Relativistic quantum mechanics of n-particle systems with cluster-separable interactions

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The question is analyzed how to describe a closed relativistic system formed by n particlelike constituents. It is proposed that to such a system there corresponds a unitary representation U of the Poincaré group being a function of $n(n-1)/2$ potentials, one for each pair, such that cluster separability holds: If the constituents are grouped into k clusters and the potentials are set to zero between constituents belonging to different clusters, then U factorizes into a tensor product of k representations, any one of them describing a closed system associated with a particular cluster of constituents. An explicit and mathematically rigorous construction is given such that these properties hold.

I. INTRODUCTION

Although any fundamental description of matter has to introduce infinitely many degrees of freedom, under normal conditions only finitely many degrees of freedom are excited, thus producing the illusion of a finite constant number of particlelike constituents. Examples of this are the electrons in an atom, the nucleons in a nucleus, and the valence quarks in a hadron. The velocities of these constituents may reach considerable values such as $v/c \approx 0.4$ in the ground state of charmonium. Considering the difficulty of the bound-state problem in relativistic quantum field theory it should be useful to have a relativistic quantum-mechanical description of an n-particle system in order to deal with the above-mentioned situations in a semiphenomenological way. From a more theoretical point of view, relativistic many-particle quantum mechanics, by questioning the widespread opinion according to which relativistic dynamics is necessarily field theory, should elucidate the actual basis of field theory.

The early works on relativistic particle mechanics[1–3] (also called theory of direct interaction) made clear that relativistic invariance alone is not a very restrictive condition and is easily satisfied by working in the center-of-mass frame. However, as Foldy[4] pointed out, a further physically indispensable property, namely, macrocausality (which is called macrolocality, cluster separability, or cluster-decomposition property as well), poses an obstinate problem in such theories. The first positive result[5] concerning the cluster separability of an n-particle system, $n>3$, refers to an expansion of the Hamiltonian (and the further operators of the theory) in powers of v/c. It shows that cluster separability can be satisfied to any order of the relativistic corrections but nothing is said about the convergence of the series. The first nonperturbative solution is due to Sokolov.[6] Although Sokolov's treatment is rather cryptic and seems unconvincing in some analytical and combinatorial points, its merit is at least that it introduces the decisive element of a solution: For any partition of the particle system into clusters, Sokolov introduces a unitary "packing operator" such that systems consisting of mutually noninteracting subsystems can be combined by an iterative procedure to form the final fully linked system. Using combinatorial results on cluster expansions,[7,8] Coester and Polyzou[9] gave a clearcut construction of an interacting n-particle system satisfying cluster separability. Whereas Sokolov uses Dirac's point form of dynamics, these authors work within the physically more reasonable instant form. In Ref. 9 the packing operators are constructed with the help of Møller operators so that asymptotic completeness is essential for the method to work. On the other hand, the Hamiltonian depends on the input data (the interaction between pairs) in such a complicated manner that it seems hopeless to prove asymptotic completeness. Already in an earlier stage, the construction is formal insofar as self-adjointness and positivity of the mass operators and the unitarity of the packing operators are not clear. Independently of Ref. 9, I gave a similar construction[10] in which the packing operators are defined kinematically so that no assumptions on the scattering behavior are necessary. As to the self-adjointness and positivity of the mass operators it is argued heuristically in Ref. 10 that these physically essential properties should hold for pair potentials that are neither too singular nor too strongly binding. In this paper, which is essentially a shortened version of Ref. 10, I use a regularization by two cutoffs (an infrared and an ultraviolet one) that preserves Poincaré invariance and cluster separability and that prevents any pathological operators from occurring during the construction. Clearly, pathologies of the unregularized system will manifest themselves after regularization as an essential dependence of physical properties on the cutoff.

Let us start with the nonrelativistic pendant of the final relativistic construction. The latter will do nothing but force relativistic kinematics on a system of finitely many (distinguishable) nonrelativistic particles that interact through pair potentials. We index the particles by the finite set J. It will be convenient to introduce some conventions and notions concerning subsets and systems of subsets of J. These sets will describe subsystems and systems of mutually noninteracting subsystems.

Convention (valid for the whole paper): Whenever the letters i,j; $\alpha,\beta,\gamma,\sigma$; ρ; $\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D}$; and $\mathscr{S}$ are used, the following declarations are implied:

(1) $i,j \in J$;
(2) $\varnothing \neq \alpha, \beta, \gamma, \sigma \subseteq J$;
(3) $\rho \subseteq J$, $1 \leq |\rho| \leq 2$ (where $|X|$ denotes the number of elements of the finite set X);
(4) $\mathscr{A}, \mathscr{B}, \mathscr{C}$, and $\mathscr{D}$ are partitions of subsets of J; and
(5) $\mathscr{S}$ is a partition of J.

Hence, $\mathscr{A}$ is a set the elements of which are nonvoid mutually disjoint subsets of J. These will be called clusters of $\mathscr{A}$. Clearly, $|\mathscr{A}|$ is the number of clusters of $\mathscr{A}$. The set of which $\mathscr{A}$ is a partition is denoted by

$$\cup \mathscr{A} \equiv \{ j \mid j \in \alpha \text{ for some } \alpha \in \mathscr{A} \} = \cup_{\alpha \in \mathscr{A}} \alpha \ . \tag{1.1}$$

Further we write $\mathscr{A} > \mathscr{B}$ is a refinement of $\mathscr{A}$, i.e.,

$$\mathscr{A} > \mathscr{B} \text{ iff } (\cup \mathscr{A} = \cup \mathscr{B}, \ \forall \beta \in \mathscr{B} \exists \alpha \in \mathscr{A} (\alpha \subseteq \beta), \text{ and } \mathscr{A} \neq \mathscr{B}) \ . \tag{1.2}$$

The finest refinement of $\mathscr{A}$ is denoted by ${}^*\mathscr{A}$; the clusters of ${}^*\mathscr{A}$ consist of single elements: $\cup {}^*\mathscr{A} = \cup \mathscr{A}$, $|{}^*\mathscr{A}| = |\cup \mathscr{A}|$. Putting

$$\mathscr{A} \wedge \mathscr{B} \equiv \{ \gamma \mid \gamma \neq \varnothing \text{ and } \exists \alpha \in \mathscr{A}, \ \beta \in \mathscr{B} (\gamma = \alpha \cap \beta) \} \tag{1.3}$$

we have for $\cup \mathscr{A} = \cup \mathscr{B}$ the following properties:

$$\mathscr{A} \geq \mathscr{A} \wedge \mathscr{B}, \quad \mathscr{B} \geq \mathscr{A} \wedge \mathscr{B}, \quad (\mathscr{A} \geq \mathscr{B} \text{ iff } \mathscr{A} \wedge \mathscr{B} = \mathscr{B}) \ . \tag{1.4}$$

A partition $\mathscr{B}$ of β induces the partition $\{\alpha\} \wedge \mathscr{B}$ of any subset α of β. We will abbreviate this partition by $\alpha \wedge \mathscr{B}$. Similarly, we will often suppress the brackets of set formation where no misunderstanding seems possible.

The state space H_α of any subsystem factorizes into the state spaces of the constituents

$$H_\alpha = \otimes_{j \in \alpha} H_j \ . \tag{1.5}$$

This induces for any $\mathscr{A}$ a tensor product

$$H_{\cup \mathscr{A}} = \otimes_{\alpha \in \mathscr{A}} H_\alpha \tag{1.6}$$

by the prescription that, for any family $(\psi_\alpha)_{\alpha \in \mathscr{A}}$ with ψ_α of the form $\otimes_{j \in \alpha} \psi_j$, we put $\otimes_{\alpha \in \mathscr{A}} \psi_\alpha \equiv \otimes_{j \in \cup \mathscr{A}} \psi_j$. Then, tensor multiplication is associative, i.e., for $\mathscr{C} = \mathscr{A} \wedge \mathscr{B}$ and for any family $(\psi_\gamma)_{\gamma \in \mathscr{C}}$, $\psi_\gamma \in H_\gamma$, we have

$$\otimes_{\gamma \in \mathscr{C}} \psi_\gamma = \otimes_{\alpha \in \mathscr{A}} (\otimes_{\gamma \in \alpha \wedge \mathscr{B}} \psi_\gamma) \ . \tag{1.7}$$

Any H_j carries a unitary projective representation $U_j = U_{\{j\}}$ of the Huygens group[11] ($\equiv$ inhomogeneous Galilei group) that describes all physical properties of the one-particle system $\{j\}$. The interaction is determined by a family $V = (V_\rho) = (V_\rho)_{\rho \subseteq J}$ of potentials, where $V_{\{i,j\}}$ describes the interaction between particle i and particle j. Excluding self-interaction, we put $V_{\{j\}} = 0$. The energy operator E_α of any subsystem α is given by

$$E_\alpha = \sum_{j \in \alpha} (m_j + \vec{\mathrm{P}}_j^2 / 2m_j) + \sum_{\rho \subseteq \alpha} V_\rho = \sum_{j \in \alpha} T_j + \sum_{\rho \subseteq \alpha} V_\rho$$
$$= E_\alpha(V) \ .$$

With any partition $\mathscr{S}$ we associate a fictitious system that is built up out of the same particles as J, but differs from J insofar as particles belonging to different clusters of $\mathscr{S}$ do not interact. What is the energy operator of that system (and its subsystems)? There are two conceptually different ways to answer this question:

(1) Since the clusters $\sigma \in \mathscr{S}$ are dynamically independent (i.e., states that describe independent subsystems at a particular time t, do so for all times) the energy is

$$E_\alpha^{\mathscr{S}} = \sum_{\beta \in \alpha \wedge \mathscr{S}} E_\beta \otimes (\text{identity on } H_{\alpha \setminus \beta}) = \sum_{\beta \in \alpha \wedge \mathscr{S}} E_\beta \ .$$

Here and in the sequel the convention is adopted that tensorial factors $\mathbb{1}$ ($\equiv$ identity operator) may be omitted if the space on which the operator acts is clear from the context.

