On the validity of various approximations for the Bethe-Salpeter equation and their WKB quantization

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The validity of the instantaneous approximation for the Bethe-Salpeter equation is questioned within the framework of the simple scalar-scalar model of Cutkosky. Detailed numerous results for various approximations are compared to the exact ones. WKB quantization is applied to these relativistic approximations. An unexpected question arises: is the currently used Bethe-Salpeter equation (i.e., the ladder approximation) well suited to describe two interacting relativistic particles?

I. INTRODUCTION

The story of a relativistic treatment for interacting particles is quite long, dense, and is far from being closed. A first step was the derivation of a full relativistic equation for two interacting particles by Bethe and Salpeter.[1] Since then, a lot of theoretical work has been devoted to the study of this equation; a complete list of references up to 1969 may be found in the paper of Nakanishi.[2] The theory was first applied to the electromagnetic interaction of the electron and the positron in positronium. Interesting discussions on this subject may be found in the review papers by Stroscio[3] and Bodwin and Yennie.[4] Very soon it became clear that an exact numerical solution of the Bethe-Salpeter (BS) equation was not possible, and people looked for some approximations. Among them one can distinguish between two families: (i) the so-called instantaneous approximation (IA) first proposed by Salpeter[5] which was widely used in the positronium description[6] and (ii) the quasipotential-approximation (QP) equations[7,12] which have forms similar to the Schrödinger or Klein-Gordon equations.

Recent progress in quark spectroscopy leads to a renewed interest in relativistic equations. It seems that nonrelativistic approaches work rather well for heavy-quark systems[8] but for light-quark systems a relativistic treatment should be more appropriate. But is it possible to apply the same kind of relativistic equations to quark systems for which the coupling constant to the gluon field is around unity as for electron systems for which the coupling constant to the photon field is around 10^{-2}?

In fact a number of equations were proposed in the past but no serious comparison among them was made in order to delimit their domain of validity. This may probably be partly due to a lack of accuracy in the numerical procedure. In this paper, owing to great care taken in numerical methods, we compare the results for several approximations in a simple model where the full relativistic BS equation can be solved. This will provide a quantitative feeling for the relativistic effects. In Sec. II the model and the different approximations are presented (BS, IA, and QP). Section III deals with semiclassical WKB quantization of the relativistic equations (IA). In Sec. IV the numerical results are discussed.

II. THE MODEL AND VARIOUS APPROXIMATIONS

The scalar-scalar model of Cutkosky[9] is relatively simple, and this is why it has been widely used for several applications.[2,10] It describes two complex scalar fields $\phi_1(x_1)$ and $\phi_2(x_2)$ of masses m_1 and m_2 interacting through a real scalar field $\psi(x)$ of mass μ with an interaction Lagrangian

$$\mathscr{L}(x)=[g_1:\phi_1^\dagger(x)\phi_1(x):+g_2:\phi_2^\dagger(x)\phi_2(x):]\psi(x)\ , \tag{2.1}$$

where g_i are the coupling constants between ϕ_i and ψ. We set

$$g_1g_2=16\pi m_1m_2\lambda\ , \tag{2.2}$$

λ being the dimensionless coupling parameter. ϕ_1 and ϕ_2 obey a Klein-Gordon-type equation, thus the free propagator is

$$G_0(x_i,y_i)=\int\frac{d^4k}{(2\pi)^4}\frac{\exp[-ik\cdot(x_i-y_i)]}{k^2-m_i^2+i\eta}\ . \tag{2.3}$$

A. The Bethe-Salpeter equation

If $|\chi\rangle$ is the state vector of a bound state of the two particles 1 and 2, the BS amplitude is defined by

$$\chi(x_1,x_2)=\langle 0\,|\,T[\phi_1(x_1)\phi_2(x_2)]\,|\,\chi\rangle\ , \tag{2.4}$$

where $|0\rangle$ is the vacuum state and T is the chronological operator. Let $\chi(p_1,p_2)$ be the Fourier transform of $\chi(x_1,x_2)$. The BS equation then can be written

$$\begin{aligned}&(p_1^2-m_1^2)(p_2^2-m_2^2)\chi(p_1,p_2)\\&\quad=\frac{-1}{(2\pi)^4}\int d^4p_1'd^4p_2'V(p_1,p_2;p_1',p_2')\chi(p_1',p_2')\end{aligned} \tag{2.5}$$

