Gauge invariance and infrared divergences in spinor quantum electrodynamics

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We apply to spinor QED a new technique developed by Bergère and Szymanowski in the case of scalar QED. This method expresses QED in terms of a manifestly gauge-independent theory. Moreover, exponentiation of the infrared divergences arises naturally.

I. INTRODUCTION

Within QED, the infrared-divergence problem was solved by Yennie, Frautschi, and Suura[1] (YFS) through a factorization of the divergent terms of scattering amplitudes. Their method is to extract the divergent part of each graph and collect these terms in an exponential. Grammer and Yennie[2] improved this technique of separation of divergent parts but they still have to examine each graph separately. Bergère and Szymanowski[3] (BS) introduced a new formulation of scalar QED which is manifestly gauge invariant and leads naturally to YFS exponentiation.

The purpose of this paper is to apply this technique to spinor QED. To cope with renormalization, the action may be regularized and counterterms included. As in Ref. 3, an examination of the renormalization procedure is not our purpose and we shall not mention these difficulties.

In Sec. II, we define notations and present the functional-integral formalism of standard spinor QED. In Sec. III, we introduce a nonlocal transformation of matter fields. It defines a transformed theory in which vertices are transverse and interacting fields exhibit the same gauge invariances as free fields. The transformed theory is equivalent to the standard one, giving the same on-mass-shell S matrix, but its Green's functions are different (off mass shell) and exhibit new infrared divergences. In Sec. IV, we eliminate these divergences through a subtraction procedure. Then we show that the transformation of Sec. III may be chosen in such a way that the subtracted theory is in fact free of any IR divergences and is manifestly gauge invariant. The exponentiation of IR-divergent terms in standard theory arises naturally when it is expressed in terms of the subtracted one.

II. THE FUNCTIONAL PRESENTATION OF THE STANDARD THEORY

A. The generating functional

In the standard theory, the Lagrangian describing the interacting system of electrons, positrons, and photons reads

$$\mathscr{L}(\bar{\psi},\psi,A_\mu)=\mathscr{L}_0(\bar{\psi},\psi)+\mathscr{L}_0(A_\mu)+\mathscr{L}_{\rm int}(\bar{\psi},\psi,A_\mu)\ , \tag{2.1}$$

where

$$\mathscr{L}_0(\bar{\psi},\psi)=\frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi-m\bar{\psi}\psi$$

is the free fermion-field Lagrangian,

$$\mathscr{L}_0(A_\mu)=-\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2$$

is the free photon-field Lagrangian, and

$$\mathscr{L}_{\rm int}(\bar{\psi},\psi,A_\mu)=-e\bar{\psi}A\!\!\!/\psi$$

is the interaction Lagrangian.

The generating functional of Green's functions is

$$Z(\bar{\eta},\eta,\chi_\mu)=\int D\bar{\psi}D\psi DA_\mu\exp\left[i\int d^4x\left[\mathscr{L}(\bar{\psi},\psi,A_\mu)-\frac{1}{2\lambda}(n_\mu A_\mu)^2+\bar{\eta}\psi+\bar{\psi}\eta+\chi_\mu A_\mu\right]\right] \tag{2.2}$$

where $\bar{\eta}$ and η are anticommuting sources for the electron-positron field, χ_μ is a source for the photon field, and

$$-(1/2\lambda)(n_\mu A_\mu)^2$$

is the usual gauge-fixing term.

In terms of the Fourier transforms of the fields,

$$\psi(x)=\frac{1}{(2\pi)^2}\int d^4p'\psi(p')e^{-ip'x}\ , \tag{2.3a}$$

$$A_\mu(x)=\frac{1}{(2\pi)^2}\int d^4k\,A_\mu(k)e^{-ikx}\ , \tag{2.3b}$$

$$\bar{\psi}(x)=\frac{1}{(2\pi)^2}\int d^4p\,\bar{\psi}(p)e^{ipx}\ . \tag{2.3c}$$

The functional integral (2.2) may be written

$$Z(\bar{\eta},\eta,\chi_\mu)=\int DA_\mu Z_0(\bar{\eta},\eta,A_\mu)\exp\left[i\int d^4x\left[\mathscr{L}_0(A_\mu)-\frac{1}{2\lambda}(n_\mu A_\mu)^2+\chi_\mu A_\mu\right]\right], \tag{2.4}$$

where $Z_0(\bar{\eta},\eta,A_\mu)$ is the partial generating functional

$$Z_0(\bar{\eta},\eta,A_\mu)=\int D\bar{\psi}D\psi\exp\left[i\int d^4p\,d^4p'\bar{\psi}(p)F^{-1}(p,p',A_\mu)\psi(p')+i\int d^4p[\bar{\psi}(p)\eta(p)+\bar{\eta}(p)\psi(p)]\right] \tag{2.5}$$

in which

$$F^{-1}(p,p',A_\mu)=D^{-1}(p)\left[\delta(p'-p)-e\int\delta^4k\,D(p)A\!\!\!/(k)\delta(p-p'-k)\right] \tag{2.6}$$

with $D^{-1}(p)=p\!\!\!/-m$ and $\delta^4k=d^4k/(2\pi)^2$.

