_\sigma}{D_0}+4\pi\alpha\int\frac{d^4k}{(2\pi)^4i}\frac{1}{k^2+i\epsilon}\times\left[\frac{N'_v}{D_0D_eD_i}+\frac{N'_{\rm sp}}{D_e^2D_i}+\frac{M'}{D_eD_i}\right]\,, \qquad (3.7a)$$

where

$$M'=2m\gamma_\nu\gamma_\sigma-4k\cdot r\gamma_\nu\not k\gamma_\sigma/(k^2+i\epsilon)\,. \qquad (3.7b)$$

B. Feynman parameter analysis

1. Initial calculation

The evaluation of the momentum space integrals occurring in (3.7) is quite straightforward, but there is one point which we should like to emphasize. That is the manner of cancellation of the infrared divergence. Consider the calculation of $F(q)$ in (3.3a). After the denominators are combined with an x integration and an integration by parts is used to change a logarithm into a denominator, we find

$$F(q)(\not q-m)=\frac{\alpha}{4\pi}\int_0^1 dx\left[\frac{3mq^2x}{q^2(x-1)+m^2}-\frac{3x(1-x)q^2\not q}{q^2(x-1)+m^2}+\mathrm{const}\right]-\delta m_2\,. \qquad (3.8a)$$

The constant is, of course, infinite, but independent of q; hence, it is removed by the mass renormalization. When $q^2=m^2$, the denominators are linear in x; however, the integrals still converge at the lower limit. It is after the mass renormalization is carried out explicitly that we find integrals which diverge at the lower limit on mass shell; this is the infrared divergence. At this stage, the expression becomes

$$F(q)(\not q-m)=\frac{\alpha}{4\pi}\int_0^1 dx\left[\frac{3m(q^2-m^2)}{q^2(x-1)+m^2}-\frac{3x(1-x)q^2(\not q-m)+3(1-x)(q^2-m^2)m}{q^2(x-1)+m^2}\right]\,. \qquad (3.8b)$$

Now we have separate infrared divergent integrals, but a cancellation occurs; and the net result is

$$F(q)=-\frac{3\alpha}{4\pi}\int_0^1 dx\left[-x+\frac{xm^2-xm(q+m)}{q^2(x-1)m^2}\right]. \tag{3.8c}$$

An important point to note is that for the actual external lines, q^2 is off the mass shell by γ^2, making these steps well defined; and the final result is insensitive to γ^2. One finds $F(r)=9\alpha/8\pi$. Had the analysis been done using a photon mass regulator, an apparently different result would have been obtained for the external line self-energy, as was pointed out by Tomozawa.[20] In that case the denominators would have been modified slightly; and if the external line were put on mass shell exactly, a different numerical value would have been obtained. On the other hand, if the electron is kept off-mass-shell, and the photon mass set equal to zero, the quoted result is obtained. Of course, any apparent difference here is compensated by differences in the other diagrams. The lesson is that one must treat these infrared sensitive terms with great care even if the Fried-Yennie gauge seems to eliminate them.

The integration of the remaining terms using Feynman parameters is straightforward. The denominators D_e and D_i are first combined using a y integration, and the result is combined with the k^2 denominators using an x integration. The only subtle feature is the presence of a (near) infrared divergence which appears in the integral with the first term of N'_{sp}. This term yields a contribution

$$\frac{\alpha}{8\pi}\gamma_\nu \not p \gamma_\sigma\int_0^1 dx\int_0^1 dy\left[\frac{r^2(2-x)}{\Delta^2}-\frac{4r^2x(1-x)(r^2+2r_0p_0y+y^2p_0^2)}{\Delta^3}\right], \tag{3.9a}$$

where

$$\Delta=x(r^2+2r_0p_0y+y^2p^2)+\gamma^2-y(p^2+2r_0p_0)\ . \tag{3.9b}$$

The trouble with (3.9) is that the combined x and y integrations can lead to a logarithmic dependence on γ, which is spurious since it cancels between the two terms. To avoid this from the start, we may integrate the second one by parts using

$$(r^2+2r_0p_0y+y^2p_0^2)dx=d\Delta+y^2\vec{p}^{\,2}dx\ ,$$

note that the contribution from the limit vanishes, and find the integral is transformed to

$$(3.9\text{a})=\frac{\alpha}{8\pi}\gamma_\nu \not p \gamma_\sigma\int_0^1 dx\int_0^1 dy\left[\frac{3xr^2}{\Delta^2}-\frac{4r^2x(1-x)y^2\vec{p}^{\,2}}{\Delta^3}\right]. \tag{3.9c}$$

In this form, the integrals cause no difficulty. Now we may summarize the complete result to this point. It takes the form

$$\text{EF}=\frac{\alpha}{4\pi}\int_0^1 dx\int_0^1 dy\left\{\left[\frac{A_3}{\Delta^3}+\frac{A_2}{\Delta^2}+\frac{B_2}{\Delta^2D_0}+\frac{B_1}{\Delta D_0}+\frac{6}{D_0}+\frac{3xm^2}{D_0\Delta(y=1)}\right]\gamma_\nu\not p\gamma_\sigma + \left[\frac{C_2}{\Delta^2}-\frac{3mx}{\Delta}+\frac{E_1}{\Delta D_0}+\frac{3mx}{\Delta(y=1)}\right]\gamma_\nu\gamma_\sigma\right\}, \tag{3.10}$$

where

$$A_3=-16x(1-x)y^2(1-y)\vec{p}^{\,2}m^2\ ,$$

$$A_2=x(6-10y+x+3xy)m^2+xy(-1-y-x+3xy)2mp_0+x^2y^2(1-y)p^2\ ,$$

$$B_2=16xy(1-x)m^2p^2+16xy^2(1-x)m^2\vec{p}^{\,2}+12x(1-x)m^3p_0+8xy(1-x)m^2p_0^2+12xy^2(1-x)mp_0p^2\ ,$$

$$B_1=x(-8+2x)m^2+xy(6-2xy)p^2+(-6+6xy+5x-2x^2y)2mp_0\ ,$$

$$C_2=(1-x)y(4x-8xy+3)mp^2+y(1-x)(-2x+3)2m^2p_0\ ,$$

$$E_1=4xmp^2+4x(1+x)m^2p_0\ .$$

2. *Transformation of the electron factor to other forms*

While the form we have arrived at directly is useful for certain calculations, it does not manifest certain properties which are desirable. For example, it does not manifestly satisfy the well-known low-energy Compton scattering theorem. According to that theorem, the electron factor should have one term which results from the static anomalous moment plus a remainder which is smaller by additional powers of momenta in the low-momenta region. The anomalous moment contribution is simply

$$\text{EF}(a_e)=\frac{\alpha}{2\pi}\frac{\gamma_\nu\not p\gamma_\sigma-p_0\gamma_\nu\gamma_\sigma}{D_0} \tag{3.11a}$$

so we write

$$\mathrm{EF}=\mathrm{EF}(a_e)+\mathrm{EF}' . \tag{3.11b}$$

The form of the second term of (3.11a) takes into account a compensation between ladder and crossed graph contributions for the hfs; other forms are possible. The explicit expression for EF′ will not be given here immediately. Instead, the method for identifying the anomalous moment terms of (3.10) will be described here, while methods for further rearrangement of EF′ will be detailed in a separate appendix. In the coefficient of $\gamma_\nu \not p \gamma_\sigma$ in (3.10), we want the part which reduces to $1/D_0$ when the momentum tends to zero. In addition to the obvious term, the last term and the B_1 term yield such contributions. The latter are easily extracted by adding and subtracting a piece from xm^2 so that a denominator Δ may be canceled, leaving a remainder with additional powers of momentum (see Appendix C). The net result is given by the first term of (3.11a) plus an expression similar to (3.10), but with a change in the definition of B_1 and the introduction of new terms of the form $1/\Delta$. The $\gamma_\nu\gamma_\sigma$ term of (3.10) is handled in a similar manner. We can now note that the whole recoil contribution from (3.11a) curiously vanishes in the order being studied. The cancellation may be most easily understood by referring to the comment just after (1.12). In the present context, the $p_0{}^2$ contribution is doubled relative to the $\vec{p}\,^2$ one; and they simply cancel.

In Appendix B in which the lepton radiative corrections are studied using formal techniques, it is shown the leading contribution for low momenta is given by the anomalous moment contribution. The purpose of that appendix is to show that the muon line does not contribute to $\ln(m_\mu/m_e)$, a result that is not yet obvious from the current expression for EF′. While it might have been better to develop those techniques into a form to be used for actual calculations, we instead rearrange EF′ further in order to manifest this behavior, which we refer to as an extended low-energy Compton scattering theorem (off-mass-shell, with general polarization, but for forward scattering only). After the anomalous moment contributions are removed, various tricks are used (as described in Appendix C) to rearrange EF′. Basically, these tricks involve various integrations by parts with respect to the x and y parameters before the p integrations are carried out. This rearrangement is not particularly useful for the numerical work to be done for the electron radiative corrections, but it seems to be essential to put the muon radiative corrections in a form which makes the numerical calculations more tractable. The rearrangement is done in two stages. After the first stage, we find

$$\mathrm{EF}'=\frac{\alpha}{4\pi}\int_0^1 dx\int_0^1 dy\left[\left[\frac{A_3}{\Delta^3}+\frac{A_2'}{\Delta^2}+\frac{B_2'}{\Delta^2 D_0}+\frac{B_1'}{\Delta D_0}\right]\gamma_\nu \not p \gamma_\sigma+\left[\frac{C_2'}{\Delta^2}-\frac{C_1}{\Delta}\right]\gamma_\nu\gamma_\sigma\right] . \tag{3.12}$$

The new coefficients are

$$A_2'=xy(7-11y-8y^2-2xy+14xy^2)p^2+xy(-5y-2x+7xy)2mp_0 ,$$

$$B_2'=(-6x+16xy-4x^2y)m^2p^2+16xy^2(1-x)m^2\vec{p}\,^2+8xy(1-x)m^2p_0{}^2+12xy^2(1-x)mp_0p^2 ,$$

$$B_1'=(6-5x-8xy+4x^2y+8xy^2-4x^2y^2)p^2 ,$$

$$C_2'=xy(6-3x-8y+2xy)mp^2+2y^2(1-x)^2(1-y)p_0p^2 ,$$

$$C_1=2(1-x)y(1-xy)p^2/m .$$

With some further tricks it is possible to transform away the terms with the D_0 denominators; this is of conceivable help for an analytic calculation since it would require one less Feynman parameter. It is also convenient for numerical analysis. The form is the same as (3.12) with new coefficients:

$$A_3\to A_3''=16xy^2(-1+x+3y-3xy+2y\ln x)m^2p^2+4xy^2[(1-x)(1-y-2y^2)-(2-8y+8y^2)\ln x](p^2)^2 ,$$

$$A_2'\to A_2''=[y(-1+6x+2x^2)+y^2(26-6y-37x-2x^2+12x^2y+16\ln x)]p^2+2xy(2-4x-5y+7xy)mp_0 ,$$

$$B_2',B_1'\to 0 .$$

These forms will be used for further analysis in Sec. IV.