where $V(p_1,p_2;p_1',p_2')$ stands for the sum of all irreducible Feynman diagrams involved in the process $1'+2'\to 1+2$. Taking advantage of the invariance under time and space translation and introducing total and relative variables, we have

$$\begin{aligned}&X=\alpha x_1+(1-\alpha)x_2,\quad x=x_1-x_2\\&P=p_1+p_2,\quad p=(1-\alpha)p_1-\alpha p_2,\quad 0\le\alpha\le 1\end{aligned} \tag{2.6}$$

$$V(p,P;p',P')=\delta(P-P')V(p,p',P)\ , \tag{2.7}$$

$$\chi(p,P)=\delta(P-K)\chi_K(p)\ . \tag{2.8}$$

The BS equation now reads

$$[(\alpha K+p)^2-m_1{}^2]\{[(1-\alpha)K-p]^2-m_2{}^2\}\chi_K(p) = \frac{-1}{(2\pi)^4}\int d^4p'V(p,p',K)\chi_K(p')\ . \tag{2.9}$$

The binding energy B (counted positively) is obtained by solving this equation in the center-of-momentum system ($\vec{K}=0$ and $K_0=m_1+m_2-B$). Because the sum of all infinitely many irreducible Feynman diagrams is not known one restricts oneself usually to the so-called ladder approximation for which V is computed by considering only one exchange between particles 1 and 2:

$$V(p,p',K)=-i\frac{16\pi m_1m_2\lambda}{(p-p')^2-\mu^2+i\eta}\ . \tag{2.10}$$

If $\mu=0$ the problem acquires a new O(4) symmetry. Since this is the only case in which the Bethe-Salpeter equation is easily solvable with sufficient accuracy we will restrict ourselves throughout this paper to the long-range interaction $\mu=0$. After performing the Wick rotation and a stereographic projection on a unit sphere in five-dimensional space, Cutkosky[9] was able to transform (2.9) into

$$(q^4+2aq^2+1)\chi(q)=\frac{\lambda}{\pi^3}\int d^4q'\frac{\chi(q')}{(q-q')^2} \tag{2.11}$$

with

$$a=2\left[1-\frac{(m_1-m_2)^2}{(m_1+m_2)^2}\right]^{-1}\left[1-\frac{K_0{}^2}{(m_1+m_2)^2}\right]-1\ . \tag{2.12}$$

In (2.11) the metric is Euclidean: $q^2=q_1{}^2+q_2{}^2+q_3{}^2+q_0{}^2$ and O(4) symmetry is transparent. Owing to this symmetry it is possible to transform (2.11) into a differential equation with only one variable (see Cutkosky's paper or the book by Itzykson and Zuber[11]). Introducing

$$\epsilon=\frac{1+a}{1-a} \tag{2.13}$$

we can write the differential equation as

$$(1-z^2)\frac{d^2g_n(z)}{dz^2}+2(n-1)z\frac{dg_n(z)}{dz}-n(n-1)g_n(z)+\frac{\lambda(1+\epsilon)}{\pi(z^2+\epsilon)}g_n(z)=0 \tag{2.14}$$

with $g_n(\pm1)=0$. This equation admits only certain values of ϵ which determine the binding energies B according to

$$B=\frac{m_1m_2}{m_1+m_2}\frac{(1+r)^2}{r}\left[1-\left[1-4\frac{r}{(1+r)^2}\frac{\epsilon}{1+\epsilon}\right]^{1/2}\right],\quad r=\frac{m_1}{m_2}\ . \tag{2.15}$$

B is a function of λ, m_1, m_2, and a new quantum number n. In fact there is also another quantum number k, but solutions for $k\neq0$ are related to relative time excitations and have been discarded in our study. There is no dependence on any quantum number of angular momentum.

In order to keep the maximum accuracy in the numerical procedure we apply the Noumerov algorithm and thus we make the following change of function:

$$g_n(z)=(1-z^2)^{(n-1)/2}y_n(z)\ . \tag{2.16}$$

Then the new differential equation, suitable for the Noumerov algorithm, is

$$\frac{d^2y_n(z)}{dz^2}=A_n(z,\epsilon)y_n(z),\quad y_n(\pm1)=0 \tag{2.17}$$

where

$$A_n(z,\epsilon)=\frac{n^2-1}{(1-z^2)^2}-\frac{\lambda(1+\epsilon)}{\pi(1-z^2)(z^2+\epsilon)}\ .$$

More details about the numerical procedure can be found in Ref. 13.