We integrate the Gaussian integral (2.5) for the anticommuting variables $\bar{\psi}$ and ψ to obtain

$$Z_0(\bar{\eta},\eta,A_\mu)=\det[F^{-1}(p,p',A_\mu)]\exp\left[-i\int d^4p\,d^4p'\bar{\eta}(p)F(p,p',A_\mu)\eta(p')\right]. \tag{2.7}$$

B. Generalized vertices and matter loops

By inverting the distribution (2.6), we define generalized vertices

$$F(p,p',A_\mu)=D(p')\delta(p'-p)+\sum_{n=1}^{\infty}\frac{(-e)^n}{n!}\int\prod_{i=1}^{n}\delta^4k_i W_{\mu_1\cdots\mu_n}(p,p',k_1,\ldots,k_n)\prod_{i=1}^{n}A_{\mu_i}(k_i)\delta\left[p-p'-\sum_{i=1}^{n}k_i\right] \tag{2.8}$$

with, for instance,

$$W_\mu(p,p',k)=-D(p)\gamma_\mu D(p'), \tag{2.9a}$$

$$W_{\mu\nu}(p,p',k_1,k_2)=D(p)\gamma_\mu D(p'+k_2)\gamma_\nu D(p')+D(p)\gamma_\nu D(p'+k_1)\gamma_\mu D(p'), \tag{2.9b}$$

etc. The generalized vertices, which are symmetric in the exchange of the photon variables $\{k,\mu\}$, are the sum of all Feynman tree graphs of the usual presentation at the corresponding order.

In the same way, the determinant in (2.7) is developed:

$$\begin{aligned}\det[F^{-1}(p,p',A_\mu)]&=\det[F^{-1}(p,p',0)]\det[F(p,p',0)F^{-1}(p',p,A_\mu)]\\&=\det[F^{-1}(p,p',0)]\exp\left[\sum_{n=1}^{\infty}\frac{(-e)^n}{n!}\int\prod_{i=1}^{n}\delta^4k_iQ_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)\prod_{i=1}^{n}A_{\mu_i}(k_i)\delta\left[\sum_{i=1}^{n}k_i\right]\right].\end{aligned} \tag{2.10}$$

The denominator of the Gell-Mann—Low formula will cancel $F^{-1}(p,p',0)$.

The $Q_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)$ is the sum, at a given order, of all fermion loops of the usual presentation. We have, for instance,

$$Q_\mu=\int d^4p\frac{4p_\mu}{p^2-m^2}=0, \tag{2.11a}$$

$$Q_{\mu\nu}=-\tfrac{1}{2}\int d^4p\,\mathrm{Tr}[D(p)\gamma_\nu D(p+k_1)\gamma_\mu+D(p)\gamma_\mu D(p+k_2)\gamma_\nu], \tag{2.11b}$$

etc.

C. The Ward-Takahashi identities

As in the case of scalar QED,[3] we easily verify that $Z_0(0,0,A_\mu)$ is invariant under the gauge transformation

$$A_\mu(k)\to A_\mu(k)-ik_\mu\alpha(k). \tag{2.12}$$

This invariance leads to

$$k_{\mu_i}Q_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)=0. \tag{2.13}$$

We also verify that $Z_0(\bar{\eta},\eta,A_\mu)$ is invariant under the gauge transformation

$$A_\mu(k)\to A_\mu(k)-ik_\mu\alpha(k),\qquad \eta\to T(\alpha)\eta, \tag{2.14}$$

where

$$T(\alpha)=\exp\left[-ie\int\delta^4k\,\alpha(k)\tau_{p'\to p'-k}\right], \tag{2.14a}$$

where $\tau_{p'\to p'-k}$ is the operator performing the indicated momentum translation which leads to

$$\begin{aligned}k_{\mu_i}W_{\mu_1\cdots\mu_n}(p,p',k_1,\ldots,k_n)&=W_{\mu_1\cdots\hat{\mu}_i\cdots\mu_n}(p,p'+k_i,k_1,\ldots,\hat{k}_i,\ldots,k_n)\\&\quad-W_{\mu_1\cdots\hat{\mu}_i\cdots\mu_n}(p-k_i,p',k_1,\ldots,\hat{k}_i,\ldots,k_n)\end{aligned} \tag{2.15}$$

at

$$p=p'+\sum_{j=1}^{n}k_j\ ,$$

where the caret means omission of the variable. The equations (2.13) and (2.15) are the Ward-Takahashi identities in the generalized-vertices formalism.