(2) Instead of synthesizing the whole system out of the clusters, we may obtain the system by switching off the interaction between different clusters. The interaction in the original system is described by the family (V_ρ) (remember $|\rho| \leq 2$) of potentials. Switching off the interaction between different $\mathscr{S}$ clusters defines the new family $(V_\rho^{\mathscr{S}})$ of potentials, where

$$V_\rho^{\mathscr{S}} \equiv \begin{cases} V_\rho & \text{if } |\rho \wedge \mathscr{S}| = 1, \\ 0 & \text{if } |\rho \wedge \mathscr{S}| = 2. \end{cases} \tag{1.8}$$

The energy corresponding to these novel potentials is then given by

$$\begin{aligned} E_\alpha(V^{\mathscr{S}}) &= \sum_{j \in \alpha} T_j + \sum_{\rho \subseteq \alpha} V_\rho^{\mathscr{S}} \\ &= \sum_{\beta \in \alpha \wedge \mathscr{S}} \left[\sum_{j \in \beta} T_j + \sum_{\rho \subseteq \beta} V_\rho \right] \\ &= \sum_{\beta \in \alpha \wedge \mathscr{S}} E_\beta(V) \\ &= E_\alpha^{\mathscr{S}}(V) \ . \end{aligned} \tag{1.9}$$

Thus, the reasonings (1) and (2) yield the same result. This expresses the *algebraic cluster separability* of a Hamiltonian with pair potentials. With the Hamiltonian we associate a representation of time translations

$$t \rightarrow \exp(itE_\alpha) \ . \tag{1.10}$$

Let us assume that any potential V_ρ commutes with momentum, angular momentum, and center-of-mass position of the free ρ system described by $\otimes_{j \in \rho} U_j$. Then (1.10) is easily extended to a (projective) representation U_α of the Huygens group by taking over the remaining nine generators from the free α system $\otimes_{j \in \alpha} U_j$. The representations U_ρ determine the potentials V_ρ and vice versa (if the U_j's are given). Therefore, we have the following situation: For any α we have a unitary representation U_α of the Huygens group, where, for $|\alpha| \geq 3$, U_α is determined by the U_ρ with $\rho \subset \alpha$, i.e.,

$$U_\alpha = f_\alpha((U_\rho)_{\rho \subset \alpha}) \ , \tag{1.11}$$

where f_α is a function. The action of partitions $\mathscr{S}$ on the Hamiltonians E_α given in (1.9) induces an action of partitions on the representations U_α:

$$U_\alpha^{\mathscr{S}} \equiv \otimes_{\beta\in\alpha\wedge\mathscr{S}} U_\beta \; . \tag{1.12}$$

The algebraic cluster-decomposition property of the Hamiltonians implies that the following diagram is commutative:

$$\begin{array}{ccc} (U_\rho)_{\rho\subset\alpha} & \xrightarrow{\;\mathscr{S}\;} & (U_\rho^{\mathscr{S}})_{\rho\subset\alpha} \\ f_\alpha \Big\downarrow & & \Big\downarrow f_\alpha \, , \\ U_\alpha & \xrightarrow{\;\mathscr{S}\;} & U_\alpha^{\mathscr{S}} = f_\alpha((U_\rho^{\mathscr{S}})_{\rho\subset\alpha}) \end{array} \tag{1.13}$$

expressing the algebraic cluster separability of the complete theory. Written in terms of infinitesimal generators, Eq. (1.13) coincides with condition (C1) of Ref. 9.

From a formal point of view, the goal of this paper may now be formulated very simply. It is to establish (by construction) the same situation with the Huygens group replaced by the Poincaré group $\mathscr{P}$. What was almost trivial now becomes a problem. The reason for this is the fact that in the Poincaré group the time translations do not commute with the rest of the group as they do in the Huygens group. The obvious way out is to let the mass operator play the same role the energy operator played in the nonrelativistic case. This amounts to considering evolution in proper time rather than in coordinate time. Now a new difficulty arises from the fact that the mass operator, rather than being a generator of the group, is a nonlinear function of the generators. Thus, the property of representations to factorize tensorially implies a more complicated property of the mass operator than being a sum over contributions from the clusters as it is the case for the energy operators. These circumstances result in a considerably more complicated form of the representations U_α. Let us write them down explicitly ($|\alpha| \geq 3$):

$$U_\alpha \equiv Z_\alpha{}^{-1} \mathrm{BT}\left[\sum_{\alpha>\mathscr{B}}^{c} Z_{\mathscr{B}} M(\bigotimes_{\beta\in\mathscr{B}} U_B) Z_{\mathscr{B}}{}^{-1}, \bigotimes_{j\in\alpha} U_j \right] Z_\alpha \, , \tag{1.14}$$

$$Z_\alpha \equiv \prod_{\alpha>\mathscr{B}}^{c} Z_{\mathscr{B}} \, , \tag{1.15}$$

$$Z_{\mathscr{A}} \equiv \left[\prod_{\mathscr{A}>\mathscr{B}}^{c} Z_{\mathscr{B}} \bigotimes_{\alpha\in\mathscr{A}} Z_{\alpha\wedge\mathscr{B}}{}^{-1} F((Z_{\alpha\wedge\mathscr{B}} \bigotimes_{\gamma\in\alpha\wedge\mathscr{B}} U_\gamma Z_{\alpha\wedge\mathscr{B}}{}^{-1})_{\alpha\in\mathscr{A}})^{-1} \right] F((Z_\alpha U_\alpha Z_\alpha{}^{-1})_{\alpha\in\mathscr{A}}) \bigotimes_{\alpha\in\mathscr{A}} Z_\alpha \, , \tag{1.16}$$

$$Z_{\mathscr{A}} \equiv \mathbb{1} \quad \text{if } \mathscr{A} =^{*} \mathscr{A} \; . \tag{1.17}$$

$M(U)$ denotes the mass operator of the representation U, BT is a function the values of which are unitary representations of the Poincaré group, and F is a function the values of which are unitary operators. The main goal of Secs. II and III is to define the function BT and F, respectively. The cluster sum $\sum^c$ and the cluster product $\prod^c$ are defined in Appendix B. Equations (1.14)–(1.17) define U_α if we know U_β for any proper subset β of α. In this situation we define U_α by (1.14) after having eliminated the Z's with the help of (1.15)–(1.17) as follows: Equation (1.16) allows us to express any $Z_{\mathscr{A}}$ by the known U_β's (only β's that are subsets of $\mathscr{A}$ clusters appear) and such $Z_{\mathscr{B}}$'s for which $\cup\mathscr{B} \subseteq \cup\mathscr{A}$ and ($|\cup\mathscr{B}| < |\cup\mathscr{A}|$ or $|\mathscr{B}| > |\mathscr{A}|$). Applying (1.16) also to these $Z_{\mathscr{B}}$'s, and so on, we finally obtain only Z's that are equal to the identity by (1.17). This defines a function $g_{\mathscr{A}}$ such that

$$Z_{\mathscr{A}} = g_{\mathscr{A}}((U_\beta)_{\beta\subset\alpha,\, \beta=\mathscr{A}\wedge\beta}) \; . \tag{1.18}$$

Together with (1.15), Eqs. (1.18) and (1.14) define a function h_α such that

$$U_\alpha = h_\alpha((U_\beta)_{\beta\subset\alpha}) \; .$$

Iterating h_α, we finally express U_α in terms of $(U_\rho)_{\rho\subset\alpha}$ thus defining the function f_α:

$$U_\alpha = f_\alpha((U_\rho)_{\rho\subset\alpha}) \; . \tag{1.19}$$

A more economical recursion scheme (based, however, on the same formulas) will be given in Sec. IV. There, it will be shown that algebraic cluster separability holds; i.e., that the diagram (1.13) is commutative with the action of partitions being defined by (1.12). Further, it will be shown that

$$U_\alpha(g) = \otimes_{j\in\alpha} U_j(g) \quad \text{for all } g\in\mathscr{E} \, , \tag{1.20}$$

where $\mathscr{E}$ is the Euclidean subgroup of the Poincaré group. Hence, interaction only affects the representation of time translations and Lorentz transformations.

The notions introduced so far formalize the idea that the dynamical behavior of the whole system should be determined by its constituents and the relations between them (it is only for simplicity that we only consider relations within pairs). The particles $j\in J$, or even the subsystems $\alpha\subset J$, are constituents of J in the sense that there is a hypothetical process [viz., going from (U_ρ) to $(U_\rho^{\mathscr{S}})$] that gives them the status of independent subentities. The question now arises whether these hypothetical subentities may exist as closed subsystems in the unchanged theory. This is to be expected if the interaction between any two particles decreases sufficiently fast as a function of the relative distance, i.e.,

$$[U_i(\vec{a}) \otimes U_j(\vec{b})][U_{\{i,j\}}(g) - U_i(g) \otimes U_j(g)] \times [U_i(\vec{a}) \otimes U_j(\vec{b})]^{-1} \to 0 \, , \tag{1.21}$$

sufficiently fast for $|\vec{a}-\vec{b}| \to \infty$, where $\vec{a}$ and $\vec{b}$ denote spatial translations and $g\in\mathscr{P}$ is arbitrary. It should be noted that the interpretation of the operator $U_i(\vec{a}) \otimes U_j(\vec{b})$ as a relative translation of the subsystems i and j [that justifies (1.21)] would hardly be consistent if (1.20) would not hold for translations. Let us now consider (Schrödinger) states ψ_1 and ψ_2 of the system α that can be prepared in a laboratory of realistic dimensions. For a partition $\mathscr{B}$ of α we choose a family $(\vec{a}_\beta)_{\beta\in\mathscr{B}}$ of translation vectors such that all vectors $\vec{a}_\beta-\vec{a}_\gamma$, $\beta\neq\gamma$, are very large compared to the dimensions of our laboratory. Then we form the states