In the limit where one mass becomes infinite, it is well known that the BS equation with the ladder approximation does not give the Klein-Gordon results—a consequence of neglecting all higher-order irreducible graphs. However, it can be shown[2,4] that for weak coupling, $\lambda\to0$, the BS binding energy agrees with the nonrelativistic (Schrödinger) limit,

$$B\underset{\lambda\to0}{\sim}[m_1m_2/(m_1+m_2)]\lambda^2/2n^2\ .$$

B. The instantaneous approximation

In more complicated cases (e.g., positronium) than the simple model considered here the BS equation—even in the ladder approximation—cannot be solved exactly. One of the most important drawbacks is the presence of a relative time corresponding to a relative energy whose physical interpretation is not quite clear. To get out of this difficulty Salpeter[5] proposed the instantaneous approximation (IA) which consists of putting $p_0=p_0'$ in $V(p,p',K)=V(p_0,\vec{p},p_0',\vec{p}\,',K)$. Since in general (i.e., for local potentials) V depends only through a term p_0-p_0' on the energies, the new interaction is thus a function of the three-vectors $\vec{p}$ and $\vec{p}\,'$ only; it is instantaneous and consequently no longer covariant. Defining

$$\phi_K(\vec{p})=\int dp_0\chi_K(p) \tag{2.18}$$

and introducing this quantity in (2.9), we get

$$\phi_K(\vec{p})=\frac{-1}{(2\pi)^4}\int d^3\vec{p}\,'V(\vec{p},\vec{p}\,',K)\phi_K(\vec{p}\,')\int dp_0[(\alpha K+p)^2-m_1{}^2+i\eta]^{-1}\{[(1-\alpha)K-p]^2-m_2{}^2+i\eta\}^{-1}\,. \tag{2.19}$$

The p_0 integration is performed either in the upper or in the lower complex half plane. In the center-of-momentum frame the final IA equation is

$$\frac{\omega_1(\vec{p})\omega_2(\vec{p})}{\pi[\omega_1(\vec{p})+\omega_2(\vec{p})]}\{K_0{}^2-[\omega_1(\vec{p})+\omega_2(\vec{p})]^2\}\phi_K(\vec{p})=\frac{i}{(2\pi)^4}\int d^3\vec{p}\,'V(\vec{p},\vec{p}\,',K)\phi_K(\vec{p}\,')\,, \tag{2.20}$$

with $\omega_i(\vec{p})=(\vec{p}^{\,2}+m_i{}^2)^{1/2}$. In order to compare to the exact BS solutions we restrict ourselves to the ladder approximation for which [cf. (2.10)]

$$V(\vec{p},\vec{p}\,',K)=\frac{i\,16\pi m_1 m_2}{(\vec{p}-\vec{p}\,')^2-i\eta}\lambda\,. \tag{2.21}$$

In the nonrelativistic limit ($|\vec{p}|\ll m_i$) (2.20) reduces to

$$\frac{\vec{p}^{\,2}}{2m}\phi(\vec{p})-\frac{\lambda}{2\pi^2}\int\frac{d^3\vec{p}\,'\phi(\vec{p}\,')}{(\vec{p}-\vec{p}\,')^2}=-B\phi(\vec{p}),\quad m=\frac{m_1m_2}{m_1+m_2}\,, \tag{2.22}$$

which is the Schrödinger equation with a Coulomb potential $-\lambda/r$ in the momentum representation. Using the angular decomposition for the Coulomb potential

$$\frac{1}{(\vec{p}-\vec{p}\,')^2}=\frac{2\pi}{pp'}\sum_{l,m}Q_l\left[\frac{p^2+p'^2}{2pp'}\right]Y_{lm}(\hat{p})Y^*_{lm}(\hat{p}\,') \tag{2.23}$$

(Q_l being the Legendre function of second kind) it is possible to find solutions of (2.20) which have a good angular momentum. The ansatz

$$\phi_{lm}(\vec{p})=\frac{\phi_l(p)}{p}Y_{lm}(\hat{p}) \tag{2.24}$$

yields

$$K_0{}^2\phi_l(p)=[\omega_1(p)+\omega_2(p)]^2\phi_l(p)-\frac{2m_1m_2\lambda}{\pi}\frac{[\omega_1(p)+\omega_2(p)]}{\omega_1(p)\omega_2(p)}\int_0^\infty Q_l\left[\frac{p^2+p'^2}{2pp'}\right]\phi_l(p')dp'\,. \tag{2.25}$$