III. THE TRANSFORMED THEORY

In this section, we express the standard theory in terms of a transformed theory. This new theory, which is equivalent to the standard one in the sense of the equivalence theorem,[3,4] is manifestly gauge invariant.

A. The transformation

We express the field $\psi(p')$ in terms of a new field $\tilde{\psi}(q')$,

$$\psi(p')=\int d^4q' S(p',q',A_\mu)\tilde{\psi}(q')\ , \tag{3.1}$$

where

$$S(p',q',A_\mu)=\delta(p'-q')-e\int \delta^4k\, S_\mu(q',k)A_\mu(k)\delta(p'-q'-k) + \frac{e^2}{2!}\int \delta^4k_1\delta^4k_2 S_{\mu\nu}(q',k_1,k_2)A_\mu(k_1)A_\nu(k_2)\delta(p'-q'-k_1-k_2)+\cdots \tag{3.2}$$

is a nonlocal transformation which may contain Dirac matrices.

The fermion Lagrangian of the standard theory may be written

$$\int d^4p\, d^4p'\bar{\psi}(p)F^{-1}(p,p',A_\mu)\psi(p')=\int d^4p\, d^4p' d^4q\, d^4q'\bar{\tilde{\psi}}(q)\bar{S}(q,p,A_\mu)F^{-1}(p,p',A_\mu)S(p',q',A_\mu)\tilde{\psi}(q') \tag{3.3}$$

in which $\bar{S}=\gamma_0 S^\dagger\gamma_0$. Equation (3.3) allows us to define

$$\tilde{F}^{-1}(q,q',A_\mu)=\int d^4p\, d^4p'\bar{S}(q,p,A_\mu)F^{-1}(p,p',A_\mu)S(p',q',A_\mu)\ . \tag{3.4}$$

We define naturally the infinite set of new vertices $\Gamma_{\{\mu\}}$ by

$$\tilde{F}^{-1}(q,q',A_\mu)=D^{-1}(q)\delta(q'-q)-\sum_{n=1}^{\infty}\frac{(-e)^n}{n!}\int\prod_{i=1}^{n}\delta^4k_i\Gamma_{\mu_1\cdots\mu_n}(q,q',k_1,\ldots,k_n)\prod_{i=1}^{n}A_{\mu_i}(k_i)\delta\left[q-q'-\sum_{i=1}^{n}k_i\right]\ . \tag{3.5}$$

These "transformed" vertices are expressed in terms of $S_{\{\mu\}}$:

$$\Gamma_\mu(q,q',k)=-\gamma_\mu-\bar{S}_\mu(q,-k)D^{-1}(q')-D^{-1}(q)S_\mu(q',k)\ , \tag{3.6a}$$

$$\begin{aligned}\Gamma_{\mu\nu}(q,q',k_1,k_2)=&-\bar{S}_{\mu\nu}(q,-k_1,-k_2)D^{-1}(q')-D^{-1}(q)S_{\mu\nu}(q',k_1,k_2)-\bar{S}_\nu(q,-k_2)\gamma_\mu-\gamma_\mu S_\nu(q',k_2)\\&-\bar{S}_\nu(q,-k_2)D^{-1}(q'+k_1)S_\mu(q',k_1)-\bar{S}_\mu(q,-k_1)\gamma_\nu-\gamma_\nu S_\mu(q',k_1)\\&-\bar{S}_\mu(q,-k_1)D^{-1}(q'+k_2)S_\nu(q',k_2)\ ,\end{aligned} \tag{3.6b}$$

etc.

We impose on the interacting transformed fields the condition that they satisfy the same gauge-invariance property as the free ones, that is to say, (3.4) has to be invariant under the gauge transformation

$$A_\mu(k)\to A_\mu(k)-ik_\mu\alpha(k)\ . \tag{3.7}$$

It is easy to prove that this condition leads to

$$k_{\mu_i}\Gamma_{\mu_1\cdots\mu_n}(q,q',k_1,\ldots,k_n)=0\ . \tag{3.8}$$

This is equivalent to the conditions

$$k_\mu S_\mu(q',k)=-1\ ,\quad k_\mu\bar{S}_\mu(q,-k)=1\ , \tag{3.9}$$

and, in general,

$$k_{\mu_i}S_{\mu_1\cdots\mu_n}(q',k_1,\ldots,k_n) =-S_{\mu_1\cdots\hat{\mu}_i\cdots\mu_n}(q',k_1,\ldots,\hat{k}_i,\ldots,k_n)\ . \tag{3.10}$$