$$\varphi_i=\otimes_{\beta\in\mathscr{B}}U_\beta(\vec{a}_\beta)\psi_i,\ \ i\in\{1,2\}\ , \tag{1.22}$$

which should be expected to describe situations for which all information can be obtained by using measuring devices that are placed in the volume obtained by applying all translation vectors $\vec{a}_\beta$ to the volume of our laboratory. These measuring devices are thus placed in widely separated laboratories L_β, $\beta\in\mathscr{B}$. One should expect

$$\langle\varphi_1\,|\,U_\alpha(g)\varphi_2\rangle\approx\langle\varphi_1\,|\,\otimes_{\beta\in\mathscr{B}}U_\beta(g)\varphi_2\rangle \quad \text{for all } g\in\mathscr{P} \tag{1.23}$$

since, otherwise, even if the measuring results obtained in different L_β's are independent [which is not the normal case for states constructed according to (1.22)] they would lose their independence in the course of time (or even after a kinematical transformation). Exact equality, however, can only be expected in the limit of infinite separation because for finite distances there are always configurations that will come close together at some time or other. Let us see how (1.23) arises:

$$\begin{aligned}\langle\varphi_1\,|\,U_\alpha(g)\varphi_2\rangle&=\langle\psi_1\,|\,\otimes_{\beta\in\mathscr{B}}U_\beta(-\vec{a}_\beta)[f_\alpha((U_\rho)_{\rho\subset\alpha})](g)\\&\qquad\times\otimes_{\beta\in\mathscr{B}}U_\beta(\vec{a}_\beta)\psi_2\rangle\\&=\langle\psi_1\,|\,[f_\alpha((U_\rho^a)_{\rho\subset\alpha})](g)\psi_2\rangle\\&=\cdots\ .\end{aligned}$$

There, we have put

$$U_\rho^a(g)\equiv\otimes_{j\in\rho}U_j(-\vec{a}_{\beta(j)})U_\rho(g)\otimes_{j\in\rho}U_j(\vec{a}_{\beta(j)})\ , \tag{1.24}$$

where $\beta(j)$ is the $\mathscr{B}$ cluster that contains j. Obviously, U_ρ^a is a representation of $\mathscr{P}$. In the last step of the above calculation we have used the fact that f_α commutes with unitary transformations. Precisely, for any family $(V_j)_{j\in\alpha}$ of unitary transformations $V_j\colon H_j\to H_j$ we have

$$\begin{aligned}(\otimes_{j\in\alpha}V_j)f_\alpha((U_\rho)_{\beta\subset\alpha})(\otimes_{j\in\alpha}V_j^{-1})\\=f_\alpha(((\otimes_{j\in\rho}V_j)U_\rho(\otimes_{j\in\rho}V_j^{-1}))_{\rho\subset\alpha})\ .\end{aligned} \tag{1.25}$$

Equation (1.21) implies $U_\rho^a\approx(U_\rho^{\mathscr{B}})^a=(U_\rho^a)^{\mathscr{B}}$. Since $f_\alpha((U_\rho)_{\rho\subset\alpha})$ is constructed out of its arguments of explicit formulas it is reasonable to expect that it depends, in some sense, continuously on its arguments such that we may continue

$$\begin{aligned}\cdots&\approx\langle\psi_1\,|\,[f_\alpha((U_\rho^{\mathscr{B}})^a_{\rho\subset\alpha})](g)\psi_2\rangle\\&=\langle\varphi_1\,|\,[f_\alpha((U_\rho^{\mathscr{B}})_{\rho\subset\alpha})](g)\varphi_2\rangle\\&=\langle\varphi_1\,|\,U_\alpha^{\mathscr{B}}(g)\varphi_2\rangle\\&=\langle\varphi_1\,|\,\otimes_{\beta\in\mathscr{B}}U_\beta(g)\varphi_2\rangle\ .\end{aligned} \tag{1.26}$$

This shows on a nonrigorous level that algebraic cluster separability implies cluster separability in the usual sense. It should be noted that algebraic cluster separability remains a reasonable concept also in cases where confining interactions make spatial separation impossible. In the sequel cluster separability will always mean algebraic cluster separability.

II. THE BAKAMJIAN-THOMAS-FOLDY CONSTRUCTION

Let U be a representation ($\equiv$ continuous unitary representation) of the Poincaré group (cf. Appendix A) acting on a Hilbert space ($\equiv$ complex separable Hilbert space) H. Then the self-adjoint operators P^0 (energy), $\vec{P}$ (momentum), $\vec{J}$ (angular momentum), and $\vec{N}$ ("boost") are determined by the equations

$$\begin{aligned}&U(a,1)=\exp(ia^0P^0-i\vec{a}\cdot\vec{P})\ ,\\&U(0,\exp(i\vec{n}\cdot\vec{\sigma}/2))=\exp(i\vec{n}\cdot\vec{J})\ ,\\&U(0,\exp(\vec{n}\cdot\vec{\sigma}/2))=\exp(i\vec{n}\cdot\vec{N})\ .\end{aligned} \tag{2.1}$$

We assume U to be positive in the sense that the self-adjoint operators P^0 and $M^2=P^\mu P_\mu$ are strictly positive (the addition "strictly" excludes zero from being an eigenvalue; no spectral gap at zero is required). Any (finite) tensor product of positive representations is easily shown to be positive. If U is irreducible it is unitary equivalent to $U_{m,s}$ with $m>0$ and $s\in\{0,\frac{1}{2},1,\dots\}$ given by

$$\begin{aligned}&U_{m,s}(a,A)\colon L^2(R^3,C^{2s+1})\to L^2(R^3,C^{2s+1})\ ,\\&(U_{m,s}(a,A)\psi)(\vec{p})=e^{ia\cdot p}[(A^{-1}p)^0/p^0]^{1/2}\\&\qquad\qquad\times D^{(s)}(R(p,A))\psi(\overrightarrow{A^{-1}p})\end{aligned} \tag{2.2}$$

where p is the four-vector $(p^0,\vec{p})=((m^2+\vec{p}^{\,2})^{1/2},\vec{p})$ and $D^{(s)}$ is the usual unitary irreducible representation of SU(2) on C^{2s+1}; the Wigner rotation $R(p,A)$ and the action of SL(2,C) on four-vectors are explained in Appendix A.

For this representation the Newton-Wigner position operator $\vec{X}$ is simply given by $\vec{X}=i\partial/\partial\vec{p}$. It is easily seen to be related to the Poincaré generators by Eq. (2.4). Since U can be decomposed into a direct integral of irreducible representations, one easily extends the definition of $\vec{X}$ in such a way that U determines a triplet $\vec{X}$ of commuting self-adjoint operators satisfying Heisenberg commutation relations with $\vec{P}$,

$$[\vec{a}\cdot\vec{X},\vec{b}\cdot\vec{P}]=i\vec{a}\cdot\vec{b}\ , \tag{2.3}$$

and satisfying Bakamjian-Thomas-Foldy[3,4] equations on a common core of $\vec{X}$:

$$\vec{N}=\tfrac{1}{2}(P^0\vec{X}+\vec{X}P^0)+\vec{P}\times\vec{S}(M+P^0)^{-1}\ ,$$

where $\vec{S}\equiv\vec{J}-\vec{X}\times\vec{P}$;

$$\vec{X}=\vec{Q}-\vec{P}\times(\vec{J}-\vec{Q}\times\vec{P})M^{-1}(M+P^0)^{-1}\ , \tag{2.4}$$

where $\vec{Q}\equiv\frac{1}{2}(P_0{}^{-1}\vec{N}+\vec{N}P_0{}^{-1})$. More information on this point is given in Refs. 12 and 13.

Since (2.3) implies, up to equivalence, the form $\vec{X}=\mathbb{1}\otimes i\partial/\partial\vec{p}$, $\vec{P}=\mathbb{1}\otimes\vec{p}$, we can factorize H into a tensor product of an "internal space" being unaffected by the operators $\vec{X}$ and $\vec{P}$ and a "center-of-mass space," where these operators act effectively. More precisely, we find a tensor product $H=H^{\text{in}}\otimes L^2(R^3)$ such that $\vec{X}$ and $\vec{P}$ act only on the second factor: $\vec{X}=\mathbb{1}\otimes i\partial/\partial\vec{p}$, $\vec{P}=\mathbb{1}\otimes\vec{p}$, whereas all internal observables (i.e., those that commute with both $\vec{X}$ and $\vec{P}$) act only on the first factor. Let us call such a tensor product a *barycentric factoring* of H (with respect to U). Since the mass M and the spin $\vec{S}$ are internal observables, they are of the form $M=M^{\text{in}}\otimes\mathbb{1}$, $\vec{S}=\vec{S}^{\text{in}}\otimes\mathbb{1}$ and $\vec{S}^{\text{in}}$ generates a representation U^{in} of SU(2) on the internal space H^{in} by

$$U^{\text{in}}(\exp(i\vec{n}\cdot\vec{\sigma}/2))=\exp(i\vec{n}\cdot\vec{S}^{\text{in}}) . \tag{2.5}$$

This representation allows us to write a factorized form of $U(g)$ for all g belonging to the Euclidean subgroup $\mathscr{E}\equiv\{(a,A)\in\mathscr{P}\mid a^0=0, A\in\text{SU}(2)\}$ of $\mathscr{P}$:

$$U(a,A)=U^{\text{in}}(A)\otimes U^{\text{c.m.}}(\vec{a},A) , \tag{2.6}$$

where $U^{\text{c.m.}}(\vec{a},A)$ is the usual action of a Euclidean transformation on one-particle momentum-space wave functions

$$(U^{\text{c.m.}}(\vec{a},A)\psi)(\vec{p})\equiv e^{-i\vec{a}\cdot\vec{p}}\psi(A^{-1}\vec{p}) .$$

There exist infinitely many barycentric factorings for U but these are all related in a canonical manner. Let $H^{\text{in}}\otimes L^2(R^3)=H=\hat{H}^{\text{in}}\hat{\otimes}L^2(R^3)$ be two barycentric factorings; then there is a unitary mapping V: $H^{\text{in}}\to\hat{H}^{\text{in}}$ that is uniquely determined by the requirement that

$$(V\psi)\hat{\otimes}\varphi=\psi\otimes\varphi \quad \text{for all } \psi\in H^{\text{in}},\ \varphi\in L^2(R^3) . \tag{2.7}$$

Clearly, V interrelates the "internal data" of U: $\hat{U}^{\text{in}}=VU^{\text{in}}V^{-1}$, $\hat{M}^{\text{in}}=VM^{\text{in}}V^{-1}$. The property of a state to factorize is independent of the barycentric factoring chosen. The physical meaning of factorizing is that measuring internal observables yields results that are statistically independent of values obtained by measuring center-of-mass observables (i.e., observables that commute with all internal observables and, hence, belong to the von Neumann algebra generated by $\vec{P}$ and $\vec{X}$).