It is not difficult to show that

$$\begin{aligned}&\phi_l(p)\underset{p\to 0}{\sim}p^{l+1}\,,\\ &\phi_l(p)\underset{p\to\infty}{\sim}p^{-l-3}\ \text{in the nonrelativistic limit}\,,\\ &\phi_l(p)\underset{p\to\infty}{\sim}p^{-l-4}\ \text{in the relativistic limit}\,.\end{aligned} \tag{2.26}$$

The function Q_l in Eq. (2.25) has a logarithmic singularity for $p'=p$. Because this singularity is moving with p, numerical integration, by a trapezoidal rule for instance, may lead to large errors. To get rid of this problem we develop $\psi_l(p)=\phi_l(p)/p^l$ in terms of spline functions. Owing to recursion formulas on the Q_l the remaining integrals can be reduced to one integral which is calculated analytically. Denoting $\psi_l(p_i)=V_i$ the original problem reduces to a standard diagonalization

$$EV_i=\sum_{j=1}^{N}\mathscr{H}_{ij}V_j \tag{2.27}$$

with a real but nonsymmetric matrix $\mathscr{H}_{ij}$, which is performed numerically without any problem. Further details concerning the numerical procedure may be found again in Ref. 13.

C. The quasipotential approximation

The quasipotential (QP) approximation provides another way to eliminate the relative time (relative energy). Instead of modifying the interaction V of the BS equation, the free propagator (2.3) is changed. One looks for a free two-particle propagator satisfying the covariance and the on-shell elastic unitary condition for all energies but eliminating the relative energy. There is a certain freedom in the choice of the propagator obeying these requirements and this explains why several quasipotential approximations have been proposed in the past. The propagator of Blankenbecler and Sugar[7] in the center-of-momentum system is

$$G_K^0(p)=\frac{\pi}{\omega_1(\vec{p})\omega_2(\vec{p})}\delta(p_0-\tfrac{1}{2}\omega_1(\vec{p})+\tfrac{1}{2}\omega_2(\vec{p}))\frac{\omega_1(\vec{p})+\omega_2(\vec{p})}{[\omega_1(\vec{p})+\omega_2(\vec{p})]^2-K_0{}^2},\quad \omega_i(\vec{p})=(\vec{p}^{\,2}+m_i{}^2)^{1/2} \tag{2.28}$$

where $p=\frac{1}{2}(p_1-p_2)$ is the relative four-momentum; the total momentum is taken to be $P=(K_0,\vec{0})$. In the case of

equal masses this reduces to

$$G_K^0(p)=\frac{2\pi}{\omega(\vec{p})}\delta(p_0)\frac{1}{4\omega(\vec{p})^2-K_0{}^2},\quad \omega(\vec{p})=(\vec{p}^{\,2}+4m^2)^{1/2} \tag{2.29}$$

where $m_1=m_2=2m$. Inserting this into Eq. (2.30) for the scattering amplitude T we have

$$T(p,q,K)=\frac{-i}{(2\pi)^4}\int V(p,p',K)G_K^0(p')T(p',q,K)d^4p' , \tag{2.30}$$

which is equivalent to the BS equation (2.9) but now the free two-particle propagator $G_K^0(p')$ is not

$$-i[(\alpha K+p')^2-m_1{}^2]^{-1}\{[(1-\alpha)K-p']^2-m_2{}^2\}^{-1}$$

but is given by (2.28) or (2.29). The δ function in $G_K^0(p')$ makes the p_0' integration trivial, putting p_0' on the mass shell and thus eliminating the relative energy as an additional variable. We obtain for equal masses

$$T(\vec{p},p_0,\vec{q},q_0,K)=\frac{-i}{(2\pi)^4}\int V(\vec{p},p_0,\vec{p}\,',0,K)\frac{2\pi}{\omega(\vec{p}\,')}\frac{1}{4\omega(\vec{p}\,')^2-K_0{}^2}T(\vec{p}\,',0,\vec{q},q_0,K)d\vec{p}\,' . \tag{2.31}$$