As in Ref. 3, we shall use transformations of the type

$$S(p',q',A_\mu)=\exp\left[-e\int\delta^4k\,\varphi_\mu(q',k)A_\mu(k)\tau_{p'\to p'-k}\right]\times\delta(p'-q') \tag{3.11}$$

in which $\varphi_\mu(q',k)$ may contain Dirac matrices and has to satisfy

$$k_\mu\varphi_\mu(q',k)=-1\ ,\quad k_\mu\bar{\varphi}_\mu(q,-k)=1\ . \tag{3.12}$$

When the gauge transformation (3.7) acts on (3.11) we have

$$S(p',q',A_\mu-ik_\mu\alpha(k))=\exp\left[-ie\int\delta^4k\,\alpha(k)\tau_{p'\to p'-k}\right]S(p',q',A_\mu)\ . \tag{3.13}$$

The partial generating functional (2.5) expressed in terms of the transformed field is

$$Z_0(\bar{\eta},\eta,A_\mu)=\int D\bar{\tilde{\psi}}D\tilde{\psi}(\det\bar{S}S)^{-1}\exp\left[i\int d^4q\,d^4q'\bar{\tilde{\psi}}(q)\tilde{F}^{-1}(q,q',A_\mu)\tilde{\psi}(q') + i\int d^4q'd^4p'[\bar{\tilde{\psi}}(q')\bar{S}(q',p',A_\mu)\eta(p')+\bar{\eta}(p')S(p',q',A_\mu)\tilde{\psi}(q')]\right], \quad (3.14)$$

where we have introduced the Jacobian of the variable transformation $(\det\bar{S}S)^{-1}$.

B. The transformed generalized vertices and spinor loops

We integrate the Gaussian integral (3.14) for the anticommuting variables $\bar{\tilde{\psi}}$ and $\tilde{\psi}$. We have

$$Z_0(\bar{\eta},\eta,A_\mu)=[\det F^{-1}(q,q',A_\mu)]\exp\left[-i\int d^4q\,d^4q'd^4p\,d^4p'\bar{\eta}(p)S(p,q,A_\mu)\tilde{F}(q,q',A_\mu)\bar{S}(q',p',A_\mu)\eta(p')\right], \quad (3.15)$$

where we used

$$(\det\bar{S}S)^{-1}\det\tilde{F}^{-1}=\det F^{-1}, \quad (3.16)$$

which exhibits that, as in Ref. 3, the matter loops are not affected by the transformation.

We invert $\tilde{F}^{-1}$ to obtain

$$\tilde{F}(q,q',A_\mu)=D(q')\delta(q'-q)+\sum_{n=1}^{\infty}\frac{(-e)^n}{n!}\int\prod_{i=1}^{n}\delta^4k_i\tilde{W}_{\mu_1\cdots\mu_n}(q,q',k_1,\ldots,k_n)\prod_{i=1}^{n}A_{\mu_i}(k_i)\delta\left[q-q'-\sum_{i=1}^{n}k_i\right], \quad (3.17)$$

which defines "transformed" generalized vertices

$$\tilde{W}_\mu(q,q',k)=D(q)\Gamma_\mu(q,q',k)D(q'), \quad (3.18a)$$

$$\tilde{W}_{\mu\nu}(q,q',k_1,k_2)=D(q)\Gamma_{\mu\nu}(q,q',k_1,k_2)D(q')+D(q)\Gamma_\nu(q,q'+k_1,k_2)D(q'+k_1)\Gamma_\mu(q'+k_1,q',k_1)D(q') +D(q)\Gamma_\mu(q,q'+k_2,k_1)D(q'+k_2)\Gamma_\nu(q'+k_2,q',k_2)D(q'), \quad (3.18b)$$

etc. These transformed generalized vertices are obviously transverse. In the transformed theory we have

$$k_{\mu_i}\tilde{W}_{\mu_1\cdots\mu_n}=0, \quad (3.19a)$$

$$k_{\mu_i}Q_{\mu_1\cdots\mu_n}=0. \quad (3.19b)$$

These equations are the Ward-Takahashi identities of the transformed theory.

The transformed generalized vertices depend on the generalized vertices of the standard theory. We have

$$F(p,p',A_\mu)=\int d^4q\,d^4q'S(p,q,A_\mu)\tilde{F}(q,q',A_\mu)\bar{S}(q',p',A_\mu), \quad (3.20)$$

which gives

$$W_\mu(p,p',k)=\tilde{W}_\mu(p,p',k)+D(p)\bar{S}_\mu(p,-k)+S_\mu(p',k)D(p'), \quad (3.21a)$$

$$W_{\mu\nu}(p,p',k_1,k_2)=\tilde{W}_{\mu\nu}(p,p',k_1,k_2)+D(p)\bar{S}_{\mu\nu}(p,-k_1,-k_2)+S_{\mu\nu}(p',k_1,k_2)D(p') + \left[S_\nu(p-k_2,k_2)\tilde{W}_\mu(p'+k_1,p',k_1)+\tilde{W}_\nu(p,p'+k_1,k_2)\bar{S}_\mu(p'+k_1,-k_2) + S_\nu(p-k_2,k_2)D(p'+k_1)\bar{S}_\mu(p'+k_2,-k_1)+\begin{bmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{bmatrix}\right]. \quad (3.21b)$$