IV. LEPTON LINE RADIATIVE CORRECTIONS—APPLICATIONS

In this section, the expressions set up in the previous section are applied to the radiative corrections for the electron and muon lines in muonium and for the electron and positron lines in positronium. While we reproduce the known α^2E_F and

$$\alpha^2\widetilde{E}_F(m_e/m_\mu)\ln(m_\mu/m_e)$$

contributions analytically for muonium, the nonlogarithmic $\alpha^2\widetilde{E}_F(m_e/m_\mu)$ terms are evaluated numerically. A possible approach to the analytic evaluation of the latter terms is suggested. The treatment of the corrections for positronium is completely numerical since the lack of a small mass ratio for an expansion makes it meaningless to extract nonrecoil terms which might be treated analytically.

The energy shift arising from corrections to the electron line in muonium is obtained by combining one of the expressions for the electron factor found in Sec. III with the muon factor (taking into account the ladder and crossed graphs), and the photon factors. (Generically, we speak in terms of radiative corrections to the electron line; radiative corrections to the muon line are obtained simply by interchanging the two masses.) The wave-function factors may be decoupled after the anomalous moment terms have been discarded since the dominant loop moments are much larger than those in the wave function, as has been described earlier. Different forms of the electron factor are convenient for different purposes, as seen below; but the general structure is the same for all the corrections. Any of the forms for the electron factor in Sec. III lead to the same type of expression for the energy shift due to radiative corrections, namely,

$$\Delta E=-i\alpha^2E_F\frac{m_em_\mu}{\pi^3}\int_0^\infty \vec{p}^{\,2}d\,|\vec{p}\,|\int_{-\infty}^{\infty}dp_0\int_0^1dx\int_0^1dy\frac{1}{(p^2)^2}[(2\vec{p}^{\,2}-3p_0^{\,2})T_1+3p_0T_2]\times\left[\frac{1}{p^2-\gamma^2-2m_\mu p_0+i\epsilon}+\frac{1}{p^2-\gamma^2+2m_\mu p_0+i\epsilon}\right], \quad (4.1)$$

where T_1 (T_2) is the coefficient of $\gamma_\nu \not{p}\gamma_\sigma$ ($\gamma_\nu\gamma_\sigma$) in any of the expressions of Sec. III.

This expression will be used first to reevaluate the nonrecoil contribution of order α^2E_F. After the preliminary work of Sec. III, the effort involved here is almost trivial and it serves as a partial check on the results of that section. Our discussion is so arranged that the reader may proceed to various depths. First we discuss this nonrecoil calculation. To understand it, the reader should verify the simplest result of Sec. III, which is (3.10) and then read Sec. IV A below. The next most important contribution is the recoil correction containing a logarithm of the mass ratio. It, too, can be arrived at most simply starting with (3.10); and it is discussed in Sec. IV B. The nonlogarithmic recoil corrections may be worked out numerically using any of the forms for the electron factor. However, in the case of radiative corrections to the muon line, it seems better to use one of the forms based on the "improved low-energy theorem"; this eliminates spurious logarithms and makes the integrals better behaved. In doing the recoil corrections to the electron line radiative corrections, it is necessary to remove the nonrecoil part either analytically or numerically. Both approaches are described in Sec. IV C, but the one actually used is numerical. The muon line corrections are straightforward and are also given in Sec. IV C. The corrections in positronium are described in Sec. IV D; it is not necessary to make a nonrecoil subtraction for them.

A. The nonrecoil contribution of order α^2E_F

The result of Kroll and Pollack[13] and Karplus, Klein, and Schwinger[13] is obtained by making nonrecoil approximations on (4.1). If we treat the muon factor following the discussion of (1.5) to (1.7), we run into the problem mentioned in Sec. I: the usual correction is multiplied by a factor m_μ/E, which implies a small recoil correction. This problem will be resolved in Sec. IV C; here we simply extract the nonrecoil term directly. To do this, we neglect $p^2-\gamma^2$ in the muon denominators of (4.1) and find for the muon factor $-2\pi i\delta(p_0)/2m_\mu$. As mentioned previously, it is best to use the most straightforward form of T_1 for this analysis. Of course, the anomalous moment contribution must first be eliminated. With p_0 fixed, the denominators become

$$D_0\rightarrow-\vec{p}^{\,2}\,, \quad (4.2a)$$

$$\Delta\rightarrow xm^2+y(1-xy)\vec{p}^{\,2}\,. \quad (4.2b)$$

At this point, the $\vec{p}$ integration is easily carried out and we are left with the Feynman parameter integral

$$\Delta E=E_F\alpha^2\frac{1}{\pi}\int_0^1dx\int_0^1dy\left\{\frac{p_1(x,y)}{[xy(1-xy)^3]^{1/2}}+\frac{p_2(x,y)}{[xy(1-xy)]^{1/2}}+\frac{3(1-x)}{2[x(1-x)]^{1/2}}\right\}, \quad (4.3)$$

where

$$p_1(x,y)=\tfrac{1}{2}y(1-y)(x-2)^2\,,$$

$$p_2(x,y)=-3+5y-\tfrac{1}{2}x-\tfrac{3}{2}xy+8y^2-16xy^2+4x^2y^2\,.$$

The x and y integrations can all be reduced to forms given in Table I and the expected result $\alpha^2E_F(\ln 2-\frac{13}{4})$ is easily obtained.

B. Radiative recoil corrections of order $\alpha^2\tilde{E}_F(m_e/m_\mu)\ln(m_\mu/m_e)$

The logarithmic dependence arises from the region where p^2 is large compared to $m_e^{\,2}$, but small compared to $m_\mu^{\,2}$. The idea for extracting this logarithmic term is simple. We examine the electron factor and try to find those terms which are a simple numerical multiple of the uncorrected electron factor. This permits us to drop many terms, verifying of course that they cannot yield logarithms. Had we not done the rearrangements of Sec. III in the original momentum space analysis, this would not have been possible. The difficulty would have shown up in separate contributions as a logarithmic divergence in the parametric integrals. In the original work of Caswell and Lepage,[5] it was shown that resulting terms with the $\ln^2(m_\mu/m_e)$ actually compensated; but it appeared quite difficult to extract the singly logarithmic terms.

TABLE I. Integrals required for the evaluation of nonrecoil contributions of order $\alpha^2 E_F$.

$$I_{mn}^d \equiv \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{x}}\int_0^1 \frac{dy}{\sqrt{y}}\,\frac{x^n y^m}{(1-xy)^{d/2}}$$

d	n	m	I_{mn}^d
1	0	0	$4\ln 2$
1	1	0	1
1	1	1	$2\ln 2-1$
1	2	0	$\frac{5}{8}$
1	1	2	$\frac{1}{4}$
1	2	2	$\frac{3}{2}\ln 2-\frac{7}{8}$
3	0	0	4
3	1	0	2
3	1	1	$4-4\ln 2$
3	2	0	$\frac{3}{2}$
3	1	2	1
3	2	2	$5-6\ln 2$

To find the terms of interest, we try to neglect m_e everywhere. This must be done with some caution since some of these terms may become divergent when the mass is neglected in the denominator. To see this, we note that in this region, Δ behaves as $-yp^2(1-xy)$, so some care is needed at the lower limit of the y integration. If there are enough powers of y in the numerator to make the integral converge, or even if it diverges logarithmically, these terms would be too small in the intermediate momentum range. The place where caution is needed is for terms with the structure m_e^2/Δ^2 (no factors of y in the numerator). For these terms the y integration produces a result with leading behavior $1/xp^2$. All other terms with a factor of m_e in the numerator may be dropped since they will converge at momenta characterized by the electron mass. One can now discover very quickly the terms of (3.10) which are capable of producing a logarithm; only the second, fourth, and fifth terms of T_1 can do so and it is easy to evaluate the coefficient by completing the elementary parametric integrations. The result is that the old integral is multiplied by $-5\alpha/4\pi$, leading to the result

$$\Delta E^M_{e-\text{line}}(\ln)=\alpha^2\widetilde{E}_F\frac{m_e}{m_\mu}\left[\frac{15}{4\pi^2}\ln\frac{m_\mu}{m_e}\right]. \qquad (4.4)$$

C. Radiative-recoil corrections from the lepton lines in muonium

We are now ready to complete the evaluation of the lepton radiative corrections, including the additive constants. Except for the removal of terms arising from the anomalous moment, (4.1) gives the complete radiative corrections to the lepton line, including the nonrecoil corrections for the electron treated in Sec. IV A. For the electron, the radiative-recoil corrections are a very small part of the complete integral. Of course, if the whole integral could be done analytically, it would likely be very easy to separate off the two parts. However, this is not practical. We must somehow arrange to separate the integrand into pieces which lead to the different types of contribution. Then it may be possible to carry out approximations which lead to the desired results. Because the answer contains logarithms of the mass ratio, this cannot be accomplished simply by expanding the integrand in inverse powers of the muon mass; a more sophisticated approach is necessary. As mentioned above, we looked into two approaches to the problem, one analytic and one numerical. The one which turned out to be more tractable is the numerical one, and we describe it first. However, the analytic attempt is also enlightening so we describe it later; conceivably, someone may succeed in pushing it to a more successful conclusion.