Let us now consider a strictly positive operator $\tilde{M}$ that commutes with $\vec{X}$ and with $\{U(g)\mid g\in\mathscr{E}\}$ or, equivalently, with $\vec{X}$, $\vec{P}$, and $\vec{J}$. There exists a unique positive representation $\tilde{U}$ the mass operator of which is $\tilde{M}$ and that satisfies

$$\vec{X}=\tilde{\vec{X}} \text{ and } U(g)=\tilde{U}(g) \quad \text{for all } g\in\mathscr{E} . \tag{2.8}$$

This is very plausible. The position operator, momentum, and angular momentum are known from (2.8); energy and "boosts" may then be defined in accordance with (2.4) as

$$\tilde{P}^0\equiv(\tilde{M}^2+\vec{P}^2)^{1/2} ,$$
$$\tilde{\vec{N}}\equiv\tfrac{1}{2}(\tilde{P}^0\vec{X}+\vec{X}\tilde{P}^0)+\vec{P}\times(\vec{J}-\vec{X}\times\vec{P})(\tilde{M}+\tilde{P}^0)^{-1} . \tag{2.9}$$

Since these equations involve unbounded operators, they are not easily seen to define self-adjoint operators. If mathematical rigor is required, it is easier to construct the finite transformation $\tilde{U}(a,A)$ directly instead of the infinitesimal ones. One then verifies (2.9) afterwards. This is indicated in Ref. 12 and carried out in detail in Ref. 10. Anyway, $\tilde{U}$ is a well-defined positive representation determined by $\tilde{M}$ and U. Let us denote it by

$$\tilde{U}=\text{BT}(\tilde{M},U) , \tag{2.10}$$

where BT stands for Bakamjian and Thomas,[3] who invented this construction in the framework of classical mechanics, where the Poincaré group is represented by canonical transformations. If $\tilde{M}$ varies over all operators allowed by the conditions stated above, the representation $\text{BT}(\tilde{M},U)$ varies over all representations satisfying (2.8). Let us call two positive representations U and $\tilde{U}$ of $\mathscr{P}$ *concentric,* in symbols $U\sim\tilde{U}$, if (2.8) holds. Concentricity is obviously an equivalence relation. The main fact on concentric representations is that a barycentric factoring with respect to one representation is also a barycentric factoring with respect to any other concentric representation. Concentric representations transformed by the same unitary transformation remain concentric.

III. A KINEMATICAL TRANSFORMATION

Let us consider a tensor product $H=\otimes_{j\in J}H_j$ of Hilbert spaces and let any H_j carry two concentric representations U_j and $\tilde{U}_j$ of $\mathscr{P}$: $U_j\sim\tilde{U}_j$. Since the relation $\sim$ involves the position operator, which is a nonlinear function of the Poincaré generators [see (2.4)], we cannot expect the tensor products $\otimes_{j\in J}U_j$ and $\otimes_{j\in J}\tilde{U}_j$ to be concentric representations. The aim of this section is to construct a unitary transformation F that achieves

$$F\otimes_{j\in J}\tilde{U}_jF^{-1}\sim\otimes_{j\in J}U_j . \tag{3.1}$$

This transformation is a cornerstone in the construction to be described in the next section.

It may be useful first to give an outline of the procedure. For every $j\in J$, we choose a barycentric factoring

$$H_j=H_j^{\text{in}}\otimes L^2(R^3) \tag{3.2}$$

with respect to U_j and, because of $U_j\sim\tilde{U}_j$, also with respect to $\tilde{U}_j$. Then, the internal mass operators M_j^{in} and $\tilde{M}_j^{\text{in}}$, and the internal SU(2) representations $U_j^{\text{in}}=\tilde{U}_j^{\text{in}}$ acting on H_j^{in} are determined as discussed in the last section. Further, we choose a tensor product of the internal spaces

$$H_*=\otimes_{j\in J}H_j^{\text{in}} . \tag{3.3}$$

Then, we construct unitary mappings W and $\tilde{W}$ from H into a common Hilbert space of H_*-valued wave functions such that

$$W\otimes_{j\in J}U_jW^{-1}\sim\tilde{W}\otimes_{j\in J}\tilde{U}_j\tilde{W}^{-1} . \tag{3.4}$$

$F=W^{-1}\tilde{W}$ is then the desired solution of (3.1). Whereas W and $\tilde{W}$ depend also on the choices (3.2) and (3.3), F depends only on the objects $\otimes_{j\in J}H_j$, $(U_j)_{j\in J}$, and $(\tilde{U}_j)_{j\in J}$. For all U_j being irreducible and of the form (2.2), the transformation W is given in Ref. 13. The generalization to reducible representations is straightforward when these are decomposed into irreducibles. However, it is not trivial to show that W is independent of the particular decomposition chosen. This independence is proved rigorously in Ref. 10 using the formalism of direct integrals. In this paper the matter will be treated on a purely calculational level using Dirac notation.

Let us now describe the construction of the W's. It suffices to consider $\tilde{W}$, since dropping the tilde everywhere will yield the construction of W; objects introduced without a tilde are *a priori* common to both cases. We choose a system $\tilde{\Omega}_j$ of internal observables that, together with $\vec{P}_j$, forms a complete system of commuting observables in H_j. Further we require that $\tilde{M}_j$ is a function, say $\tilde{\mu}_j$, of $\tilde{\Omega}_j$ (a simple but unnecessarily restrictive way to assure this is to choose $\tilde{M}_j$ as a member of $\tilde{\Omega}_j$). Any operator $A\in\tilde{\Omega}_j$ is of the form $A^{\text{in}}\otimes\mathbb{1}$ with respect to the tensor product $H_j=H_j^{\text{in}}\otimes L^2(R^3)$ and the operators A^{in} form a complete set of commuting observables $\tilde{\Omega}_j^{\text{in}}$ in H_j^{in}. Let $\tilde{\Sigma}_j$ be the common spectrum of $\tilde{\Omega}_j^{\text{in}}$ and let $|\tilde{\omega}_j\rangle$, $\tilde{\omega}_j\in\tilde{\Sigma}_j$, be a basis of generalized eigenvectors:

$$f(\tilde{\Omega}_j^{\text{in}})\,|\tilde{\omega}_j\rangle=f(\tilde{\omega}_j)\,|\tilde{\omega}_j\rangle$$

for all $\tilde{\omega}_j\in\tilde{\Sigma}_j$ and all Borel functions $f\colon\tilde{\Sigma}_j\to C$. In particular,

$$\tilde{M}_j^{\text{in}}\,|\tilde{\omega}_j\rangle=\tilde{\mu}_j(\tilde{\omega}_j)\,|\tilde{\omega}_j\rangle\ . \qquad (3.5)$$

Orthogonality and completeness of the $|\tilde{\omega}_j\rangle$ are expressed by

$$\int|\tilde{\omega}_j\rangle d\tilde{\omega}_j\langle\tilde{\omega}_j|=\mathbb{1},\quad \langle\tilde{\omega}_j\,|\,\tilde{\omega}_j'\rangle=\delta(\tilde{\omega}_j,\tilde{\omega}_j')\ , \qquad (3.6)$$

where $d\tilde{\omega}_j$ denotes the volume element of a measure on the spectrum $\tilde{\Sigma}_j$ and the δ function refers to integration with respect to $d\tilde{\omega}_j$:

$$f(\tilde{\omega}_j)=\int d\tilde{\omega}_j'\delta(\tilde{\omega}_j,\tilde{\omega}_j')f(\tilde{\omega}_j')\ . \qquad (3.7)$$

Since U_j^{in} commutes with $\tilde{M}_j^{\text{in}}$, we have

$$\begin{aligned}0&=\langle\tilde{\omega}_j\,|\,[M_j^{\text{in}},U_j^{\text{in}}(R)]\,|\,\tilde{\omega}_j'\rangle\\&=[\tilde{\mu}_j(\tilde{\omega}_j)-\tilde{\mu}_j(\tilde{\omega}_j')]\langle\tilde{\omega}_j\,|\,U_j^{\text{in}}(R)\,|\,\tilde{\omega}_j'\rangle\ .\end{aligned} \qquad (3.8)$$

A common basis of generalized eigenvectors of $\tilde{\Omega}_j$ and $\vec{P}_j$ is now given by

$$|\tilde{\omega}_j,\vec{p}_j\rangle=|\tilde{\omega}_j\rangle\otimes|\vec{p}_j\rangle\ . \qquad (3.9)$$