Now, it is necessary to discuss briefly the ultraviolet behavior of the transformed generalized vertices. Equations (3.12) show that $S_{\mu_1\cdots\mu_n}$ behaves as $1/k^n$ when all k's go to infinity. In (3.21a), if q' is fixed, $D(q)\bar{\varphi}_\mu(q,-k)\sim 1/k^2$, $\varphi_\mu(q',k)D(q')\sim 1/k$, and $W_\mu\sim 1/k$. We infer that $\tilde{W}_\mu\sim 1/k$ and behaves like W_μ. More generally, $\tilde{W}_{\{\mu\}}$ behaves like $W_{\{\mu\}}$. Thus, the ultraviolet divergences of the transformed theory are not worse than the divergences of the standard one. Finally it can be proved, as in Ref. 3, that the transformed theory and the standard theory are equivalent, giving the same S matrix.

C. The transformation and Dirac matrices

The condition (3.12a) is not sufficient to determine completely φ_μ. Its transverse part is really undetermined. A proper definition of the longitudinal component of φ_μ allows us to restrict the dependence on the Dirac matrices to the part of φ_μ which does not act to make the vertices transverse. Let us give a useful example.

In practical applications, we often have to calculate $\Gamma_\mu(q,q',k)u(q')$ where $u(q')$ is a solution of the Dirac equation $D^{-1}(q')u(q')=0$. The calculations are simplified if we have $\Gamma_\mu u(q')=0$. Equations (3.6a) and (3.12) lead to

$$\Gamma_\mu u(q')=[-\gamma_\mu - D^{-1}(q)\varphi_\mu]u(q') , \tag{3.22}$$

which vanishes when

$$\varphi_\mu u(q')=-D(q'+k)\gamma_\mu u(q') .$$

This can be satisfied, for example, by taking

$$\varphi_\mu=\frac{-2q'_\mu - \not{k}\gamma_\mu}{2k\cdot q'+k^2} \tag{3.23a}$$

with $q=q'+k$ and $q^2\neq m^2$.

φ_μ may be split into a longitudinal part φ^l_μ and a transverse part φ^t_μ by

$$\varphi^l_\mu=-\frac{2q'_\mu+k_\mu}{2k\cdot q'+k^2} , \tag{3.23b}$$

$$\varphi^t_\mu=i\frac{k_\nu\sigma_{\nu\mu}}{2k\cdot q'+k^2} . \tag{3.23c}$$

When $k\to 0$, we verify

$$\varphi^l_\mu\sim-\frac{q'_\mu}{k\cdot q'} . \tag{3.24}$$

We shall see, in Sec. IV, that this behavior clarifies the IR divergences. When $k\to\infty$ we verify

$$\varphi^l_\mu\sim-\frac{k_\mu}{k^2} \tag{3.25a}$$

and

$$\varphi^t_\mu\sim ik_\nu\frac{\sigma_{\nu\mu}}{k^2} . \tag{3.25b}$$

S becomes a unitary (for the Dirac scalar product) transformation ($\bar{S}=S^{-1}$).

On the other hand, for any φ_μ, we may define a transverse component by

$$\varphi^t_\mu=\left[g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\varphi_\nu . \tag{3.26a}$$

Then, because of (3.12), we have

$$\varphi^l_\mu=-\frac{k_\mu}{k^2} , \tag{3.26b}$$

which does not contain Dirac matrices (this splitting is not the same as the splitting performed on the preceding example). Thus, we have

$$\varphi_\mu A_\mu=-\frac{k_\mu A^l_\mu}{k^2}+\varphi^t_\mu A^t_\mu , \tag{3.27}$$

where similarly

$$A^t_\mu=\left[g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]A_\nu ,$$

$$A^l_\mu=\frac{k_\mu k_\nu}{k^2}A_\nu ,$$

and the transformation may be written

$$S=T^{-1}\left[-i\frac{k_\mu A_\mu}{k^2}\right]S(p',q',A^t) , \tag{3.28}$$

where $T(-ik_\mu A_\mu/k^2)$ is a gauge transformation in which

$$\alpha(k)=-i\frac{k_\mu A_\mu(k)}{k^2} .$$

By (3.27), to take A^t_μ is equivalent to taking S with φ^t_μ, which gives transverse vertices $\Gamma_{\mu_1\cdots\mu_n}$ for $n>1$, while for $n=1$, $k_\mu\Gamma_\mu=-\not{k}\neq 0$. Similarly, to perform $T^{-1}(-ik_\mu A_\mu/k^2)$ is equivalent to taking S with φ^l_μ, which makes Γ_μ transverse.