We want to identify a piece of the integrand of (4.1) which gives directly the results of Sec. IV A, yet leaves a remainder which is easy to handle numerically. One way would be first to carry out the p_0 integration analytically. One of the poles will turn out to be very close to zero. We could subtract from the residue of that pole a contribution which leads precisely to the nonrecoil result. This would lead to a rather cumbersome integrand, but the method seems feasible. Instead of this, we find another integrand which is structurely very close to that of (4.1) in the momentum range near the electron mass, but which reproduces the same nonrecoil result. To attain this result, we first rotate the p_0 contour ($p_0 \to ip_4$); no poles or cuts are crossed in this process. At this stage, the muon factor takes the form

$$\text{MF}=\frac{2p_E^2}{(p_E^2)^2+4m_\mu^2 p_4^2}, \qquad (4.5)$$

where

$$p_E^2=\vec{p}^{\,2}+p_4^2 .$$

We now see that for $m_\mu \to \infty$ (4.5) produces a strong peak at $p_4=0$, corresponding to the δ function which gave the nonrecoil result. To exploit this, we note that when we set $p_4=0$ everywhere except where it appears explicitly multiplying m_μ^2 in the denominator of (4.5), precisely the same result is produced as with a δ function. Thus, we have only to subtract the integrand obtained in this way to remove the nonrecoil result. The resulting integrand will then be strongly suppressed in the region which originally gave the large contribution. This approach has been used with several of the forms of T_1 and T_2 developed in Sec. III.

The four-dimensional integral is evaluated numerically with the adaptive Monte Carlo programs SHEP and VEGAS[22] after changing variables to reduce the $|\vec{p}\,|$ and p_4 integrations to a finite range. The result for the integral is

$$E^{M}_{e-\text{line}}=\alpha^2\widetilde{E}_F\frac{m_e}{m_\mu}(5.361\pm0.058) \quad (4.6a)$$

$$=\alpha^2\widetilde{E}_F\frac{m_e}{m_\mu}\left[\frac{15}{4\pi^2}\ln\frac{m_\mu}{m_e}+3.335\pm0.058\right]. \quad (4.6b)$$

Note that the integral evaluated includes the complete logarithm. We have not attempted to identify and subtract the part of the integrand which produces the logarithm. Thus, the second form of (4.6) results from subtracting the result of the previous section. While we have not identified the logarithm in the numerical work, the mass ratio was varied, and the results are compatible with (4.6b).

Now we describe an attempt at an analytic approach which at first looks promising, but which we subsequently abandoned. We follow the sequence (1.5) to (1.7). The δ function now has the denominator $2E$ rather than $2m_\mu$, producing a factor $m_\mu/E=1-m_e/E$ multiplying the original nonrecoil correction. The $-m_e/E$ piece of this factor represents a recoil correction, as described in the Introduction. We now discuss the role of the other terms in (1.7). The bracket of the first term of (1.7) may be approximated by $-2\pi i\delta'(p_0)$. The factor it multiplies may be treated in either of two ways. One is to use it to cancel D_0 where appropriate and then evaluate the p_0 integration. The result is again a three-dimensional integral very similar in structure to the one evaluated in the nonrecoil calculation. It is, of course, straightforward, but somewhat tedious, to evaluate. The result is $\alpha^2\widetilde{E}_F(m_e/m_\mu)(7\ln2-\frac{1}{4})$. The other way, which is intuitively more instructive, is to split up the two pieces of the numerator factor D_0. The $2m_ep_0$ term converts δ' into $-\delta$, and results in a contribution which precisely cancels the one which arose originally from the correction factor to the nonrecoil calculation. The other term can be regarded as corresponding to the correction found by putting the muon on mass shell rather than simply setting $p_0=0$. The evaluation of this term leads to integrals similar to those outlined in the discussion of the nonrecoil correction, but the details are a bit more tedious. The result turns out to be $\alpha^2\widetilde{E}_F(m_e/m_\mu)(6\ln2+3)$.

The last term of (1.7) still contains the logarithmic contributions; we have not found it convenient to evaluate this term either numerically or analytically. Because of the preceding steps, we now have a denominator $(p_0+i\epsilon)$ which prevents the contour rotation used with the previous forms. For this reason, the use of (1.7) was abandoned and the numerical method described earlier was used instead. Originally, we had expected a smaller result than (4.6). The argument was that the nonlogarithmic terms associated with the logarithm should be of order unity. Instead, they turn out to be of order π^2 if we normalize to the coefficient of the logarithm. Perhaps this has to do with the fact that there were originally $\ln^2(m_\mu/m_e)$ terms associated with the separate graphs, and it is not unusual for π^2 to accompany such terms. A particularly simple term is considered in Appendix D, and it does yield a contribution with a π^2 factor. In any case, the additive constants lead to a quite significant contribution to the theory.

After the preceding discussion, the muon line contribution seems almost trivial. There is no point in separating a nonrecoil contribution for the muon; all contributions have a recoil factor after the anomalous moment term is removed from the muon factor. Actually, the numerical analysis is not quite trivial. Had one used the first form derived for the muon factor, separated contributions would have contained spurious $\ln(m_e/m_\mu)$ and $\ln^2(m_e/m_\mu)$ contributions. In any case, separate contributions are numerically very large and many significant figures are lost when they are added. The way to avoid this is to use the forms which incorporate the "improved low-energy theorem." When this is done, the integrals are tractable using the same numerical techniques mentioned earlier. The result for the muon leg is

$$E^{M}_{\mu-\text{line}}=\alpha^2\widetilde{E}_F\frac{m_e}{m_\mu}(-1.0372\pm0.0091)\,. \quad (4.7)$$

D. Radiative-recoil corrections for positronium

As in the case of vacuum polarization for positronium or the muon-line contribution to muonium, there is no point in splitting the correction into nonrecoil and recoil pieces. There is not much to be said about the calculation. The numerical methods are the same as those used earlier, and the result is

$$\Delta E^{\text{Ps}}_{e+,e-\text{lines}}=\alpha^2E^{\text{Ps}}_F(-1.787\pm0.004)\,, \quad (4.8)$$

where E^{Ps}_F is the Fermi splitting, *not* including the annihilation contribution.

V. SUMMARY AND DISCUSSION

In this section we collect together all the known contributions to the ground-state hyperfine splitting in muonium and positronium and list the most important terms remaining to be calculated. We use the constants

$$\alpha^{-1}=137.035\,963(15)\,,$$

$$R_\infty=10\,973\,731.528\,7(113)m^{-1}\,,$$

$$c=2.997\,924\,58\times10^{10}\text{cm/sec}\,,$$

$$\mu_p/\mu_B=0.001\,521\,032\,202\,1(152)\,,$$

$$\mu_\mu/\mu_p=3.183\,345\,473(948)\,,$$

$$a_e=0.001\,159\,652\,200(40)\,,$$

and

$$m_\mu/m_e=206.768\,262\,0(617)(\text{Ref.}\,23)\,.$$

Then we find the Fermi splitting for muonium

$$E_F=\tfrac{16}{3}\alpha^2R_\infty\frac{\mu_\mu}{\mu_B}\left[1+\frac{m_e}{m_\mu}\right]^{-3}$$

$$=4\,459\,034.6(1.6)\text{ kHz}\,. \quad (5.1)$$

The 1.6-kHz error quoted for E_F, which is the largest uncertainty in muonium hyperfine splitting, is the combination of the 0.30-ppm uncertainty in μ_μ/μ_B, or equivalently the muon mass, and the 0.22-ppm uncertainty in the square of the fine-structure constant. We define the analogous splitting for positronium as

$$E_F^{\rm Ps}=\tfrac{2}{3}\alpha^2 R_\infty$$
$$=116\,792.42(3)\ \text{MHz} \qquad (5.2)$$

to which must be added the contribution from one-photon annihilation included below.

At this point, we should like to stress that the inclusion of the muon anomaly in (5.1) is a carryover from the definition of the Fermi splitting in hydrogen, $E_F^{\rm H}$. There the magnetic moment of the proton is quite different from that of a pointlike Dirac particle, so E_F should be defined with the factor $(1+a_P)$. For the case of muonium, where a_μ is a small radiative correction, the definition of E_F becomes to some degree a matter of choice.[24] Because an important subset of the corrections, namely, the Breit correction and all nonrecoil corrections, depend on the interaction of the electron with an external magnetic field proportional to the total magnetic moment of the muon, we choose to define E_F in the conventional way. [For positronium, where the electron and positron stand on an equal footing, we have chosen not to include the anomalous moments in the definition (5.2)]. On the other hand, the conventional definition is not appropriate for recoil corrections since the picture of the muon as providing a static magnetic field breaks down. This is why we have been expressing the recoil corrections in terms of $\widetilde{E}_F$, defined in (1.2b). The theoretical prediction for the muonium hfs is now expressed as[25]

$$\nu=[1+\tfrac{3}{2}\alpha^2+a_e+\epsilon_1+\epsilon_2+\epsilon_3 +\sigma_1+\delta_\mu(1+a_\mu)^{-1}]E_F\ , \qquad (5.3)$$

where

$$\epsilon_1=\alpha^2(\ln 2-\tfrac{5}{2})\ ,$$
$$\epsilon_2=-\frac{8\alpha^3}{\pi}\ln\alpha(\ln\alpha-\ln 4+\tfrac{281}{480})\ ,$$
$$\epsilon_3=\frac{\alpha^3}{\pi}(15.38\pm0.29)\ ,$$
$$\sigma_1=\frac{\alpha^3}{\pi}D_1\ ,$$
$$\delta_\mu=-\frac{3\alpha}{\pi}\frac{m_e m_\mu}{m_\mu{}^2-m_e{}^2}\ln\frac{m_\mu}{m_e} -\frac{\alpha^2 m_e m_\mu}{(m_e+m_\mu)^2}(2\ln\alpha+8\ln 2-3\tfrac{11}{18}) +\left[\frac{\alpha}{\pi}\right]^2\frac{m_e}{m_\mu}\left[-2\ln^2\frac{m_\mu}{m_e}+\tfrac{13}{12}\ln\frac{m_\mu}{m_e}+(18.18\pm0.63)\right].$$

Note that the factor $(\tfrac{13}{12})$ in δ_μ has previously appeared[7] as $(\tfrac{31}{12})$; the difference arises from the inclusion of $(1+a_\mu)^{-1}$ with the recoil terms. The uncalculated term σ_1 arises from the representative graphs shown in Fig. 6 in the nonrecoil approximation. If we estimate the uncertainty arising from our ignorance of the constant D_1 as 1 kHz, the present theoretical prediction becomes

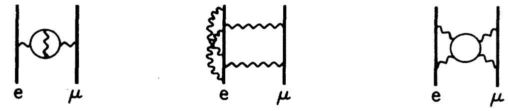

FIG. 6. Representative graphs of uncalculated $\alpha^3 E_F$ contributions to muonium hfs.

$$\nu=4\,463\,304.7(1.9)\ \text{kHz} \qquad (5.4)$$

which is to be compared with the experimental value[1]

$$\nu=4\,463\,302.88(16)\ \text{kHz}\ . \qquad (5.5)$$

The 1.9-kHz uncertainty in the theoretical value has three components: the 1.6-kHz uncertainty in E_F; the 1-kHz uncertainty estimated for D_1; and a 0.2-kHz uncertainty from the error estimates for the numerical integrations in the present work and Ref. 8.