Putting also p_0 and q_0 on the mass shell, i.e., $p_0=0$, $q_0=0$, this is an equation for $T_{\text{on shell}}(\vec{p},\vec{q},K)=T(\vec{p},0,\vec{q},0,K)$. From this we can deduce

$$\frac{\omega(\vec{p})}{2\pi}[K_0{}^2-4\omega(\vec{p})^2]\phi_K(\vec{p})=\frac{i}{(2\pi)^4}\int V_{\text{on shell}}(\vec{p},\vec{p}\,',K)\phi_K(\vec{p}\,')d^3\vec{p}\,' . \tag{2.32}$$

This coincides with the IA equation (2.20) for $m_1=m_2$ because for equal masses the on-shell potential and the instantaneous one are identical. For $m_1\neq m_2$ one also gets an equation like (2.20) but here $V_{\text{on shell}}(\vec{p},\vec{p}\,',K)$ is a complicated nonlocal potential.

The QP approximation discussed by Todorov[12] seems to us quite simple and attractive. Starting from a three-dimensional Lippmann-Schwinger-type equation for the scattering amplitude and demanding covariance and the on-shell unitary condition he obtains a free propagator proportional to $K_0{}^{-1}(\vec{p}^{\,2}-b^2-i\eta)^{-1}$ where b^2 is defined by

$$4K_0{}^2b^2(K_0{}^2)=K_0{}^4-2(m_1{}^2+m_2{}^2)K_0{}^2+(m_1{}^2-m_2{}^2)^2 \tag{2.33}$$

and is negative for bound-state problems.

In the static limit where one mass goes to infinity the Todorov QP equation yields the Klein-Gordon equation. In the case of the scalar-scalar model studied here the QP equation is very similar to the Schrödinger equation:

$$2K_0(\vec{p}^{\,2}-b^2)\phi(\vec{p})=\frac{-i}{(2\pi)^3}\int V(\vec{p},\vec{p}\,',K)\phi(\vec{p}\,')d^3\vec{p}\,' . \tag{2.34}$$

Inserting (2.21) for V and taking the Fourier transform one obtains

$$\left[-\nabla^2-b^2-\frac{2m_1m_2\lambda}{K_0}\frac{1}{r}\right]\phi(\vec{r})=0 . \tag{2.35}$$

Equation (2.35) differs from the nonrelativistic Schrödinger equation only in the constant factors

$$\lambda\to\frac{m_1m_2}{K_0m}\lambda \text{ and } B\to-\frac{b^2}{2m} . \tag{2.36}$$

The Schrödinger energy levels being $B=m\lambda^2/(2n^2)$ one obtains

$$-\frac{b^2}{2m}=m\left[\frac{m_1m_2}{K_0m}\right]^2\frac{\lambda^2}{2n^2} . \tag{2.37}$$

Substituting b^2 from (2.33) and solving for $K_0{}^2$ gives

$$K_0{}^2=m_1{}^2+m_2{}^2+2m_1m_2\left[1-\frac{\lambda^2}{n^2}\right]^{1/2} . \tag{2.38}$$

Introducing $r=m_1/m_2$ we get for the binding energy the analytical expression

$$\frac{B}{m}=\frac{1+r}{r}\left\{1+r-\left[1+r^2+2r\left[1-\frac{\lambda^2}{n^2}\right]^{1/2}\right]^{1/2}\right\} . \tag{2.39}$$

III. WKB QUANTIZATION

The QP equation (2.35) having Schrödinger form is best suited for WKB quantization. Because WKB quantization of the Schrödinger equation with a Coulomb potential gives the exact energy levels and (2.35) differs only by the substitution (2.36) WKB quantization of (2.35) leads to the exact energy values according to (2.39).

Applying semiclassical quantization to the instantaneous equation (2.20) is less trivial. Transforming (2.20) into the coordinate representation and making the ansatz

$$\phi_{lm}(\vec{r})=\frac{\phi_l(r)}{r}Y_{lm}(\hat{r}) \tag{3.1}$$

we get

$$\frac{\omega_1(\partial_r)\omega_2(\partial_r)}{\omega_1(\partial_r)+\omega_2(\partial_r)}\{K_0{}^2-[\omega_1(\partial_r)+\omega_2(\partial_r)]^2\}\phi_l(r) = -2m_1m_2\lambda\frac{1}{r}\phi_l(r) \quad (3.2)$$

with

$$\omega_i(\partial_r)=\left[m_i{}^2-\hbar^2\frac{\partial^2}{\partial r^2}+\hbar^2\frac{l(l+1)}{r^2}\right]^{1/2},$$

where we put back all $\hbar$'s. To apply the WKB approximation to (3.2) we have to make the Langer modification which consists of replacing $\hbar^2 l(l+1)$ by $\hbar^2(l+\frac{1}{2})^2$, that is, in semiclassical theory the angular momentum becomes $L=\hbar(l+\frac{1}{2})$.