We also easily prove that the special transformation

$$\varphi_\mu=-\frac{k_\mu}{k^2} \tag{3.29}$$

is a unitary transformation which leads to a transformed theory with only one nonvanishing vertex, the transverse vertex:

$$\Gamma_\nu=-\gamma_\mu\left[g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right] . \tag{3.30}$$

This theory is, in fact, the standard one in which A_μ is expressed in the Lorentz gauge ($k_\mu A_\mu=0$). This interesting result shows that any Feynman diagram of the standard theory in the Lorentz gauge may be rewritten in terms of transverse vertices. In an arbitrary gauge, we would have to sum over all diagrams, at a given order, to exhibit transversality.

More generally, the projector on the longitudinal part may always be taken free of Dirac matrices, in such a way that the fermion spin does not appear in the part of φ_μ which is useful for us.

IV. THE SUBTRACTED THEORY

The S transformation does not make worse the superficial degree of UV divergence because φ_μ behaves like $1/k$, but this behavior is disastrous when $k\to 0$. As in Ref. 3, we define a subtraction to regularize the integrals. This operation induces a new subtracted theory which will not be equivalent to the preceding ones because the transformation is too singular. We shall see that this subtracted theory, which is free of any IR divergence, describes a different physical system, related to the original one by a simple exponential factor containing all the infrared divergences.

A. The subtracted transformation

We define the subtracted transformation

$$S_\gamma(p',q',A_\mu)=\exp\left[-e\int\delta^4k\,\varphi_\mu(q',k)A_\mu(k)[\tau_{p'\to p'-k}-\gamma(k)]\right]\delta(p'-q') , \tag{4.1}$$

where $\gamma(k)$ is any function sufficiently decreasing at $k\to\infty$, and satisfying $\gamma(0)=1$ and $\gamma(k)=\gamma^{\dagger}(-k)$.

A gauge transformation acts on S_γ as

$$S_\gamma(p',q',A_\mu-ik_\mu\alpha(k))=e^{ie\alpha_\gamma}T(\alpha)S_\gamma(p',q',A_\mu)\ , \tag{4.2}$$

where α_γ is a constant phase: $\alpha_\gamma=\int\delta^4k\,\alpha(k)\gamma(k)$. Thus, the subtraction does not alter the properties of the S transformation, that is to say, the vertices will still be transverse.

As in Sec. III, this transformation S_γ defines a new field,

$$\psi(p')=\int d^4q' S_\gamma(p',q',A_\mu)\widetilde{\psi}_\gamma(q')\ , \tag{4.3}$$

and subtracted vertices

$$\widetilde{F}_\gamma{}^{-1}(q,q',A_\mu)=\int d^4p\,d^4p'\bar{S}_\gamma(q,p,A_\mu)F^{-1}(p,p',A_\mu)S_\gamma(p',q',A_\mu)\ , \tag{4.4a}$$

$$\widetilde{F}_\gamma{}^{-1}(q,q',A_\mu)=D^{-1}(q)\delta(q'-q)-\sum_{n=1}^{\infty}\frac{(e)^n}{n!}\prod_{i=1}^{n}\delta^4k_i\Gamma^\gamma_{\mu_1\cdots\mu_n}\prod_{i=1}^{n}A_{\mu_i}(k_i) \tag{4.4b}$$

with, for instance,

$$\Gamma^\gamma_\mu(q,q',k)=\Gamma_\mu(q,q',k)\delta(q-q'-k)+[D^{-1}(q)\varphi_\mu(q',k)+\bar{\varphi}_\mu(q,-k)D^{-1}(q')]\gamma(k)\delta(q'-q)\ , \tag{4.5a}$$

$$\begin{aligned}\Gamma^\gamma_{\mu\nu}(q,q',k_1,k_2)=\;&\Gamma_{\mu\nu}(q,q',k_1,k_2)\delta(q-q'-k_1-k_2)-(\not{k}_1\varphi_{\mu\nu}+\bar{\varphi}_\nu\not{k}_1\varphi_\mu+\bar{\varphi}_\nu\gamma_\mu+\gamma_\mu\varphi_\nu+\Gamma_{\mu\nu})\gamma(k_1)\delta(q-q'-k_2)\\&-(\not{k}_2\varphi_{\mu\nu}+\bar{\varphi}_\mu\not{k}_2\varphi_\nu+\bar{\varphi}_\mu\gamma_\nu+\gamma_\nu\varphi_\mu+\Gamma_{\mu\nu})\gamma(k_2)\delta(q-q'-k_1)\\&+(\Gamma_{\mu\nu}+\bar{\varphi}_\mu\gamma_\nu+\gamma_\nu\varphi_\mu+\bar{\varphi}_\nu\gamma_\mu+\gamma_\mu\varphi_\nu+\not{k}_1\varphi_{\mu\nu}+\not{k}_2\varphi_{\mu\nu}+\bar{\varphi}_\nu\not{k}_1\varphi_\mu+\bar{\varphi}_\mu\not{k}_2\varphi_\nu)\gamma(k_1)\gamma(k_2)\delta(q-q')\end{aligned} \tag{4.5b}$$