While the agreement between theory and experiment is satisfactory, it is of considerable interest to have the experimental error in the muon mass reduced and the calculation of D_1 performed. To illustrate the situation another way, it is interesting to use muonium hyperfine splitting to provide an independent determination of the fine-structure constant α. We find

$$\alpha_{\rm hfs}{}^{-1}=137.035\,991(20) \qquad (5.6)$$

as compared with values inferred from the electron $g-2$,[26] the Josephson junction,[27] and the quantized Hall effect[28]

$$\alpha_{g-2}{}^{-1}=137.035\,993(10)\ ,$$
$$\alpha_J{}^{-1}=137.035\,963(15)\ , \qquad (5.7)$$
$$\alpha_H{}^{-1}=137.035\,968(23)\ .$$

A factor of 2 or 3 improvement of the error in (5.6) will be useful in checking the consistency with the other α involving QED theory, α_{g-2}. Agreement would probably indicate some as yet unknown systematic effect in the Josephson junction experiment, as it is unlikely that a breakdown of QED would affect muonium and the electron anomaly in the same way. Agreement between the Josephson effect α and that from hfs, on the other hand, would be a signal of a breakdown of QED in the electron anomaly. Perhaps such a breakdown would show up first in the anomaly, where momenta of the order of the electron mass dominate, rather than in muonium where the important momenta are smaller by a factor of α.

The situation for positronium can be described more briefly since the theory is less advanced than in muonium. The weighted average of the experimental measurements

$$\nu=203\,388.5(1.0)\ \text{MHz} \qquad (5.8)$$

(Ref. 29) is to be compared with the present theoretical expression

FIG. 7. Uncalculated contributions to positronium hfs at order $\alpha^2 E_F^{\mathrm{Ps}}$.

$$\nu=\alpha^2 R_\infty\left[\frac{2}{3}+\frac{1}{2}-\frac{\alpha}{\pi}(\ln 2+\tfrac{16}{9})+\tfrac{5}{12}\alpha^2\ln\alpha^{-1}+K\alpha^2+K'(\alpha)\alpha^3\right], \qquad (5.9)$$

where $K'(\alpha)$ is expected to contain $\ln^2(\alpha)$. In this paper we have determined a part of K $[-1.148(3)]$ giving

$$\nu=203\,389.6 \text{ MHz}. \qquad (5.10)$$

To complete the evaluation of K, the contributions associated with Fig. 7 must be evaluated. Since the nominal order of these terms is 11 MHz, no comparison can yet be made between theory and experiment.

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APPENDIX A: USE OF A SPECIAL GAUGE TO SIMPLIFY BOUND-STATE PROBLEMS

When a virtual photon spans across several Coulomb interactions (either as an exchanged photon or one associated with the electron or muon line), it sometimes happens that the final order in α becomes independent of the number of Coulomb interactions (above some minimum). Thus, it can become necessary to sum over an infinite number of Coulomb interactions, possibly with the use of some closed form expression. It is also possible that for a fixed number of Coulomb exchanges, the order in α can be smaller than the correct order obtained by summing the series. One way to avoid the latter problem is to use a special gauge now known as the Fried-Yennie gauge.[19] The purpose of this appendix is to give a simple discussion of both phenomena and to determine which diagrams contribute to the order of interest in the present paper. Our aim is to obtain radiative-recoil contributions which are accurate to order $\alpha^2(m_e/m_\mu)E_F$. As a first step, we investigate the nonrecoil corrections to the same order in α. Then we describe briefly how recoil modifies the analysis. For comparison between theory and experiment, one needs nonrecoil terms with one more power of α. We do not study these in detail; but they are discussed elsewhere.[8,14,15] It will be clear from the following discussion that these require a sum over states and they cannot be obtained simply from kernels of finite order.

We fix our attention on radiative corrections to the electron line for the hyperfine splitting. Clearly, the argument can be adapted to other situations. We assume that the virtual photon spans the transverse photon which gives the hfs along with a number of Coulomb interactions. The result is correct also if the photon does not span the transverse photon, provided there are zero or one Coulomb interactions between the two structures (with more than one Coulomb interaction, second-order perturbation theory in the hfs and the Lamb shift yields a result of order $\alpha^3 E_F$). We concentrate on the nonrecoil radiative corrections here, and describe at the end briefly how recoil affects the details. The methods described here are for the purpose of estimating orders of different graphs; they are not generally useful for actual calculations. For this discussion, we ignore the distinction between m_e and m_r.

An electron propagator, as modified by virtual photon emission, is given by

$$\frac{m_e+\gamma_0(E'-k_0)-\vec{\gamma}\cdot(\vec{p}-\vec{k})}{k^2-2k_0E'+2\vec{k}\cdot\vec{p}-\vec{p}^{\,2}-\gamma^2+i\epsilon}, \qquad (\mathrm{A1})$$

where $-\vec{p}$ labels the muon momentum. Where the photon emission occurs, the electron propagator not containing the k dependence is absorbed into the definition of the wave function. Our aim is to examine the nonrelativistic region of the integral ($\vec{p}\sim\alpha m_e; k\sim\alpha^2 m_e$) to see how many inverse powers of α are produced by the integral in various situations. Certain types of contribution will be characterized by the relativistic region ($\vec{p},k\sim m_e$). For example, if we close the k_0 contour in the lower half plane, the enclosed poles in (A1) are located at

$$k_0=E'+[m_e{}^2+(\vec{p}-\vec{k})^2]^{1/2}.$$

These poles can give only a relativistic contribution. On the other hand, the photon enclosed pole is at $k_0=|\vec{k}|$, and it may yield a nonrelativistic contribution. Our strategy is the following. We examine the photon pole contributions alone. If the resulting integral converges in the nonrelativistic region [note that convergence is reduced by setting $k^2=0$ in the denominator of (A1)], we may estimate its order of magnitude by simple dimensional arguments. If the resulting integral does not converge, we know that the electron mass sets the scale and estimate its size accordingly.

Let us suppose that there are n Coulomb interactions in addition to the transverse photon. Consider factors of (A1) which are not immediately adjacent to the wave function or the hfs interaction. The dominant contribution from the numerator of such a factor will be associated with the large-large matrix element, which is approximately $2m_e$. One numerator adjacent to the hfs should be a $-\vec{\gamma}\cdot\vec{p}$ in order that the combination couple large components to large components and produce a contribution to the hfs. A numerator adjacent to the wave function will be one or the other of these depending on whether the polarization index is temporal or spatial (also on whether the large or small component of the wave function is being considered). For uniformity, we factor $2m_e$ out of each electron numerator producing an overall factor for the kernel

$$(\text{overall factor of kernel alone})=\frac{\gamma^{n+2}}{m_\mu}\ . \tag{A2}$$

The denominator m_μ comes from the spin reduction in the muon line. This expression does not directly give a correct indication of the order of magnitude of the contribution (for one thing, it is not dimensionally correct): some of the numerators will turn out to be smaller in order of magnitude than m_e, and the integrations will introduce powers of γ and m_e. This is what we have to analyze. Now the electron factors are replaced by

$$\frac{[m_e+\gamma_0(E'-k)-\vec{\gamma}\cdot\vec{p}+\vec{\gamma}\cdot\vec{k}]/2m_e}{-2kE'+2\vec{k}\cdot\vec{p}-\vec{p}^{\,2}-\gamma^2+i\epsilon}\ , \tag{A3}$$

where $k=|\vec{k}|$ at the photon pole.

Consider a numerator adjacent to a wave function [with $p\to p_e=(E',\vec{p}_e)$, the wave-function momentum]; it may be rearranged using

$$(m_e+\not{p}_e-\not{k})\gamma_\mu/2m_e=[2p_{e\mu}-\gamma_\mu(\not{p}_e-m_e)-\not{k}\gamma_\mu]/2m_e\ . \tag{A4}$$

The factor $(\not{p}_e-m_e)/2m_e$ adjacent to the wave function is approximately equivalent to $-(\vec{p}_e^{\,2}+\gamma^2)/4m_e^{\,2}$ times the nonrelativistic wave function.

Let us first examine the situation where the kernel is dominated by the relativistic region, either by considering an electron pole contribution or for some other reason. Then if we may ignore the wave-function momenta inside the kernel, the separate wave-function integrations produce a factor γ^3. The kernel integration makes up the dimensions by producing a factor $m_e^{\,-n-3}$, so the final order turns out to be

$$\text{relativistic order}=\frac{\gamma^{n+2+3}}{m_\mu m_e^{\,n+3}}=\frac{\alpha^{n+5}m_e}{m_\mu}\ . \tag{A5}$$

For our present purposes, we may ignore all such terms with $n>1$. Had we taken into account corrections from the wave-function momenta inside the kernel, the result would have been of higher order in α. Thus, even for $n=1$, we may ignore such terms.