Equation (3.2) does not have the Schrödinger form. To show that, nevertheless, the WKB approximation is applicable we make the ansatz

$$\phi_l(r)=\exp\left[\frac{i}{\hbar}\sigma(r)\right] \quad (3.3)$$

and remark that

$$\hbar^n\frac{\partial^n}{\partial r^n}\phi_l(r)=\exp\left[\frac{i}{h}\sigma(r)\right]\{[i\sigma'(r)]^n+O(\hbar)\} . \quad (3.4)$$

Restricting ourselves to the lowest-order approximation in $\hbar$ we have for any function $f(r)$ that contains *no inverse powers of* $\hbar$

$$\hbar^2\frac{\partial^2}{\partial r^2}[f(r)\phi_l(r)]=f(r)\hbar^2\frac{\partial^2}{\partial r^2}\phi_l(r)+O(\hbar) , \quad (3.5)$$

i.e., $\partial/\partial r$ and $f(r)$ commute in this approximation. Developing the left-hand side of (3.2) with the ansatz (3.3) in a power series and using (3.5) it is easy to see that

$$\frac{\omega_1(\sigma')\omega_2(\sigma')}{\omega_1(\sigma')+\omega_2(\sigma')}\{K_0{}^2-[\omega_1(\sigma')+\omega_2(\sigma')]^2\} = -2m_1m_2\lambda\frac{1}{r} \quad (3.6)$$

with

$$\omega_i(\sigma')=\left[m_i{}^2+\hbar^2\frac{(l+\frac{1}{2})^2}{r^2}+\sigma'(r)^2\right]^{1/2} .$$

Solving for $\sigma'(r)$ the wave function becomes, in lowest order of the WKB approximation,

$$\phi_l(r)=\exp\left[\int_{r_0}^{r}\frac{i}{\hbar}\sigma'(r)dr\right] . \quad (3.7)$$

Because of the centrifugal potential $L^2/r^2=\hbar^2(l+\frac{1}{2})^2/r^2$, which is present even for s waves, the problem has two classical turning points r_1,r_2 and continuity of the wave function demands

$$\int_{r_1}^{r_2}\sigma'(r)dr=(n+\tfrac{1}{2})\pi\hbar , \quad (3.8)$$

which is the usual WKB quantization condition. Equation (3.6) may be solved for σ' analytically for $m_1=m_2=2m$ or for $m_2\to\infty$.

(*a*) $m_1=m_2=2m$. Equation (3.6) now reads

$$\omega(\sigma')[K_0{}^2-4\omega(\sigma')^2]=-16m^2\lambda/r . \quad (3.9)$$

Solving for ω gives

$$\omega(\sigma')=\omega(r)=\begin{cases}\dfrac{K_0}{2\sqrt{3}}\left\{\left[\dfrac{\tilde{r}}{r}+\left[\dfrac{\tilde{r}^2}{r^2}-1\right]^{1/2}\right]^{1/3}+\left[\dfrac{\tilde{r}}{r}-\left[\dfrac{\tilde{r}^2}{r^2}-1\right]^{1/2}\right]^{1/3}\right\} & \text{for } r<\tilde{r} ,\\ \dfrac{K_0}{\sqrt{3}}\cos\left[\frac{1}{3}\arccos\left[\dfrac{\tilde{r}}{r}\right]\right] & \text{for } r\geq\tilde{r}\end{cases} \quad (3.10)$$

with $\tilde{r}=2m^2\lambda 12^{3/2}/K_0{}^3$. For $r\geq\tilde{r}$ there exist two other real solutions of (3.9) but only the one given above matches continuously with the solution for $r<\tilde{r}$. Between the turning points r_1 and r_2

$$\sigma'(r)=[\omega(r)^2-(l+\tfrac{1}{2})^2/r^2-4m^2]^{1/2} \quad (3.11)$$

is real (we again put $\hbar=1$) and the integral in (3.8) is calculated numerically.