in which $\varphi_{\mu\nu}=\frac{1}{2}(\varphi_\mu\varphi_\nu+\varphi_\nu\varphi_\mu)$. We easily verify that these vertices are transverse. As in Sec. III, the fermion loops are those of the standard theory.

We define subtracted generalized vertices

$$F(p,p',A_\mu)=\int d^4q\,d^4q' S_\gamma(p,q,A_\mu)\widetilde{F}_\gamma(q,q',A_\mu)\bar{S}_\gamma(q',p',A_\mu) \tag{4.6}$$

and

$$\widetilde{F}_\gamma(q,q',A_\mu)=D(q')\delta(q-q')+\sum_{n=1}^{\infty}\frac{(-e)^n}{n!}\int\prod_{i=1}^{n}\delta^4k_i W^\gamma_{\mu_1\cdots\mu_n}\prod_{i=1}^{n}A_{\mu_i}(k_i) \tag{4.7}$$

with, for instance,

$$\widetilde{W}^\gamma_\mu(q,q',k)=D(q)\Gamma^\gamma_\mu(q,q',A_\mu)D(q')\ , \tag{4.8a}$$

$$\begin{aligned}\widetilde{W}^\gamma_{\mu\nu}(q,q',k_1,k_2)=\;&D(q)\Gamma^\gamma_{\mu\nu}(q,q',k_1,k_2)D(q')+\int d^4p\,D(q)\Gamma^\gamma_\mu(q,p,k_1)D(p)\Gamma^\gamma_\nu(p,q',k_2)D(q')\\&+\int d^4p\,D(q)\Gamma^\gamma_\nu(q,p,k_2)D(p)\Gamma^\gamma_\mu(p,q',k_1)D(q')\ .\end{aligned} \tag{4.8b}$$

B. Cancellation of IR divergences[5]

The elimination of off-mass-shell IR divergences proceeds as in scalar QED.[3] On the mass shell, a naive power counting shows that the dangerous generalized vertices are the vertices built only with the one-photon vertex Γ_μ.

From (3.22), (3.23a), and (3.24), we conclude that a φ_μ satisfying

$$\lim_{k\to 0}\varphi_\mu\sim-\frac{q'_\mu}{k\cdot q'} \tag{4.9a}$$

or

$$\lim_{k\to 0}\bar{\varphi}_\mu\sim\frac{q_\mu}{k\cdot q} \tag{4.9b}$$

gives a finite generalized vertex at $k\to 0$ and avoids the IR dangerous terms. Thus, the subtracted theory is free of any IR divergence. Again, we notice that the Dirac matrices do not appear in the part of φ_μ which is useful for us.

C. The standard theory in terms of the subtracted theory

The result just obtained shows that we may always choose φ_μ free of Dirac matrices and calculations are greatly simplified. Then, from (3.20) and (4.6) we have

$$\widetilde{F}(q,q',A_\mu)=\exp\left[-i\int\delta^4k\,j^\gamma_\mu(q,q',k)A_\mu(k)\right]\times\widetilde{F}_\gamma(q,q',A_\mu) \tag{4.10}$$

in which

$$j^\gamma_\mu(q,q',k)=ie[\varphi_\mu(q,k)+\varphi_\mu(q',-k)]\gamma(k)\ . \tag{4.11}$$

If φ_μ satisfies (4.9), j^γ_μ behaves like

$$ie\left[\frac{q'_\mu}{k\cdot q'}-\frac{q_\mu}{k\cdot q}\right]$$

at the limit $k\to 0$. This is the classical current created by a particle of charge e, which is suddenly accelerated from q_μ to q'_μ. The transformed theory describes that electron which would be described by the subtracted one, but interacting, in addition, with a current j^γ_μ. When $k\to 0$, this current is the classical current created by the scattered electron.

We may conclude that the subtraction cancels, from the physical system described by the transformed theory, the classical behavior of the scattered fermions. As the subtracted theory is free of IR divergences, this exhibits that it is this classical behavior which produces IR divergences.