To study the nonrelativistic region, we should incorporate the wave-function dependence in the discussion. For the large components, these produce the factor (to adequate approximation)

$$\frac{\gamma^5}{(\vec{p}_e^{\,\prime 2}+\gamma^2)^2(\vec{p}_e^{\,2}+\gamma^2)^2}\ . \tag{A6}$$

The complete overall factor is now γ^5 times (A2), and the integral must have dimensions $(\text{mass})^{-n-5}$. We cannot simply say we expect it to give a factor γ^{-n-5} because there are explicit factors of m_e in the various terms of the integrand, as well as in the electron denominator. To pursue our previously outlined strategy, we make the variable changes

$$\vec{p}=\gamma\vec{t}\ \text{for each}\ \vec{p}\ \text{or}\ \vec{p}_e\ ,$$
$$k^\mu=\gamma^2\xi^\mu/m_e\ , \tag{A7}$$

and factor out γ^2 from each electron denominator, each Coulomb denominator, and the transverse photon denominator, γ^4 from each wave-function denominator, and γ^3 from each d^3p. The photon integration d^3k/k yields $\gamma^4/m_e^{\,2}$ times $d^3\xi/\xi$. A power of momentum supplied by the muon line yields a factor of γ. At this stage we find

$$(\text{overall factor after rescaling})=\frac{\gamma^4}{m_\mu m_e^{\,2}}\ . \tag{A8}$$

Notice that the n dependence has now dropped out and that the integral is dimensionless. At this point (A3) becomes

$$-\frac{\frac{1}{2}[1+\gamma_0(1-\frac{1}{2}\alpha^2-\alpha^2\vec{\xi})-\alpha\vec{\gamma}\cdot\vec{t}+\alpha^2\vec{\gamma}\cdot\vec{\xi}]}{(2\xi+\vec{t}^{\,2}+1-2\alpha\vec{\xi}\cdot\vec{t}-i\epsilon)}$$
$$\approx-\frac{\frac{1}{2}(1+\gamma_0)}{2\xi+t^2+1}\ . \tag{A9}$$

The first term of (A4) is 1 for $\mu=0$; let us concentrate on this contribution for both numerators next to the wave function. This gives the leading order, and we discuss other types of contributions later. The approximation suggested by the right-hand side of (A9) is actually too drastic. To obtain an hfs contribution, one electron numerator adjacent to the transverse photon interaction must be $-\vec{\gamma}\cdot\vec{t}$, introducing an extra factor of α. The integral is now dimensionless; if it converges satisfactorily as $\alpha\to0$, it will give a pure number of order unity. Thus the leading order is expected to be

$$(\text{expected nonrelativistic order})=\frac{\alpha^5 m_e^{\,2}}{m_\mu}\ . \tag{A10}$$

In all other numerators only the $(1+\gamma_0)/2$ terms are retained as $\alpha\to0$. Note that this is one order in α less than the terms we seek.

To test the assumption about convergence of the integrals, imagine carrying out the $\vec{t}$ integrations first, keeping the denominator on the left-hand side of (A9). One might do this by using a parametrization to combine the denominators. The result will take the form

$$\int d(\text{par})\int_0^\infty\frac{\xi\,d\xi}{[(2\xi+1)R+\alpha^2S\xi^2]^{(1/2)n+2+(0,1/2,\ \text{or}\ 1)}}\ , \tag{A11}$$

where R and S are combinations of parameters. The different exponents correspond to whether the external momenta have been decoupled from the integrations over kernel variables (0 when both are neglected, $\frac{1}{2}$, or 1 when one or none are neglected). The worst case is when both are neglected, and we see that the ξ integration converges at the upper limit when $\alpha\to0$ provided $n\geq1$. For $n=0$, we might expect that the integral could produce $\ln\alpha$.

Convergence of the integral in this way does signify that the nonrelativistic region dominates and the estimate (A10) is correct. Note that this result is independent of n

(above a certain limit) and is of lower order than the terms we seek. Before discussing how to deal with this complication, let us discuss some of the terms we have ignored to see whether they are important. Consider the changes when the 0 index is replaced by the space index in the first terms of (A4). The overall factor in (A10) is multiplied by α^2 making the result smaller than the order of interest here, provided the integral corresponding to (A11) converges. Now there are two more powers of $\vec{t}$ in the numerator, but neither wave function decouples from the kernel integration. The result is that the higher order is valid for $n \geq 1$. We may neglect these terms. If we take a cross term (with $\mu = 0$) between the first and second terms of (A4), the result is similar except that one wave function may now decouple from the kernel integration; and now it is possible that for $n = 1$ the integral could yield $\ln\alpha$. With the second term of (A4) used twice, the nonrelativistic order given by (A10) is multiplied by α^4. However, this is now valid only for $n \geq 2$ (possibly with $\ln\alpha$). For smaller n, we must revert to the relativistic order given by (A5), but with two extra powers of α because the wave functions do not decouple. These terms are then quite negligible. A power of α^2 in the numerator [including one from the last term of (A4)] increases the order by a factor of α^2, but requires that n be increased by 2 if the relativistic region is to be avoided. Examining all the terms in this way, one finds that except for the leading $\mu = 0$ terms all others are of higher order than of present interest for $n > 1$.

So far most terms can be neglected for $n > 1$. Only the first terms of (A4) contribute for all values of n to an order which is *one factor of α less* than the order under consideration. We now show that such terms are spurious and can be precisely compensated by an appropriate choice of gauge. To this end, we make the replacement

$$\frac{g_{\mu\nu}}{k^2+i\epsilon} \to \frac{g_{\mu\nu}}{k^2+i\epsilon} + 2\beta \frac{k_\mu k_\nu}{(k^2+i\epsilon)^2} . \quad \text{(A12)}$$

The leading term from the factors (A4) multiplied into (A12) gives

$$\frac{p' \cdot p}{k^2+i\epsilon} + 2\beta \frac{k \cdot p' k \cdot p}{(k^2+i\epsilon)^2} = m_e^2 \left[\frac{1}{k^2+i\epsilon} + 2\beta \frac{k_0^2}{(k^2+i\epsilon)^2} \right] - \frac{\vec{p}\,' \cdot \vec{p}}{k^2+i\epsilon} - 2\beta m_e \frac{k_0 \vec{k} \cdot (\vec{p} + \vec{p}\,')}{(k^2+i\epsilon)^2} + 2\beta \frac{\vec{k} \cdot \vec{p}\,' \vec{k} \cdot \vec{p}}{(k^2+i\epsilon)^2} . \quad \text{(A13)}$$

As in the previous discussion, only the term proportional to m_e^2 can contribute to the order of interest. This term may be rearranged

$$m_e^2 \left[\frac{1}{k^2+i\epsilon} - \beta k_0 \frac{\partial}{\partial k_0} \frac{1}{k^2+i\epsilon} \right] = m_e^2 \left[\frac{1+\beta}{k^2+i\epsilon} - \beta \frac{\partial}{\partial k_0} \frac{k_0}{k^2+i\epsilon} \right] . \quad \text{(A14)}$$

Now consider the effect of integrating by parts. When $\partial/\partial k_0$ acts on the electron factor, the derivatives of k_0^2 or the numerator k_0 are less important than the derivative of $-2E'k_0$ (ultimately by a relative factor α^2). After the integration by parts, we may close the contour below and again pick up the dominant pole at $k_0 = |\vec{k}|$. The result at this stage has the structure (for the dominant term)

$$\int_0^\infty k\,dk \left[1+\beta+\beta k \frac{\partial}{\partial k} \right] \text{EF} = \int_0^\infty k\,dk (1+\beta-2\beta) \text{EF} . \quad \text{(A15)}$$

Obviously, this term cancels for $\beta = 1$ (the Fried-Yennie gauge[19]). There are some necessary comments to be made about (A15). Originally the k_0 differentiation acted only on a certain dependence in the electron factor. Now the k derivative acts on all k dependence. The extra terms differentiated correspond to corrections with two additional powers of α, hence they may be neglected for our present considerations.

Our conclusion is that for the nonrecoil correction of order $\alpha^2 E_F$ it is adequate to keep only one Coulomb interaction in the Fried-Yennie gauge. For the contributions arising from the anomalous moment of the electron, which have one factor α less, it is sufficient to use only the transverse photon interaction. Proceeding to the radiative-recoil corrections, we know that there must be at least two photons exchanged and that the loop momentum between them will be (at least) relativistic for the electron. Hence, the electron propagator between these two photons will not yield inverse powers of α. In the dimensional discussion, there are now two powers of m_μ in the denominator, and it turns out that if we go through the same analysis as before the order in α is increased by 1. Because the terms with any number of Coulomb interactions are now just of the order of interest, it is still convenient to use the Fried-Yennie gauge in order to avoid having to deal with an infinite set of graphs.

Having limited the necessary contributions to the graphs of Fig. 5, we note that one final simplification occurs. It can easily be seen that since the contributions we seek are now in the relativistic momentum region that the wave-function momenta inside the kernels may be ignored. This permits the decoupling of the wave-function integrations from those inside the kernel.

APPENDIX B: FORMAL TECHNIQUES FOR THE ANALYSIS OF RADIATIVE CORRECTIONS TO THE MUON LINE

As we found in the treatment of the electron radiative corrections, it is important to consider together the complete set of diagrams which are gauge invariant in the virtual photon. The method used in Sec. III relied heavily on Feynman diagram techniques. Another approach will be described here. The conclusion will be that the dominant term for small momentum is the anomalous moment term and that other contributions will have too many powers of momentum to produce $\ln(m_\mu/m_e)$ terms in radiative corrections to the muon line. This an extension of the usual low-energy Compton scattering theorem to off-mass-shell amplitudes.