The representation $\tilde{U}_j$ can be written in the form

$$\langle\tilde{\omega}_j,\vec{p}_j\,|\,U_j(a,A)\psi\rangle=e^{ia\cdot\tilde{p}_j}[(A^{-1}\tilde{p}_j)^0/\tilde{p}_j^0]^{1/2}\int d\tilde{\omega}_j'\langle\tilde{\omega}_j\,|\,U_j^{\text{in}}(R(\tilde{p}_j,A))\,|\,\tilde{\omega}_j'\rangle\langle\tilde{\omega}_j',\overrightarrow{A^{-1}p_j}\,|\,\psi\rangle\ , \qquad (3.10)$$

where $\tilde{p}_j=([\tilde{\mu}_j(\tilde{\omega}_j)^2+\vec{p}_j^{\,2}]^{1/2},\vec{p}_j)$. We are now in the position to write the defining formula for $\tilde{W}$:

$$\tilde{W}\colon\bigotimes_{j\in J}H_j\to L^2\left[K^{(J)}\times R^3,\delta\left[\textstyle\sum_j\vec{k}_j\right]\prod_j d\vec{k}_j d\vec{p},H_*\right],$$

$$(\tilde{W}\psi)((\vec{k}_j)_{j\in J},\vec{p})=\int(\tilde{m}/\tilde{p}^0)^{1/2}\prod_j d\tilde{\omega}_j(\tilde{p}_j^0/\tilde{k}_j^0)^{1/2}\otimes_j U_j^{\text{in}}(R(\tilde{p}_j,A(\tilde{p}))^{-1})\,|\,\tilde{\omega}_j\rangle\langle(\tilde{\omega}_j)_{j\in J},(\vec{\tilde{p}}_j)_{j\in J}\,|\,\psi\rangle\ , \qquad (3.11)$$

where

$$K^{(J)}\equiv\{(\vec{k}_j)_{j\in J}\in R^{3J}\,|\,\textstyle\sum_j\vec{k}_j=\vec{0}\} \qquad (3.12)$$

and where the quantities $\tilde{m}$, $\tilde{p}$, $\tilde{p}_j,\tilde{k}_j^0$ on the right-hand side (RHS) are given by $(\vec{k}_j)$ and $\vec{p}$ on the left-hand side (LHS) and the integration variables $\tilde{\omega}_j$ through the following chain of formulas:

$$\begin{aligned}\tilde{k}_j^0&=[\tilde{\mu}_j(\tilde{\omega}_j)^2+\vec{k}_j^{\,2}]^{1/2}\ ,\\ \tilde{m}&=\textstyle\sum_j\tilde{k}_j^0\ ,\\ \tilde{p}&=((\tilde{m}^2+\vec{p}^{\,2})^{1/2},\vec{p})\ ,\\ \tilde{p}_j&=A(\tilde{p})\tilde{k}_j=A(\tilde{p})(\tilde{k}_j^0,\vec{k}_j)\ .\end{aligned} \qquad (3.13)$$

These relations express a simple kinematical situation. Interpret the $\tilde{p}_j$ as four-momenta of free classical particles, the mass of particle j being $\tilde{\mu}_j(\tilde{\omega}_j)$; then the $\tilde{k}_j$ are the momenta of the particles relative to the rest frame of the particle system. In fact, we have, due to $\sum_j\vec{k}_j=\vec{0}$,

$$\textstyle\sum_j\tilde{p}_j=A(\tilde{p})(\sum_j\tilde{k}_j^0,\sum_j\vec{k}_j)=A(\tilde{p})(\tilde{m},\vec{0})=\tilde{p}\ ,$$

so that $\tilde{p}$ is the total momentum of the system and $\tilde{m}$ is its mass. Hence, $A(\tilde{p})^{-1}$ is the boost that transforms to the rest frame. In the course of showing that $\tilde{W}$ has the desired properties, the reasons for the particular form of $\tilde{W}$ will become clear. For a connection to the theory of induced representations see Ref. 13. The numerical factor $(\tilde{m}/\tilde{p}^0)\prod_j(\tilde{p}_j^0/\tilde{k}_j^0)$ is chosen such that $\tilde{W}$ is isometric. That $\tilde{W}$ is bijective is easily seen from the fact that the transformation

$$K^{(J)}\times R^3\to R^{3J},\quad((\vec{k}_j),\vec{p})\to(\vec{\tilde{p}}_j)$$

[for fixed $\tilde{\mu}_j(\tilde{\omega}_j)$] is bijective. The inverse transformation is given by the following chain of formulas:

$$\tilde{p}_j^0=[\tilde{\mu}_j(\tilde{\omega}_j)^2+\vec{p}_j^{\,2}]^{1/2},\quad\tilde{p}=\textstyle\sum_j\tilde{p}_j\ ,$$

$$\tilde{k}_j=A(\tilde{p})^{-1}\tilde{p}_j,\quad\vec{p}=\vec{\tilde{p}},\quad\vec{k}_j=\vec{\tilde{k}}_j\ .$$

Thus, $\tilde{W}$ is unitary. Let us now show that the definition (3.11) does not depend on the basis chosen. For notational convenience, we consider the definition of W instead of $\tilde{W}$. We choose a new basis $|\eta_j\rangle$ (associated with a new complete set of commuting observables). Then we have

$$M_j^{\rm in}|\omega_j\rangle=\mu_j(\omega_j)|\omega_j\rangle\ ,$$
$$M_j^{\rm in}|\eta_j\rangle=\nu_j(\eta_j)|\eta_j\rangle\ , \tag{3.14}$$

and

$$|\omega_j\rangle=\int d\eta_j\,|\eta_j\rangle\langle\eta_j|\omega_j\rangle\ . \tag{3.15}$$

From

$$\langle\eta_j|M_j^{\rm in}|\omega_j\rangle=\mu_j(\omega_j)\langle\eta_j|\omega_j\rangle$$
$$=\nu_j(\eta_j)\langle\eta_j|\omega_j\rangle$$

we conclude

$$[\nu_j(\eta_j)-\mu_j(\omega_j)]\langle\eta_j|\omega_j\rangle=0\ . \tag{3.16}$$

Introducing (3.15) in (3.11) we obtain

$$(W\psi)((\vec{k}_j),\vec{p})=\int (m/p^0)^{1/2}\prod_j d\omega_j d\eta_j d\eta'_j(p_j^0/k_j^0)^{1/2}\langle\eta_j|\omega_j\rangle\langle\omega_j|\eta'_j\rangle\langle(\eta'_j),(\vec{p}_j)|\psi\rangle$$
$$\times\otimes_j U_j^{\rm in}(R(p_j,A(p))^{-1})|\eta_j\rangle\ .$$

The factors $\langle\eta_j|\omega_j\rangle$, together with (3.16), imply that the integrand vanishes at all points at which $\mu_j(\omega_j)$ differs from $\nu_j(\eta_j)$. Thus, we may replace $\mu_j(\omega_j)$ by $\nu_j(\eta_j)$. Therefore, the quantities m, p^0, p_j, and k_j, that depend via (3.13) on $(\vec{k}_j)$, $\vec{p}$, and $(\mu_j(\omega_j))$ may be considered to depend on (η_j) rather than on (ω_j). Looking at the expressions this way, ω_j occurs only in $\int d\omega_j\,|\omega_j\rangle\langle\omega_j|$ and we can perform these integrations using the first equation of (3.6) so that we have

$$(W\psi)((\vec{k}_j),\vec{p})=\int (m/p^0)^{1/2}\prod_j d\eta_j d\eta'_j(p_j^0/k_j^0)^{1/2}\langle\eta_j|\eta'_j\rangle\langle(\eta'_j),(\vec{p}_j)|\psi\rangle$$
$$\times\otimes_j U_j^{\rm in}(R(p_j,A(p))^{-1})|\eta_j\rangle\ .$$

Performing the η'_j integration we obtain

$$\int (m/p^0)^{1/2}\prod_j d\eta_j(p_j^0/k_j^0)^{1/2}\otimes_j U_j^{\rm in}(R(p_j,A(p))^{-1})|\eta_j\rangle\langle(\eta_j),(\vec{p}_j)|\psi\rangle\ ,$$

as it should be.

Let us now consider the transformed representation

$$\tilde{U}'\equiv\tilde{W}\otimes_j\tilde{U}_j\tilde{W}^{-1}\ . \tag{3.17}$$

We will see that

$$(\tilde{U}'(a,A)\psi)((\vec{k}_j),\vec{p})=\int (\prod_j d\tilde{\omega}_j)e^{ia\cdot\tilde{p}}[(A^{-1}\tilde{p})^0/\tilde{p}^0]^{1/2}\otimes_j U_j^{\rm in}(R(\tilde{p},A))|\tilde{\omega}_j\rangle\langle(\tilde{\omega}_j)|\psi((R(\tilde{p},A)^{-1}\vec{k}_j),\overrightarrow{A^{-1}p})\rangle\ . \tag{3.18}$$

We have

$$(\tilde{W}\otimes_j\tilde{U}_j(a,A)\psi)((\vec{k}_j),\vec{p})=\int (\tilde{m}/\tilde{p}^0)^{1/2}\prod_j d\tilde{\omega}_j(\tilde{p}_j^0/\tilde{k}_j^0)^{1/2}\otimes_j U_j^{\rm in}(R(\tilde{p}_j,A(\tilde{p}))^{-1})|\tilde{\omega}_j\rangle e^{ia\cdot\Sigma_j\tilde{p}_j}\prod_j[(A^{-1}\tilde{p}_j)^0/\tilde{p}_j^0]^{1/2}$$
$$\times\int d\tilde{\omega}'_j\langle\tilde{\omega}_j|U_j^{\rm in}(R(\tilde{p}_j,A))|\tilde{\omega}'_j\rangle\langle(\tilde{\omega}'_j),\overrightarrow{A^{-1}\tilde{p}_j})|\psi\rangle=\cdots\ .$$