Any Green's function which describes a scattering of $2m$ fermions and n photons in the transformed theory may be written

$$\tilde{G}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)=\frac{1}{Z(0,0,\chi_\mu)}\bigg|_{\chi_\mu=0}\prod_{\substack{\text{external}\\ \text{photon}}}\left[-i\frac{\delta}{\delta\chi_\mu}\right]i^m\sum_{\substack{\text{perm}\\ \text{of } p'_i}}\sigma\prod_{i=1}^{m}\tilde{F}\left[p_i,p'_i,-i\frac{\delta}{\delta\chi_\mu}\right]\det F^{-1}\left[-i\frac{\delta}{\delta\chi_\mu}\right]$$
$$\times\exp\left[-\tfrac{1}{2}\int d^4k\,\chi_\mu(k)\Pi_{\mu\nu}(k)\chi_\nu(-k)\right]\bigg|_{\chi_\mu=0} \tag{4.12a}$$

and we will note

$$\tilde{G}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)=\tilde{G}_{\mu_1\cdots\mu_n}(\chi_\mu)\,|_{\chi=0}\,, \tag{4.12b}$$

where $\Pi_{\mu\nu}(k)$ is the photon propagator[3] and σ is the parity of the permutation.

Using (4.10), one finds that

$$\tilde{G}_{\mu_1\cdots\mu_n}(\chi_\mu)=\exp\left[-\int\delta^4k\,J^\gamma_\mu\frac{\delta}{\delta\chi_\mu}\right]\tilde{G}^{\,\gamma}_{\mu_1\cdots\mu_n}(\chi_\mu)\,, \tag{4.13}$$

where $J^\gamma_\mu=\sum_{i=1}^{m}j^\gamma_\mu(p_i,p'_i,k)$. In (4.13), the exponential is a translation operator on χ_μ. Thus, we finally have

$$\tilde{G}_{\mu_1\cdots\mu_n}(\chi_\mu)\,|_{\chi_\mu=0}=\tilde{G}^{\,\gamma}_{\mu_1\cdots\mu_n}(\chi_\mu)\,|_{\chi_\mu=-[1/(2\pi)^2]J^\gamma_\mu}\,. \tag{4.14}$$

As in scalar QED,[3] we may write the scattering matrix of the standard theory in terms of the scattering matrix of the subtracted theory:

$$S_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)=\left[S^\gamma_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)+\frac{e}{(2\pi)^2}\sum_{i=1}^{n}J^\gamma_\alpha\Pi_{\alpha\mu_i}S^\gamma_{\mu_1\cdots\hat{\mu}_i\cdots\mu_n}+\cdots\right]$$
$$\times\exp\left[\frac{e^2}{2}\int\frac{d^4k}{(2\pi)^4}J^\gamma_\mu(k)\Pi_{\mu\nu}(k)J^\gamma_\nu(-k)\right] \tag{4.15}$$

in which

$$S^\gamma_{\mu_1\cdots\mu_n}=[\tilde{G}^{\,\gamma}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)+e\int\delta^4k\,J^\gamma_\alpha\Pi_{\alpha\mu}\tilde{G}^{\,\gamma}_{\mu\mu_1\cdots\mu_n}+\cdots]^{(a)}_{(\text{ms})}\,, \tag{4.16}$$

where (a) and (ms) mean amputated and on mass shell. This result is formally equivalent to the BS one.

The expansion (4.15) has a finite number of terms. Each term represents the current—external-photon interaction. The expansion (4.16) has a finite number of terms at each order of perturbation which represent the current-vertices interaction. All the IR divergences are in the exponential term of (4.15) which represents the current-current interaction.

The BS method which is applied here to spinor QED leads to exponentiation of IR divergences in a simpler and more natural way than the YFS technique.

V. CONCLUSION

We exhibited that the fermion spin is not relevant for both making the theory manifestly gauge invariant and exponentiating the infrared divergences. Thus, it was obvious that our results would have to be formally identical to those of Bergère and Szymanowski. Applications to Bloch-Nordsieck cross sections, to Kinoshita-Lee-Nauenberg probabilities, and to amplitudes between coherent states would be calculated as BS did it.

Now, we shall attempt to generalize this technique to QCD. It is a more difficult problem because the gluon field is self-interacting and thus the transformation has to act upon it.

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[5]This claim rests on the work of Bergère and Szymanowski. Their argument rests on the pole-factorization property which apparently fails in QED (Refs. 6 and 7). To define the S matrix, they institute the pole-factorization property by introducing a photon mass $\mu \neq 0$ which eliminates the infrared divergences and take the limit $\mu \rightarrow 0$ at the end of the computations. The connection of the results obtained by this set of manipulations to $\mu = 0$ QED is not established. Since the artificial $\mu \neq 0$ theories used in this procedure are not bona fide Lagrangian theories, the unitarity of the theory obtained by taking the limit $\mu \rightarrow 0$ is open to question.

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