Our approach uses formal techniques similar to those developed by Erickson and Yennie,[30] and Fox and Yennie.[30] We start by defining a coordinate space operator

$$\Pi_\mu = p_\mu + i\partial_\mu - a_\mu e^{-q\cdot x} - b_\mu e^{+iq\cdot x} , \tag{B1}$$

where p_μ is the external muon four-momentum $(m_\mu, 0)$. To illustrate the use of this operator, we note that the one-loop muon factor may be written (throughout this appendix, m is the muon mass)

$$\begin{aligned}\frac{\partial^2}{\partial a_\alpha \partial b_\beta}&\left[(\not{p}-m)\frac{1}{\not{\Pi}-m}(\not{p}-m)\right]\Bigg|_{a=b=0}\\ &=(\not{p}-m)\left[\frac{1}{\not{\Pi}-m}\left\{\gamma_\alpha e^{-iq\cdot x}\frac{1}{\not{\Pi}-m}\gamma_\beta e^{+iq\cdot x}+\gamma_\beta e^{+iq\cdot x}\frac{1}{\not{\Pi}}\gamma_\alpha e^{-i\vec{q}\cdot\vec{x}}\right\}\frac{1}{\not{\Pi}-m}\right]_{a=b=0}(\not{p}-m)\\ &=\gamma_\alpha\frac{1}{\not{p}-\not{q}-m}\gamma_\beta+\gamma_\beta\frac{1}{\not{p}+\not{q}-m}\gamma_\alpha \end{aligned} \tag{B2}$$

since ∂_μ on the extreme right or left gives 0.

It is straightforward to check that the complete set of radiative corrections to the muon line (including *both* external line self-energy diagrams) is given by

$$\frac{\partial^2}{\partial a_\alpha \partial b_\beta}\left\{(\not{p}-m)\frac{1}{\not{\Pi}-m}\left[4\pi\alpha\int\frac{d^4k}{(2\pi)^4 i(k^2-\lambda^2)}\gamma^\mu\frac{1}{\not{\Pi}-\not{k}-m}\gamma_\mu-\delta m^{(2)}\right]\frac{1}{\not{\Pi}-m}(\not{p}-m)\right\}\Bigg|_{a=b=0} , \tag{B3}$$

where it is also understood that the final expression is to be taken between large components of the muon spinor. Of course, nothing would be gained by first carrying out the differentiations in (B3) since that would simply reproduce the sum of contributions from six Feynman diagrams. Instead we try to rearrange the expression in square brackets, taking into account that it has noncommuting pieces. We expect the result to be expressible in the form ($\tilde{\Sigma}$ corresponds to the square brackets)

$$\begin{aligned}\tilde{\Sigma}(\Pi)=(\not{\Pi}-m)B+(\not{\Pi}-m)^2F(\not{\Pi})\\ +\text{terms from noncommutativity} .\end{aligned} \tag{B4}$$

The first two terms are the same expressions which would be obtained for Π_μ a c number. We will first dispense with them. If the B term is inserted in (B3), it gives a result which is B times (B2). In the usual language, this is just the consequence of having two, rather than one, external electron self-energies. For proper counting, it should be discarded. The F term from (B4) gives a vanishing contribution because when it is inserted in (B3), the $1/(\not{\Pi}-m)$ poles are canceled, and the final result vanishes when taken between large components.

Now turn to the terms arising from noncommutativity, which we refer to as *field-strength* terms by analogy to earlier work. If any of these have a factor of $(\not{\Pi}-m)$ to the right or left, it may be discarded since the operations implicit in (B3) will give a vanishing result. Thus, we are able to drop all terms where a $(\not{\Pi}-m)$ acts to the right or left whether they involve field strengths or not. This corresponds, in some way, to compensations between different graphs. When this has been done the terms kept, which we call $\Sigma_{\rm FS}$,

$$\Sigma \rightarrow \Sigma_{\rm FS} \tag{B5}$$

will have at least one factor of a or b because of the commutators. This means that at least one of the derivatives must act on $\Sigma_{\rm FS}$. Consequently we are left only with modified vertex and spanning photon diagrams, the effects from self-energy diagrams having been absorbed into these structures.

For the present, we shall proceed a little more generally than is necessary for our immediate purposes since the results may be useful in other analyses. Later we shall make some approximations specific to our current problem ($q \lesssim m$).

The first step is to rationalize the propagator in (B3)

$$\frac{1}{\not{\Pi}-\not{k}-m}=\frac{1}{k^2-2k\cdot\Pi+\not{\Pi}^2-m^2}(\not{\Pi}-\not{k}+m) . \tag{B6}$$

This may also be written with the operators in the reverse order, and it is often advantageous to average over the two orderings. There are now many options which may be followed. Having tried several, we select the following. Combine the denominator in (B6) with $1/k^2$ by use of the standard Feynman trick; this is legitimate since one of the denominators is a c number. At this stage we have

$$\tilde{\Sigma}^{(2)}=4\pi\alpha\int\frac{d^4k}{(2\pi)^4 i}\int_0^1 dz\,\gamma_\mu\frac{1}{D^2}(\not{\Pi}-k+m)\gamma^\mu-\delta m^{(2)} , \tag{B7a}$$

where

$$D \equiv (k - z\Pi)^2 + z(1-z)(\not{\Pi}^2 - m^2) + z^2\mathcal{M} - z^2 m^2 - (1-z)\lambda^2 \tag{B7b}$$

and

$$\mathcal{M} = \not{\Pi}^2 - \Pi^2 = q\not{a}e^{-iq\cdot x} - q\not{b}e^{iq\cdot x} . \tag{B7c}$$

We also use

$$(\not{\Pi} + m)\gamma^\mu = 2\Pi^\mu , \tag{B8}$$

where we have dropped terms proportional to $\not{\Pi} - m$ on the outside. Also

$$\frac{1}{D^2}\not{k} = -\frac{1}{2}\frac{\partial}{\partial k^\sigma}\frac{1}{D}\gamma^\sigma + \frac{1}{D}z\Pi_\sigma\frac{1}{D}\gamma^\sigma . \tag{B9}$$

The first term on the RHS may be dropped since it is a perfect derivative. Thus, the integrand of (B7a) may be rewritten

$$\gamma_\mu\frac{1}{D^2}\Pi^\mu + \Pi^\mu\frac{1}{D^2}\gamma_\mu - \frac{1}{2}\gamma_\mu\frac{1}{D}z\Pi_\sigma\frac{1}{D}\gamma^\sigma\gamma^\mu - \frac{1}{2}\gamma_\mu\gamma^\sigma\frac{1}{D}z\Pi_\sigma\frac{1}{D}\gamma^\mu , \tag{B10a}$$

where we have used symmetrical ordering. Next we notice

$$\gamma_\mu\frac{1}{D} = \frac{1}{D}\gamma_\mu - \frac{1}{D}[2z\not{\Pi}_\mu, \not{\Pi}]\frac{1}{D} . \tag{B10b}$$

This permits us to bring γ_μ through the D, leaving explicit field strength terms. Whenever there is a factor of $\not{\Pi}$ on the outside, it may be replaced by m. The algebraic reduction of (B10a) in which all γ's are brought next to their natural partners is straightforward, but lengthy. We need retain terms with at most two commutators. The complete result is

$$(\text{B10a}) = I_0 + I_1 + I_2 , \tag{B11a}$$

where

$$I_0 = \frac{2m(1+z)}{D^2} , \tag{B11b}$$

$$I_1 = \frac{1}{D}2z(\not{\Pi} - m)\frac{1}{D} - 2z\left[\left\{\frac{1}{D}[\Pi_\mu, \not{\Pi}]\frac{1}{D^2} + \frac{1}{D^2}[\Pi_\mu, \not{\Pi}]\frac{1}{D}\right\}, \Pi^\mu\right] + \frac{4z^2}{D}\left[\mathcal{M}\frac{1}{D}\not{\Pi} + \not{\Pi}\frac{1}{D}\mathcal{M}\right]\frac{1}{D} + \frac{2z^2}{D}\left\{[\Pi_\mu, \not{\Pi}]\frac{1}{D}\Pi^\mu - \Pi^\mu\frac{1}{D}[\Pi_\mu, \not{\Pi}]\right\}\frac{1}{D} , \tag{B11c}$$

$$I_2 = \frac{4z^3}{D}\left\{[\Pi_\mu, \not{\Pi}]\frac{1}{D}\not{\Pi}\frac{1}{D}[\Pi^\mu, \not{\Pi}] - [\Pi_\mu, \not{\Pi}]\frac{1}{D}[\Pi^\mu, \not{\Pi}]\frac{1}{D}\not{\Pi} - \not{\Pi}\frac{1}{D}[\Pi_\mu, \not{\Pi}]\frac{1}{D}[\Pi^\mu, \not{\Pi}]\right\}\frac{1}{D} + \frac{2z^3}{D}\left\{\mathcal{M}\frac{1}{D}\Pi_\sigma\frac{1}{D}[\Pi^\sigma, \not{\Pi}] - \mathcal{M}\frac{1}{D}[\Pi^\sigma, \not{\Pi}]\frac{1}{D}\Pi_\sigma + \Pi_\sigma\frac{1}{D}[\Pi^\sigma, \not{\Pi}]\frac{1}{D}\mathcal{M} - [\Pi^\sigma, \not{\Pi}]\frac{1}{D}\Pi_\sigma\frac{1}{D}\mathcal{M} + \Pi_\sigma\frac{1}{D}[\Pi^\sigma, \not{\Pi}] - [\Pi^\sigma, \not{\Pi}]\frac{1}{D}\mathcal{M}\frac{1}{D}\Pi_\sigma\right\}\frac{1}{D} , \tag{B11d}$$

the decomposition here is into the number of explicit powers of the field strength. Of course, I_0 and I_1 will be expanded further when appropriate. At most, two powers can contribute to our final expression.

Now let us discuss simplifications which are possible for muon-leg radiative corrections to the muonium hyperfine splitting. Here we are interested in the momentum range $q \lesssim m$. To develop a logarithm of the mass ratio from this range, it is necessary that the muon factor behave like q/m_μ^2. This immediately eliminates all the contributions from I_2 since they have at least two factors of q in the numerator, permitting the electron mass to be neglected. We should note, and dispense with, a possible subtlety at this point. When the k and z integrations are carried out, the result may contain a factor of $\ln(q_0/m_\mu)$. However, because of the factor of q^2, the integration scale is set by m_μ and the final result is nonlogarithmic.