Owing to the factors $\langle\tilde{\omega}_j|U_j^{\rm in}(R(p_j,A))|\tilde{\omega}'_j\rangle$, Eq. (3.8) allows us to replace $\tilde{\mu}_j(\tilde{\omega}_j)$ by $\tilde{\mu}_j(\tilde{\omega}'_j)$. Hence, $\tilde{\omega}_j$ effectively appears only in $\int d\tilde{\omega}_j\,|\tilde{\omega}_j\rangle\langle\tilde{\omega}_j|$ and these integrations can be carried out to yield (after changing the name of the remaining integration variables from $\tilde{\omega}'_j$ to $\tilde{\omega}_j$)

$$\cdots=\int (\tilde{m}/\tilde{p}^0)^{1/2}e^{ia\cdot\tilde{p}}\prod_j d\tilde{\omega}_j[(A^{-1}\tilde{p}_j)^0/\tilde{k}_j^0]^{1/2}$$
$$\times\otimes_j U_j^{\rm in}[R(\tilde{p}_j,A(\tilde{p}))^{-1}R(\tilde{p}_j,A)]|\tilde{\omega}_j\rangle\langle(\tilde{\omega}_j),(\overrightarrow{A^{-1}\tilde{p}_j})|\psi\rangle. \tag{3.19}$$

On the other hand we have

$$(\tilde{U}'(a,A)\tilde{W}\psi)((\vec{k}_j),\vec{p})=\int \prod_j d\tilde{\omega}_j e^{i\tilde{p}\cdot a}[(A^{-1}\tilde{p})^0/\tilde{p}^0]^{1/2}\otimes_j U_j^{\rm in}(R(\tilde{p},A))|\tilde{\omega}_j\rangle\langle(\tilde{\omega}_j)|(W\psi)((R(\tilde{p},A)^{-1}\vec{k}_j),\overrightarrow{A^{-1}\tilde{p}})\rangle$$
$$=\int (\prod_j d\tilde{\omega}_j d\tilde{\omega}'_j)e^{i\tilde{p}\cdot a}[(A^{-1}\tilde{p})^0/\tilde{p}^0]^{1/2}\otimes_j U_j^{\rm in}(R(\tilde{p},A))|\tilde{\omega}_j\rangle$$
$$\times(\hat{m}/\hat{p}^0)^{1/2}\prod_j(\hat{p}_j^0/\hat{k}_j^0)^{1/2}\langle\tilde{\omega}_j|U_j^{\rm in}(R(\hat{p}_j,A(\hat{p}))^{-1})|\tilde{\omega}'_j\rangle\langle(\tilde{\omega}'_j),(\vec{\hat{p}}_j)|\psi\rangle$$
$$=\cdots\ ,$$

where the quantities with carets are formed from $(R(\tilde{p},A)^{-1}\vec{k}_j)$, $\overrightarrow{A^{-1}\tilde{p}}$, and $(\tilde{\omega}'_j)$ by the same formulas as the quantities with tildes from $(\vec{k}_j)$, $\vec{p}$, and $(\tilde{\omega}_j)$ [see (3.13)]. The comment interrupting the last calculation applies verbatim and yields

$$\cdots=\int(\prod_j d\tilde{\omega}_j)e^{i\tilde{p}\cdot a}[(A^{-1}\tilde{p})^0/\tilde{p}^0]^{1/2}(\hat{m}/\hat{p}^0)^{1/2}\prod_j(\hat{p}_j^0/\hat{k}_j^0)^{1/2}\otimes_j U_j^{\rm in}(R(\tilde{p},A)R(\hat{p}_j,A(\hat{p}))^{-1})|\tilde{\omega}_j\rangle\langle(\tilde{\omega}_j),(\vec{\hat{p}}_j)|\psi\rangle=\cdots\ .$$

Since $[R(p,A)^{-1}\vec{k}_j]^2=\vec{k}_j^{\,2}$, we have $\hat{k}_j^0=\tilde{k}_j^0$, hence $\hat{m}=\tilde{m}$ and $\hat{p}=A^{-1}\tilde{p}$. Further we have

$$\widetilde{p}_j=A(\widetilde{p})\widetilde{k}_j=A(A^{-1}\widetilde{p})R(\widetilde{p},A)^{-1}\widetilde{k}_j=A(A^{-1}\widetilde{p})[A(\widetilde{p})^{-1}AA(A^{-1}\widetilde{p})]^{-1}=A^{-1}A(\widetilde{p})\widetilde{k}_j=A^{-1}\widetilde{p}_j$$

and

$$\cdots=\int(\prod_j d\widetilde{\omega}_j)e^{i\widetilde{p}\cdot a}(\widetilde{m}/\widetilde{p}^0)^{1/2}\Pi_j[(A^{-1}\widetilde{p}_j)^0/\widetilde{k}_j^0]^{1/2}$$
$$\times\otimes_j U_j^{\text{in}}(R(\widetilde{p},A)R(A^{-1}\widetilde{p}_j,A(A^{-1}\widetilde{p}))^{-1})\,|\,\widetilde{\omega}_j\rangle\langle(\widetilde{\omega}_j),\overrightarrow{A^{-1}\widetilde{p}_j}\,|\,\psi\rangle\ . \quad (3.20)$$

Thus, (3.19) and (3.20) are equal if for all $j\in J$

$$R(\widetilde{p}_j,A(\widetilde{p}))^{-1}R(\widetilde{p}_j,A)=R(\widetilde{p},A)R(A^{-1}\widetilde{p}_j,A(A^{-1}\widetilde{p}))^{-1}\ ,$$

which is in fact correct [see (A1)]. Thus, (3.18) is established. We now compare $\widetilde{U}'$ with U'. Let (a,A) belong to the Euclidean subgroup $\mathscr{E}$. Then we have

$$\overrightarrow{A^{-1}\widetilde{p}}=A^{-1}\vec{p},\ \ (A^{-1}\widetilde{p})^0=p^0\ ,$$
$$R(\widetilde{p},A)=A=R(\widetilde{p}_j,A),\ \ e^{i\widetilde{p}\cdot a}=e^{-i\vec{p}\cdot\vec{a}}\ .$$

Therefore, no tilde appears on the RHS of (3.18), implying $\widetilde{U}'(g)=U'(g)$ for all $g\in\mathscr{E}$. Further, we easily deduce from (3.18) that the Newton-Wigner position operator of $\widetilde{U}'$ is $i\partial/\partial\vec{p}$. Again, no tilde is involved, so that the Newton-Wigner operators coincide for $\widetilde{U}'$ and U'. Thus, (3.4) is shown. Moreover, we see that the canonical tensor factoring

$$L^2(K^{(J)}\times R^3,H_*)=L^2(K^{(J)},H_*)\otimes L^2(R^3)$$

is a barycentric factoring for both $\widetilde{U}'$ and U'. What remains to be shown is that $W^{-1}\widetilde{W}$ does not depend on the barycentric factorings chosen in the construction. Choosing other barycentric factorings and another tensor product $\overline{H}_*$ of internal spaces determines a unitary transformation V: $H_*\to\overline{H}_*$ such that for the corresponding transformation

$$V\otimes\mathbb{1}\colon L^2(K^{(J)}\times R^3,H_*)\to L^2(K^{(J)}\times R^3,\overline{H}_*)$$

we have $\overline{W}=(V\otimes\mathbb{1})W$, $\overline{\widetilde{W}}=(V\otimes\mathbb{1})\widetilde{W}$. Thus, in forming F, the factor V cancels out.

Finally, we should write the complete formula for F:

$$F=\int\left[\prod_j d\omega_j d\widetilde{\omega}_j d\vec{k}_j\right]d\vec{p}\,\delta\left[\sum_j\vec{k}_j\right](\widetilde{m}p^0/m\widetilde{p}^0)^{1/2}\prod_j(\widetilde{p}_j^0k_j^0/p_j^0\widetilde{k}_j^0)^{1/2}$$
$$\times\langle\omega_j\,|\,U_j^{\text{in}}(R(p_j,A(p))R(\widetilde{p}_j,A(\widetilde{p}))^{-1})\,|\,\widetilde{\omega}_j\rangle\,|\,(\omega_j),(\vec{p}_j)\rangle\langle(\widetilde{\omega}_j),(\vec{p}_j)\,|\ . \quad (3.21)$$

This is easily verified by showing that this formula implies

$$\langle\psi\,|\,F\varphi\rangle=\langle W\psi\,|\,\widetilde{W}\varphi\rangle\ .$$

Having given the construction of F, we will study the dependence of F on the input data of the construction, i.e., we consider F as a function. The input data are (a) a family $(H_j)_{j\in J}$ of Hilbert spaces and a tensor product $\otimes_{j\in J}H_j$ of these spaces (note that specifying a tensor product does not mean only to specify the Hilbert space $\otimes_{j\in J}H_j$ but also how to form products $\otimes_{j\in J}\psi_j$ for $\psi_j\in H_j$), and (b) two concentric families $(\widetilde{U}_j)_{j\in J}$ and $(U_j)_{j\in J}$ of positive representations of $\mathscr{P}$ defined on the H_j's. Therefore, we write

$$F((\widetilde{U}_j)_{j\in J},(U_j)_{j\in J},\otimes_{j\in J}H_j)\ . \quad (3.22)$$

By $(W^{-1}\widetilde{W})^{-1}=\widetilde{W}^{-1}W$ we have

$$F((\widetilde{U}_j)_{j\in J},(U_j)_{j\in J},\otimes_{j\in J}H_j)$$
$$=F((U_j)_{j\in J},(\widetilde{U}_j)_{j\in J},\otimes_{j\in J}H_j)^{-1}\ . \quad (3.23)$$