It is not difficult to see that for $K_0/m<4$ corresponding to $B>0$, r_2 remains finite and $r_1>0$ so that the integral in (3.8) is finite. For $B\to 0$, i.e., $K_0/m\to 4$, one has $r_2\to\infty$ and the integrand behaves asymptotically as $1/r$, thus the integral becomes infinite. This shows that the system has infinitely many bound states. The importance of the Langer modification $l(l+1)\to(l+\frac{1}{2})^2$ even for s waves may be estimated from the numerical results (for $n=1$, $l=0$, $\lambda=0.1$ the exact value for B/m is 0.4966×10^{-2}; the WKB value with the Langer modification is 0.4967×10^{-2}, without the modification 0.4710×10^{-2}).

(*b*) $m_2\to\infty$. In this case (3.6) reduces to

$$\omega_1(\sigma')[(m-B)-\omega_1(\sigma')]=-m\lambda/r , \quad (3.12)$$

which yields

$$\sigma'(r)=\left\{\left\{\frac{m-B}{2}+\left[\left[\frac{m-B}{2}\right]^2+\frac{m\lambda}{r}\right]^{1/2}\right\}^2-\frac{(l+\frac{1}{2})^2}{r^2}-m^2\right\}^{1/2} . \quad (3.13)$$

The integration in (3.8) is again performed numerically.

WKB quantization of the BS equation (2.9) seems to be

TABLE I. Calculated energies for the first three bound states ($n=1,2,3$) for three different coupling parameters λ. The upper row refers to the limit $m_1/m_2 \to 0$ and the lower one to $m_1=m_2$. The various approximations BS, IA, WKB, QP, NR are explained in the text.

		BS	IA ($l=0$)	IA ($l=1$)	IA WKB ($l=0$)	IA WKB ($l=1$)	QP	NR
$\lambda=\frac{1}{137.036}$	$n=1$	0.2544×10^{-4}	0.2659×10^{-4}		0.2662×10^{-4}		0.2663×10^{-4}	0.2663×10^{-4}
		0.2544×10^{-4}	0.2659×10^{-4}		0.2662×10^{-4}		0.2663×10^{-4}	
	$n=2$	0.6366×10^{-5}	0.6616×10^{-5}	0.6645×10^{-5}	0.6656×10^{-5}	0.6656×10^{-5}	0.6656×10^{-5}	0.6656×10^{-5}
		0.6366×10^{-5}	0.6616×10^{-5}	0.6645×10^{-5}	0.6656×10^{-5}	0.6656×10^{-5}	0.6656×10^{-5}	
	$n=3$	0.2830×10^{-5}	0.2941×10^{-5}	0.2954×10^{-5}	0.2958×10^{-5}	0.2958×10^{-5}	0.2958×10^{-5}	0.2958×10^{-5}
		0.2830×10^{-5}	0.2941×10^{-5}	0.2954×10^{-5}	0.2958×10^{-5}	0.2958×10^{-5}	0.2958×10^{-5}	
$\lambda=0.1$	$n=1$	0.3696×10^{-2}	0.4926×10^{-2}		0.4920×10^{-2}		0.5013×10^{-2}	0.5×10^{-2}
		0.3698×10^{-2}	0.4966×10^{-2}		0.4967×10^{-2}		0.5016×10^{-2}	
	$n=2$	0.9292×10^{-3}	0.1235×10^{-2}	0.1245×10^{-2}	0.1239×10^{-2}	0.1247×10^{-2}	0.1251×10^{-2}	0.125×10^{-2}
		0.9293×10^{-3}	0.1239×10^{-2}	0.1247×10^{-2}	0.1245×10^{-2}	0.1249×10^{-2}	0.1251×10^{-2}	
	$n=3$	0.4136×10^{-3}	0.5498×10^{-3}	0.5537×10^{-3}	0.5523×10^{-3}	0.5545×10^{-3}	0.5557×10^{-3}	0.5556×10^{-3}
		0.4136×10^{-3}	0.5512×10^{-3}	0.5543×10^{-3}	0.5539×10^{-3}	0.5551×10^{-3}	0.5557×10^{-3}	
$\lambda=1$	$n=1$	0.1649	0.3421		0.3367		1	0.5
		0.1684	0.4131		0.3995		1.1716	
	$n=2$	0.4244×10^{-1}	0.1002	0.1067	0.0997	0.1063	0.1340	0.125
		0.4267×10^{-1}	0.1082	0.1174	0.1068	0.1171	0.1363	
	$n=3$	0.1898×10^{-1}	0.4733×10^{-1}	0.4962×10^{-1}	0.4731×10^{-1}	0.4954×10^{-1}	0.5719×10^{-1}	0.5556×10^{-1}
		0.1902×10^{-1}	0.4970×10^{-1}	0.5253×10^{-1}	0.4940×10^{-1}	0.5248×10^{-1}	0.5761×10^{-1}	

quite difficult if not impossible because of the four-dimensional integral and the appearance of the relative energy.