We can argue away a great number of the remaining terms without actually calculating them. Some principles we use are the following. (a) To produce an hfs contribution a term must have an even number of $\vec{\gamma}$ matrices. (b) A factor of q_α or q_β multiplied into the electron factor will, because of gauge invariance, produce a zero contribution.

Start with the second term of I_1, which has a double commutator structure. In D, first rewrite the combination

$$z(1-z)(\not{\Pi}^2 - m^2) + z^2\mathcal{M} = z(1-z)(\Pi^2 - m^2) + z\mathcal{M} \tag{B12}$$

and expand up the $z\mathcal{M}$ keeping the zeroth and first powers. In either case, one derviative must act on $[\Pi_\mu, \not{\Pi}]$ which is proportional to a or b. With zero powers of $\mathcal{M}$, the other derivative must act on $1/(\not{\Pi} - m)$ in order to produce an hfs contribution. In that case, the double commutator structure yields a factor of q^2 so that no logarithmic terms can be produced. [Again, $\ln(q_0/m_\mu)$ from k and z integrations causes no problem.] With one power of $\mathcal{M}$, the other derivative must act on the $\mathcal{M}$. This term is identically zero, even though the k and z integrations produce inverse powers of q, because the external factors of Π^μ become p^μ and the two terms cancel.

The fourth term of I_1 is almost a double commutator structure, but the D's are distributed differently so the discussion must be modified slightly. We again expand up $z\mathcal{M}$. With one power of $\mathcal{M}$, the result is no longer identically zero, but the extra factor of z means that the k and z integrations produce $\ln(q_0/m_\mu)$ rather than inverse powers of q so no logarithmic terms are produced. With zero powers of $\mathcal{M}$, the two terms would cancel except for the arrangement of factors (Π_μ is p_μ or $p_\mu \pm q_\mu$, depending on location). The near cancellation means that the result of k and z integrations will have additional powers of q [possibly with $\ln(q_0/m_\mu)$] so that logarithmic terms cannot be produced.

The remaining terms of I_1 may be considerably simplified. By the same type of arguments, $\not\Pi$ in the third form may be replaced by m [its commutator with D produces higher powers of q which prevent $\ln(m_\mu/m_e)$]. Continuing with the third term, we may next neglect $z^2\mathcal{M}$ in the denominator since the correction has too many powers of q. Having done that, we may neglect $z(1-z)(\not\Pi^2-m^2)$ also since the correction produces a factor of $(\not\Pi^2-m^2)$ which may be dropped, or commutator terms which cannot give $\ln(m_\mu/m_e)$. It is useful to define a succession of approximations to D:

$$D'=(k-z\Pi)^2+z(1-z)(\not\Pi^2-m^2)-z^2m^2-(1-z)\lambda^2\,, \tag{B13a}$$

$$D''=(k-z\Pi)^2-z^2m^2-(1-z)\lambda^2\,. \tag{B13b}$$

To summarize the approximations to the third term of I_1: Replace $\not\Pi$ by m and D by D''. We will retain this contribution.

Turning to the first term of I_1, we expand up powers of $z^2\mathcal{M}$:

$$2z\frac{1}{D}(\not\Pi-m)\frac{1}{D}=2z\frac{1}{D'}(\not\Pi-m)\frac{1}{D'}-2z^2\left\{\frac{1}{D'}\mathcal{M}\frac{1}{D'}\left[\not\Pi,\frac{1}{D'}\right]+\left[\frac{1}{D'},\not\Pi\right]\frac{1}{D'}\mathcal{M}\frac{1}{D'}-z^2\frac{1}{D'}\mathcal{M}\frac{1}{D'}(\not\Pi-m)\frac{1}{D'}\mathcal{M}\frac{1}{D'}\right\}. \tag{B14a}$$

No higher powers of $\mathcal{M}$ occur to second order in the field strengths. The whole { } has too many powers of q to produce $\ln(m_\mu/m_e)$. The first term of (B14a) may be expanded up in a similar way:

$$2z\frac{1}{D'}(\not\Pi-m)\frac{1}{D'}=2z\frac{1}{D''}\left[\not\Pi,\frac{1}{D''}\right]-z(1-z)\left(\left[\frac{1}{D''},\not\Pi^2\right]\frac{1}{D''}\left[\not\Pi,\frac{1}{D''}\right]+\left[\frac{1}{D''},\not\Pi\right]\frac{1}{D''}\left[\not\Pi^2,\frac{1}{D''}\right]-z(1-z)\left[\frac{1}{D''},\not\Pi^2\right]\frac{1}{D''}(\not\Pi-m)\frac{1}{D''}\left[\not\Pi^2,\frac{1}{D''}\right]\right). \tag{B14b}$$

Again, the term in the large parentheses cannot produce $\ln(m_\mu/m_e)$. (To see this, it is helpful to think through the structure of the integral; basically, there are too many q's in the numerator and the z integration causes no difficulty.)

The first term of (B14b) may now be dispensed with. We observe

$$[D'',\not\Pi]=-z[\Pi_\mu,\not\Pi](k^\mu-z\Pi^\mu)-z(k^\mu-z\Pi^\mu)[\Pi_\mu,\not\Pi]\,.$$

Now consider what happens when the derivatives act on the resulting expression. To produce enough γ matrices, only one derivative can act on it. But then the k integration would vanish by symmetry except for the fact that different denominators have different values of momenta ($\Pi^\mu \to p^\mu$ or $p^\mu \pm q^\mu$). This means that after integration $k^\mu - z\Pi^\mu \to q^\mu$ and no logarithms occur.

Summary of I_1 after permissible approximations:

$$I_1 \to 4z^2 m\left[\frac{1}{D''}\mathcal{M}\frac{1}{D''^2}+\frac{1}{D''^2}\mathcal{M}\frac{1}{D''}\right]. \tag{B15}$$

Now turn to I_0, which is in some ways more subtle since it still contains divergences. First expand in powers of $z^2\mathcal{M}$, dropping $(z^2\mathcal{M})^2$ terms which, by oft-repeated arguments, cannot produce $\ln(m_\mu/m_e)$:

$$I_0 \cong \frac{2m(1+z)}{D'^2}-2m(1+z)z^2\left[\frac{1}{D''^2}\mathcal{M}\frac{1}{D''}+\frac{1}{D''}\mathcal{M}\frac{1}{D''^2}\right]. \tag{B16a}$$

Note that in the terms involving $\mathcal{M}$ we have approximated D' by D''. In the first term if we could shift k^μ by the operator $z\Pi^\mu$, we could then set $\not\Pi^2-m^2=0$ and obtain $\delta m^{(2)}$ which is canceled by the subtraction. This suggests expansion in powers of $(\not\Pi^2-m^2)$:

$$\frac{2m(1+z)}{D'^2}=\frac{2m(1+z)}{D''^2}-2m(1-z^2)z\left[\frac{1}{D''^2}\left[\Pi^2,\frac{1}{D''}\right]+\left[\frac{1}{D''},\Pi^2\right]\frac{1}{D''^2}\right]$$
$$+2m(1+z)(1-z)^2z^2\left[\left[\frac{1}{D''},\Pi^2\right]\frac{1}{D''^2}\left[\Pi^2,\frac{1}{D''}\right]+\text{etc.}\right]. \qquad \text{(B16b)}$$

The second term here is similar to the first term of (B14a). The discussion is a little more delicate because of the additional denominator, but the conclusion is the same: No $\ln(m_\mu/m_e)$ is produced. Similarly, the last term produces no $\ln(m_\mu/m_e)$.

The first term of (B16b) is now easily dispensed with. Suppose one derivative acts on it,

$$\frac{\partial}{\partial a^\alpha}\frac{1}{D''}=-\frac{1}{D''}(+2k_\alpha e^{-iq\cdot x}z-z^2\Pi_\alpha e^{-iq\cdot x}-z^2e^{-iq\cdot x}\Pi_\alpha)\frac{1}{D''}. \qquad \text{(B17)}$$

After the k integration, the result will be $\propto q_\alpha$ and the contribution vanishes using principle (b). If two derivatives act, no hfs contribution can be produced. Thus, the derivatives do not act on D'' and $\Pi^\mu\to p^\mu$ or $p^\mu\pm q^\mu$. In either case, the k integration can be shifted and the result is canceled by the $\delta m^{(2)}$ substraction.

At this stage we are left with (B15) plus part of (B16a):

$$I_0+I_1\approx 2mz^2(1-z)\left[\frac{1}{D''^2}\mathscr{M}\frac{1}{D''}+\frac{1}{D''}\mathscr{M}\frac{1}{D''^2}\right]. \qquad \text{(B18)}$$

If two derivatives act on this, the argument of the preceding paragraph shows that the result vanishes. But then, with one derivative, the Π^μ's in the denominator take on the values p^μ or $p^\mu\pm q^\mu$. If we neglect q^μ in the denominator, the k integration can be shifted and the integral is precisely the one giving the anomalous moment to this order.

APPENDIX C: SOME TECHNIQUES FOR REARRANGING THE PARAMETER INTEGRALS IN SEC. III

The purpose of these techniques is to treat apparently dissimilar terms together in such a way that the small-momentum behavior can be analyzed and displayed. In the following, the symbol $\doteq$ represents equivalence under the combined x and y integrations; the expressions differ only by integrations by parts.