Hence, $F=\mathbb{1}$ if the families $(\widetilde{U}_j)_{j\in J}$ and $(U_j)_{j\in J}$ coincide. Let $(\widetilde{U}_j)_{j\in J}$, $(U_j)_{j\in J}$,$\otimes_{j\in J}H_j$ and $(\widetilde{U}'_j)_{j\in J}$,$(U'_j)_{j\in J}$, $\otimes'_{j\in J}H'_j$ be two isomorphic data, which means that there are unitary transformations V_j: $H_j\to H'_j$ such that

$$V_j\widetilde{U}_jV_j^{-1}=\widetilde{U}'_j\ \text{and}\ V_jU_jV_j^{-1}=U'_j\ .$$

Then the unitary transformation V,

$$V\colon\otimes_{j\in J}H_j\to\otimes'_{j\in J}H'_j\ ,\ \ V\otimes_{j\in J}\psi_j\equiv\otimes'_{j\in J}V_j\psi_j\ ,$$

satisfies

$$VF((\widetilde{U}_j)_{j\in J},(U_j)_{j\in J},\otimes_{j\in J}H_j)V^{-1}$$
$$=F((\widetilde{U}'_j)_{j\in J},(U'_j)_{j\in J},\otimes'_{j\in J}H'_j)\ . \quad (3.24)$$

The possibility of changing the tensor product will not be used in the following. It is considered here in order to indicate the functorial nature of F. Equation (3.24) follows from the basis independence of the construction because expressing the LHS with respect to bases $|\,\omega_j,\vec{p}_j\rangle$ and $|\,\widetilde{\omega}_j,\vec{p}_j\rangle$ [see (3.21)] and the RHS with respect to the bases $\otimes_jV_j\,|\,\omega_j,\vec{p}_j\rangle$, $\otimes_jV_j\,|\,\widetilde{\omega}_j,\vec{p}_j\rangle$ results in identical expressions.

IV. CONSTRUCTION OF THE INTERACTING REPRESENTATIONS

We assume that the one-particle Hilbert spaces H_j, $j\in J$, are given together with the tensor products $H_\alpha=\otimes_{j\in\alpha}H_j$ and $H_{\cup\mathscr{A}}=\otimes_{\alpha\in\mathscr{A}}H_\alpha$ (remember the notation introduced at the beginning of Sec. I) such that the associative law (1.7) is satisfied. In the sequel we will be concerned with constructing positive representations U_α, where the positive representations U_ρ (remember $|\rho|\le 2$)

play the role of an input. Let us fix the convention that for every operator to be constructed, a lower index indicates both the space on which it acts and the U_ρ out of which it is constructed. For instance, U_α and $Z_{\mathscr{A}}$ will be defined on H_α and $H_{\cup\mathscr{A}}$, respectively, and will be constructed out of $(U_\rho)_{\rho\subseteq\alpha}$ and $(U_\rho)_{\rho\subseteq\cup\mathscr{A}}$. Therefore, U_α should, properly, not be considered a fixed representation but a representation-valued function of the variables U_ρ, $\rho\subseteq\alpha$. This point of view is essential for the algebraic cluster separability to make sense, as is clear from Sec. I. Representations belonging to disjoint subsystems will often have to be multiplied tensorially. We abbreviate

$$U_{\mathscr{A}}\equiv\otimes_{\alpha\in\mathscr{A}}U_\alpha \ . \tag{4.1}$$

The representations $U_{\{i,j\}}$, which describe the interaction between pairs, may, for instance, be of the form $U_{\{i,j\}}=\mathrm{BT}(M_{ij},U_i\otimes U_j)$, where $M_{ij}=M(U_i\otimes U_j)+v_{ij}(\,|\vec{\mathrm{R}}_i-\vec{\mathrm{R}}_j\,|\,)$ and where the operator $\vec{\mathrm{R}}_i$ (Ref. 13) describes the position of particle i relative to the center of mass of the system $\{i,j\}$. That M_{ij} be self-adjoint and positive poses rather weak conditions on the potential functions v_{ij}. This construction guarantees

$$U_{\{i,j\}}\sim U_i\otimes U_j=U_{*\{i,j\}} \tag{4.2}$$

and it is only this property of $U_{\{i,j\}}$ that will be needed in the sequel. This property implies no restriction; any mass spectrum and any scattering matrix that result from any relativistic two-particle theory can also be produced by one that satisfies (4.2).

Let us introduce the construction of the many-particle representations for a four-particle system, because this is the simplest case that shows all features of the general one. The general n-particle system will then be treated more concisely. Let us put $J=\{1,2,3,4\}$. We have to construct the representations $U_{\{i,j,k\}}$ and $U_{\{1,2,3,4\}}$ out of $U_{\{i\}}$ and $U_{\{i,j\}}$. The first step is, of course, the construction of the three-particle representations. Nevertheless, we turn to the construction of $U_{1234}\equiv U_{\{1,2,3,4\}}$ assuming the $U_{ijk}\equiv U_{\{i,j,k\}}$ to be given. The construction of the latter representation out of the U_ρ will then be obvious. It is a natural idea[14] to use representations with spectator particles as the building stones of the construction. Since the three-particle representations are given, one might think of using all representations of the form $U_{ijk}\otimes U_l=U_{\{\{i,j,k\},\{l\}\}}\equiv U_{ijk/l}$, which contain one spectator particle. However, trying to satisfy cluster separability rather inevitably leads to using all representations $U_{\mathscr{S}}$ on equal footing. In our case, we have the 14 representations

$$\begin{aligned}&U_{1/234},U_{2/134},U_{3/124},U_{4/123}\ ;\\ &U_{12/34},U_{13/24},U_{14/23}\ ;\\ &U_{12/3/4},U_{13/2/4},U_{14/2/3},U_{23/1/4},U_{24/1/3},U_{34/1/2}\ ;\\ &U_{1/2/3/4}\ .\end{aligned} \tag{4.3}$$

In the nonrelativistic system of Sec. I, to any representation $U_{\mathscr{S}}$ there corresponds the representation $U_J^{\mathscr{S}}$ in (1.12). The Hamiltonian $E^{\mathscr{S}}$ of this representation contains only potentials for those pairs ρ that are not separated by $\mathscr{S}$. For instance,

$$\begin{aligned}E^{13/24}&=T_1+T_2+T_3+T_4+V_{13}+V_{24}\\ &=E^{1/2/3/4}+V_{12}+V_{24}\ .\end{aligned} \tag{4.4}$$

Can the total Hamiltonian

$$E=\textstyle\sum_{j\in J}T_j+\sum_{\rho\subset J}V_\rho \tag{4.5}$$

be written as a linear combination of the $E^{\mathscr{S}}$, where $\mathscr{S}$ ranges over the 14 proper partitions of $\{1,2,3,4\}$? Indeed,

$$\begin{aligned}E=&E^{1/234}+E^{2/134}+E^{3/124}+E^{4/123}+E^{12/34}+E^{13/24}+E^{14/23}\\ &-2(E^{12/3/4}+E^{13/2/4}+E^{14/2/3}+E^{23/1/4}+E^{24/1/3}+E^{34/1/2})+6E^{1/2/3/4}\ .\end{aligned} \tag{4.6}$$

One easily checks that in this linear combination each T_j is multiplied by $4+3-2\times6+6=1$ and each V by $3-2=1$, which shows that (4.6) is true. In a similar way we can express any $E^{\mathscr{S}}$ as a linear combination of Hamiltonians that belong to partitions finer than $\mathscr{S}$. For instance,

$$E^{1/234}=E^{1/2/34}+E^{1/3/24}+E^{1/4/23}-2E^{1/2/3/4}\ . \tag{4.7}$$

The intention is now to combine the mass operators of the representations (4.3) in a similar way as the Hamiltonians are combined in (4.6). The linear combinations occurring in (4.6) and (4.7) are studied for arbitrary finite J in Appendix B. With the notion of a cluster sum $\sum^c$, introduced there, we write (4.6) and (4.7) as

$$E=\textstyle\sum^c_{J>\mathscr{S}}E^{\mathscr{S}}\ \text{and}\ E^{1/234}=\sum^c_{1/234>\mathscr{S}}E^{\mathscr{S}}\ . \tag{4.8}$$

It should be noted that the cluster sum, unlike the usual sum, depends not only on the terms in the sum but also on how they are indexed by partitions. The ansatz

$$M(U_{1234})=\textstyle\sum^c_{1234>\mathscr{S}}M(U_{\mathscr{S}}) \tag{4.9}$$

now looks natural, not only from the analogy used above, but also with respect to the nonrelativistic limit. It is, however, insufficient for the following reason: The only way at our disposal to form a representation of $\mathscr{P}$ out of its mass operator M is the Bakamjian-Thomas-Foldy construction (2.10). In this method we need an auxiliary representation (the second argument of the function BT) from which the operators $\vec{\mathrm{P}}$, $\vec{\mathrm{J}}$, and $\vec{\mathrm{X}}$ have to be taken and to which the new representation is concentric. Since any asymmetry in handling the particles would cause serious difficulties, this auxiliary representation has to be the free one $U_1\otimes U_2\otimes U_3\otimes U_4=U_{*J}$. Hence, only the mass operators of representations that are concentric to U_{*J} can easily be combined to yield a new representation, and the latter will also be concentric to U_{*J}. At this point, Sokolov's idea of packing operators[15] comes into play. In the present setting these are unitary operators $Z_{\mathscr{A}}$ that achieve

$$Z_{\mathscr{A}} \otimes_{\alpha \in \mathscr{A}} U_\alpha Z_{\mathscr{A}}^{-1} \sim \otimes_{\alpha \in \mathscr{A}} (\otimes_{j \in \alpha} U_j) . \quad (4.10)$$

One easily deduces from Theorem 1 in Ref. 13 that the only theory satisfying both (4.10) with $Z_{\mathscr{A}} = \mathbb{1}$ and cluster separability is the free one. Hence, we cannot avoid introducing the packing operators. If these operators are known, we define U_{1234} by

$$Z_J M(U_J) Z_J^{-1} \equiv \sum_{J > \mathscr{S}}^c Z_{\mathscr{S}} M(U_{\mathscr{S}}) Z_{\mathscr{S}}^{-1} , \quad (4.11)$$

$$U_J \equiv Z_J^{-1} \mathrm{BT}(Z_J M(U_J) Z_J^{-1}, \otimes_{j \in J} U_j) Z_J , \quad (4.12)$$

thus modifying the ansatz (4.9) by the packing operators.