IV. RESULTS AND DISCUSSION

All the equations studied here give binding energies B proportional to the reduced mass m. Thus, the interesting quantity to study is the ratio B/m.

The nonrelativistic (NR) Schrödinger equation gives the well-known binding energies $B/m=\lambda^2/2n^2$. This is the only case for which all recoil effects are fully included in the reduced mass only. It is interesting to see that only the IA energies depend on the angular momentum l while this is not true for the extreme cases BS and NR, as well as for QP. However, in the limit of a very weak coupling constant, the nonrelativistic limit $\lambda^2/2n^2$ is obtained regardless of the approximation used.

In Table I we present the values of B/m for the approximations studied [BS, IA, IA with WKB quantization, QP (Todorov), and NR]. We select three values for the coupling constant λ, one typical for a photon exchange $\lambda=1/137.036$, one typical for gluon exchange $\lambda=1$, and an intermediate one $\lambda=0.1$. We study the first three states ($n=1,2,3$), two different angular momenta $l=0$ and $l=1$, and report the results for a system with an infinite mass $r=0$ (upper row) and one with equal masses $r=1$ (lower row).

The numerical accuracy is 10^{-3} for weak λ, 10^{-4} for $\lambda=1$ for IA, and 10^{-4} for BS and IA with WKB quantization, while the QP and NR values are obtained analytically. Interesting features emerge from this table.

The recoil effects outside the reduced mass are of minor importance; they increase with λ and are maximum for the instantaneous approximation. The full relativistic BS case deviates more and more from the Schrödinger results with increasing coupling: the difference is already 6% for $\lambda=1/137.036$, 30% for $\lambda=0.1$, and for $\lambda=1$ BS and NR differ by a factor 3. BS always gives less binding energy than NR. The instantaneous approximation (IA) also gives less binding energy than NR but the effect is much less important; as for BS it increases with λ and is more pronounced for low values of angular momentum, but the discrepancy between IA and NR is never greater than 30%. On the other hand, the quasipotential binding energy is greater than the corresponding nonrelativistic one but it is always very close to the latter (besides for $\lambda=1$) since the difference is of order λ^4. The QP equation giving the Klein-Gordon equation for $m_2\to\infty$, the case of $r=0$ (upper row) gives at the same time the Klein-Gordon binding energies.

The WKB approximation for IA gives binding energies very close to the exact IA values. The difference is about 1% or less except for the ground state for strong coupling, $\lambda=1$, where the discrepancy is 3%. The WKB values show the same variation of the binding energies with the angular momentum as the exact IA calculations. Also the recoil effects outside the reduced mass are very well reproduced. It has been seen that it is absolutely necessary to make the Langer modification $l(l+1)\to(l+\frac{1}{2})^2$ even for s waves. It seems that the WKB quantization provides a good approximation also to relativistic equations that do not have the Schrödinger form but may be written with a three-dimensional local interaction potential. Because evaluating an integral is considerably simpler than solving a differential or integral equation it should be encouraging to apply WKB quantization to other relativistic problems.

From this model study, we may conclude that the effect of taking an instantaneous approximation in the Bethe-Salpeter equation is very important; it can affect the result by a factor 2 or more if the coupling constant is large and it always gives more binding. However, the Klein-Gordon equation (QP values for $r=0$) or Dirac equation in more realistic situations is known to provide binding energies for small r very close to the experimental ones; BS and Klein-Gordon energies differ by a factor 3 for large couplings; this seems to indicate that the usual form of the BS equation (with the ladder approximation and free propagators) is inadequate for a correct description of two interacting particles, even for weak coupling.

It is remarkable to see that the instantaneous approximation seems to compensate to a great extent the effect of higher-order diagrams neglected in the ladder approximation.

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