1. The terms with y fixed may be put on a common footing with other terms with the help of

$$\frac{1}{\Delta(y=1)}\doteq\frac{\partial}{\partial y}\frac{y^n}{\Delta}=\frac{ny^{n-1}}{\Delta}+\frac{y^n[(1-x)2mp_0+(1-2xy)p^2]}{\Delta^2}. \qquad \text{(C1)}$$

2. To identify the anomalous moment terms, which have the pure denominator D_0, we use

$$\frac{xm^2}{D_0\Delta}=\frac{1}{D_0}+\frac{y(1-x)}{\Delta}+\frac{xy(1-y)p^2}{\Delta D_0}. \qquad \text{(C2)}$$

3. Some terms with different powers of $1/\Delta$ must be brought together in order to manifest low-momentum cancellations. One convenient way is through an identity obtained by considering (for $m,n\neq 0$)

$$0\doteq\frac{\partial}{\partial x}\frac{x^n-x^m}{\Delta^n}=\frac{nx^{n-1}-mx^{m-1}}{\Delta^n}-\frac{n(x^{n-1}-x^{m-1})[\Delta+yD_0]}{\Delta^{n+1}}$$

which gives

$$\frac{(m-n)x^{m-1}}{\Delta^n}\doteq\frac{-ny(x^{n-1}-x^{m-1})D_0}{\Delta^{n+1}}. \qquad \text{(C3)}$$

A special case results from the limit $m\to n$

$$\frac{x^{n-1}}{\Delta^n}\doteq\frac{nx^{n-1}y\ln xD_0}{\Delta^{n+1}}. \qquad \text{(C4)}$$

APPENDIX D: ANALYTIC EVALUATION OF $\alpha^2 m_e/m_\mu E_F$ CONSTANTS

In Sec. IV C the numerical evaluation of constant $\alpha^2m_e/m_\mu E_F$ contributions was presented. It was mentioned there that the breakup of the muon propagator given in (1.7) was useful in the analytic evaluation of these constants. In particular, the second term in this order of m_e/m_μ becomes $-2\pi i\delta'(p_0)/2E$, and yields $\alpha^2m_e/m_\mu E_F(7\ln 2-\frac{1}{4})$ in a calculation very similar to that which gives the nonrecoil term. The third term turns out to be less trivial, which we demonstrate in this appendix by calculating a part of its contribution to the constant.

The term we choose to concentrate on is the last term of (3.10) which is somewhat simplified by having the parameter y fixed to be 1. Its contribution to hyperfine splitting is

$$\Delta E=-\frac{9i\alpha^2m_e{}^2m_\mu E_F}{\pi^3}\int_0^\infty \vec{p}\,^2d|\vec{p}|\int_{-\infty}^{\infty}\frac{p_0dp_0}{(p^2)^2}\int_0^1\frac{x\,dx}{\Delta(y=1)}\left[\frac{1}{p^2-2m_\mu p_0+i\epsilon}+\frac{1}{p^2+2m_\mu p_0+i\epsilon}\right]. \qquad \text{(D1)}$$

The term in large parentheses is split up as described before [see (1.7)] and we isolate the term

$$\frac{1}{p^2-2m_\mu p_0+i\epsilon}+\frac{1}{p^2+2m_\mu p_0+i\epsilon}\to-\frac{p^2}{2m_\mu{}^2(p_0+i\epsilon)^2}+O\left[\frac{m_e}{m_\mu}\right], \tag{D2}$$

$$\Delta E_3=-\frac{9i\alpha^2 m_e{}^2E_F}{2m_\mu\pi^3}\int_0^\infty \vec{p}^{\,2}d\,|\vec{p}\,|\int_{-\infty}^\infty\frac{dp_0}{p^2(p_0+i\epsilon)}\int_0^1\frac{x\,dx}{(1-x)[p^2+2m_ep_0-xm_e^2/(1-x)]}\,. \tag{D3}$$

Rather than directly rotate $p_0\to ip_4$ we first combine p^2 and $\Delta(y=1)$ with a Feynman parameter ρ:

$$\Delta E_3=-\frac{9i\alpha^2 m_e{}^2E_F}{2m_\mu\pi^3}\int_0^\infty \vec{p}^{\,2}d\,|\vec{p}\,|\int_0^1\frac{x\,dx}{1-x}\int_0^1 d\rho\int_{-\infty}^\infty\frac{dp_0}{p_0+i\epsilon}\frac{1}{[(p_0+m_e\rho)^2-\vec{p}^{\,2}-m_e{}^2\rho^2-\rho x m_e{}^2/(1-x)]^2}\,. \tag{D4}$$

At this point we introduce the Wick rotation in the form $p_0\to-m_e\rho+ip_4$ and scale out m_e to find

$$\Delta E_3=-\frac{9\alpha^2E_F}{\pi^3}\frac{m_e}{m_\mu}\int_0^\infty \vec{p}^{\,2}d\,|\vec{p}\,|\int_0^\infty dp_4\int_0^1\frac{x\,dx}{1-x}\int_0^1\rho\,d\rho\frac{1}{p_4{}^2+\rho^2}\frac{1}{[p_4{}^2+\vec{p}^{\,2}+\rho^2+\rho x/(1-x)]^2}\,. \tag{D5}$$

Finally gathering together the denominators with a Feynman parameter y allows the p and p_4 integrations to be performed with the result

$$\Delta E_3=-\frac{9\alpha^2E_F}{8\pi^2}\frac{m_e}{m_\mu}\times\int_0^1 x\,dx\int_0^1 d\rho\int_0^1\frac{dy}{\sqrt{y}}\frac{1}{\rho(1-x)+xy}\,. \tag{D6}$$

This integral can be evaluated with the final result

$$\Delta E_3=-\frac{15}{16}\alpha^2\frac{m_e}{m_\mu}E_F\,. \tag{D7}$$

Note the absence of π^2 in this expression, which suggests that this sort of term will be numerically significant. Unfortunately, while it is always possible to generate parameter integrals such as (D6), we have not been able to evaluate analytically in a systematic way the four-dimensional forms we have examined, but suspect that with enough effort this could be accomplished.

[1]F. G. Mariam *et al.*, Phys. Rev. Lett. 49, 993 (1982).
[2]G. P. Lepage, Phys. Rev. A 16, 863 (1977).
[3]G. T. Bodwin and D. R. Yennie, Phys. Rep. 43C, 267 (1978).
[4]G. T. Bodwin, D. R. Yennie, and M. Gregorio, Phys. Rev. Lett. 41, 1088 (1978).
[5]W. E. Caswell and G. P. Lepage, Phys. Rev. Lett. 41, 1092 (1978).
[6]G. T. Bodwin, D. R. Yennie, and M. Gregorio, Phys. Rev. Lett. 48, 1799 (1982).
[7]E. A. Terray and D. R. Yennie, Phys. Rev. Lett. 48, 1803 (1982); J. R. Sapirstein, E. A. Terray, and D. R. Yennie, *ibid.* 51, 982 (1983).
[8]J. R. Sapirstein, Phys. Rev. Lett. 51, 985 (1983).
[9]M. Gell-Mann and M. L. Goldberger, Phys. Rev. 96, 1433 (1954); F. E. Low, *ibid.* 96, 1428 (1954).
[10]J. Schwinger, Proc. Natl. Acad. Sci. USA 37, 452 (1951); 37, 455 (1951); E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951); E. E. Salpeter, *ibid.* 87, 328 (1952); H. Grotch and D. R. Yennie, Rev. Mod. Phys. 41, 350 (1969).
[11]D. R. Yennie, G. T. Bodwin, and M. A. Gregorio (unpublished).
[12]It might appear that the $\frac{3}{2}\alpha^2$ mentioned after (1.2) should be $\frac{3}{2}\alpha'^2$, producing a correction whose order is of interest. However, a refinement of the analysis (Ref. 11) shows that the correction is really of relative order $\alpha^2 m_e{}^2/m_\mu{}^2$, which is too small to be of present interest.
[13]N. Kroll and F. Pollack, Phys. Rev. 84, 597 (1951); R. Karplus, A. Klein, and J. Schwinger, *ibid.* 84, 597 (1951).
[14]A. J. Layzer, Nuovo Cimento 33, 1538 (1964); and D. E. Zwanziger, *ibid.* 34, 77 (1964).
[15]S. J. Brodsky and G. W. Erickson, Phys. Rev. 148, 26 (1966).
[16]G. P. Lepage (private communication).
[17]G. J. Gounaris and J. J. Sakurai, Phys. Rev. Lett. 21, 244 (1968).
[18]T. H. Bauer *et al.*, Rev. Mod. Phys. 50, 261 (1978).
[19]H. M. Fried and D. R. Yennie, Phys. Rev. 112, 1391 (1958).
[20]However, it was pointed out by Tomozawa [Ann. Phys. (N.Y.) 128, 491 (1980)] that very near the mass shell the propagator and vertex have rapid, but finite variations. The scale of this region is characterized by $\delta p^2\sim m\lambda$, where λ is the photon mass. Thus, we will avoid actually going to mass shell until such terms compensate as they must since $\lambda\to 0$.
[21]Once the Fried-Yennie gauge has been used to justify Fig. 5 as the complete set, we find that we can introduce another type of infrared cutoff, neglect the wave-function momenta inside the kernel, put the external lines on mass shell, and do the calculation in any convenient gauge. The result is independent of gauge and has no infrared singularity since it amounts to a contribution to forward elastic scattering at zero energy. However, the Fried-Yennie gauge does seem to make some of the calculational details simpler so we shall use it in our presentation.
[22]SHEP: A. Dufner, in Proceedings of the Colloquium on Computational Methods in Theoretical Physics, Marseille, 1970 (unpublished); VEGAS: G. P. Lepage, J. Comput. Phys. 27, 192 (1978).
[23]Preliminary results from the 1983 Least Squares Adjustment of the Fundamental Constants (private communication from B. N. Taylor).
[24]We would like to thank R. Cohen for bringing this point to

our attention.

[25]A constant $(\alpha/\pi)(Z\alpha)^2 E_F \frac{64}{225}$ arising from vacuum polarization (Ref. 15) has been added to the self-energy contribution calculated in Ref. 8.

[26]T. Kinoshita and W. B. Lindquist, Phys. Rev. Lett. 47, 1573 (1981).

[27]E. R. Williams and P. T. Olsen, Phys. Rev. Lett. 42, 1575 (1979).

[28]D. C. Tsui *et al.*, Phys. Rev. Lett. 48, 3 (1982).

[29]A. P. Mills, Jr. Phys. Rev. A 27, 262 (1982); A. P. Mills, Jr., and G. M. Bearman, Phys. Rev. Lett. 34, 246 (1978); P. O. Egan, V. W. Hughes, and M. H. Yani, Phys. Rev. A 15, 251 (1977).

[30]G. E. Erickson and D. R. Yennie, Ann. Phys. (N.Y.) 35, 271 (1965); 35, 447 (1965); J. A. Fox and D. R. Yennie, *ibid.* 81, 438 (1973).