Let us now study how the packing operators $Z_{\mathscr{A}}$ can be determined. As for the U_α, the procedure will be in steps that increase the number of particles by one. By (4.2), the starting definition

$$Z_{\mathscr{A}} \equiv \mathbb{1} \ \text{ if } \ |\cup \mathscr{A}| \le 2 \quad (4.13)$$

satisfies (4.10). How to proceed to larger $\mathscr{A}$ is suggested by considering cluster separation. Let us write $U_{\mathscr{A}}^{\mathscr{S}}$ and $Z_{\mathscr{A}}^{\mathscr{S}}$ for the objects constructed with the input $(U_\rho^{\mathscr{S}})_{\rho \subseteq \cup \mathscr{A}}$ instead of $(U_\rho)_{\rho \subseteq \mathscr{A}}$, where

$$U_\rho^{\mathscr{S}} \equiv U_{\rho \wedge \mathscr{S}} . \quad (4.14)$$

Cluster separability is satisfied if and only if $U_\alpha^{\mathscr{S}} = U_{\alpha \wedge \mathscr{S}}$ for every α and $\mathscr{S}$. Owing to (1.7), this is equivalent to

$$U_{\mathscr{A}}^{\mathscr{S}} = \otimes_{\alpha \in \mathscr{A}} U_\alpha^{\mathscr{S}} = \otimes_{\alpha \in \mathscr{A}} \otimes_{\beta \in \alpha \wedge \mathscr{S}} U_\beta = \otimes_{\gamma \in \mathscr{A} \wedge \mathscr{S}} U_\gamma = U_{\mathscr{A} \wedge \mathscr{S}} . \quad (4.15)$$

The defining property (4.10) for $Z_{\mathscr{A}}$ implies $Z_{\mathscr{A}}^{\mathscr{S}} U_{\mathscr{A}}^{\mathscr{S}} Z_{\mathscr{A}}^{\mathscr{S}\,-1} \sim U_{*\mathscr{A}}$. Hence, $Z_{\mathscr{A}}^{\mathscr{S}} Z_{\mathscr{A} \wedge \mathscr{S}}^{-1}$ has to commute with $\vec{\mathrm{P}}$, $\vec{\mathrm{J}}$, and $\vec{\mathrm{X}}$ of the free representation $U_{*\mathscr{A}}$. It will turn out that the simplest way of assuring this is in fact possible:

$$Z_{\mathscr{A}}^{\mathscr{S}} = Z_{\mathscr{A} \wedge \mathscr{S}} . \quad (4.16)$$

We now assume that $Z_{\mathscr{A}}$ and $U_{\mathscr{A}}$ are already constructed for $|\cup \mathscr{A}| \le 3$ such that (4.10), (4.13), (4.15), and (4.16) are satisfied. We then have to construct $Z_{\mathscr{A}}$ for $|\cup \mathscr{A}| = 4$. Let us first consider the case $|\mathscr{A}| \ge 2$. Obviously, $Z_{\mathscr{A}}$ satisfies (4.10) if and only if the auxiliary operator

$$Y_{\mathscr{A}} \equiv Z_{\mathscr{A}} \otimes_{\alpha \in \mathscr{A}} Z_\alpha^{-1} \quad (4.17)$$

satisfies

$$Y_{\mathscr{A}} \otimes_{\alpha \in \mathscr{A}} (Z_\alpha U_\alpha Z_\alpha^{-1}) Y_{\mathscr{A}}^{-1} \sim \otimes_{\alpha \in \mathscr{A}} U_{*\alpha} . \quad (4.18)$$

It is for analyzing this equation that the kinematical transformation F in Sec. III was introduced. Remembering $Z_\alpha U_\alpha Z_\alpha^{-1} \sim U_{*\alpha}$ (by assumption, because $|\alpha| \le 3$) we see that

$$Y_{\mathscr{A}} = F((Z_\alpha U_\alpha Z_\alpha^{-1})_{\alpha \in \mathscr{A}}, (U_{*\alpha})_{\alpha \in \mathscr{A}}, \otimes_{\alpha \in \mathscr{A}} H_\alpha) \quad (4.19)$$

is a solution of (4.18). Since F will always occur in the form that the second and third argument is determined by the first one, we will write only the first argument. More precisely, let $(\tilde{U}_\alpha)_{\alpha \in \mathscr{A}}$ be any family satisfying $\tilde{U}_\alpha \sim U_{*\alpha}$; then we put

$$F((\tilde{U}_\alpha)_{\alpha \in \mathscr{A}}) \equiv F((\tilde{U}_\alpha)_{\alpha \in \mathscr{A}}, (U_{*\alpha})_{\alpha \in \mathscr{A}}, \otimes_{\alpha \in \mathscr{A}} H_\alpha) . \quad (4.20)$$

In order to find the most general solution of (4.18), we introduce a further auxiliary operator $E_{\mathscr{A}}$ through

$$E_{\mathscr{A}} \equiv Y_{\mathscr{A}} F((Z_\alpha U_\alpha Z_\alpha^{-1})_{\alpha \in \mathscr{A}})^{-1} = Z_{\mathscr{A}} \otimes_{\alpha \in \mathscr{A}} Z_\alpha^{-1} F((Z_\alpha U_\alpha Z_\alpha^{-1})_{\alpha \in \mathscr{A}})^{-1} . \quad (4.21)$$

One easily sees that (4.10) is satisfied if and only if

$$E_{\mathscr{A}} \text{ commutes with } \vec{\mathrm{P}}, \ \vec{\mathrm{J}}, \text{ and } \vec{\mathrm{X}} \text{ of } U_{*\mathscr{A}} . \quad (4.22)$$

Let us now look for the consequences of (4.16) on $Y_{\mathscr{A}}$ and $E_{\mathscr{A}}$,

$$\begin{aligned} Y_{\mathscr{A}}^{\mathscr{S}} &\equiv Z_{\mathscr{A}}^{\mathscr{S}} \otimes_{\alpha \in \mathscr{A}} (Z_\alpha^{\mathscr{S}})^{-1} \\ &= Z_{\mathscr{A} \wedge \mathscr{S}} \otimes_{\alpha \in \mathscr{A}} Z_{\alpha \wedge \mathscr{S}}^{-1} \\ &= Z_{\mathscr{A} \wedge \mathscr{S}} \otimes_{\alpha \in \mathscr{A}} \otimes_{\beta \in \alpha \wedge \mathscr{S}} Z_\beta^{-1} Y_{\alpha \wedge \mathscr{S}}^{-1} \\ &= Z_{\mathscr{A} \wedge \mathscr{S}} \otimes_{\gamma \in \mathscr{A} \wedge \mathscr{S}} Z_\gamma^{-1} \otimes_{\alpha \in \mathscr{A}} Y_{\alpha \wedge \mathscr{S}}^{-1} \\ &= Y_{\mathscr{A} \wedge \mathscr{S}} \otimes_{\alpha \in \mathscr{A}} Y_{\alpha \wedge \mathscr{S}}^{-1} . \end{aligned} \quad (4.23)$$

For a long time, I tried to show that this equation is satisfied for the particular solution (4.19). Finally it turned out that for $|\cup \mathscr{A}| = 4$ this is generally only the case in a free theory. Hence, one has to expect $E_{\mathscr{A}} \neq \mathbb{1}$ whenever $|\cup \mathscr{A}| \ge 4$. (For $|\cup \mathscr{A}| \le 3$ one finds $E_{\mathscr{A}} = \mathbb{1}$; this is the most important simplification occurring for three-particle systems.) As a consequence of (4.23), we find for $E_{\mathscr{A}}^{\mathscr{S}}$

$$E_{\mathscr{A}}^{\mathscr{S}} = Z_{\mathscr{A} \wedge \mathscr{S}} \otimes_{\alpha \in \mathscr{A}} Z_{\alpha \wedge \mathscr{S}}^{-1} \times F((Z_{\alpha \wedge \mathscr{S}} U_{\alpha \wedge \mathscr{S}} Z_{\alpha \wedge \mathscr{S}}^{-1})_{\alpha \in \mathscr{A}})^{-1} . \quad (4.24)$$

This relation is easily fulfilled with the help of the cluster product introduced in Appendix B. We put

$$E_{\mathscr{A}} = \prod_{\mathscr{A} > \mathscr{B}}^c E_{\mathscr{A}}^{\mathscr{B}} , \quad (4.25)$$

where $E_{\mathscr{A}}^{\mathscr{B}}$ is defined for any $\mathscr{B}$ with $\cup \mathscr{B} \subseteq \cup \mathscr{A}$ by Eq. (4.24) with $\mathscr{S}$ replaced by $\mathscr{B}$. Looking at $\ln E_{\mathscr{A}}^{\mathscr{S}}$ as a function of $\mathscr{S}$, we see that (4.25) means that the $\mathscr{A}$-connected part of this function vanishes. Since it is only the $\mathscr{A}$-connected part that cannot be determined by recursion relations, (4.25) is a natural definition. If every $E_{\mathscr{A}}^{\mathscr{B}}$ on the RHS of (4.25) satisfies (4.22) the same is true for $E_{\mathscr{A}}$ as a consequence of (4.25). An analogous statement would not hold for the definition $Y_{\mathscr{A}} \equiv \prod_{\mathscr{A} > \mathscr{B}}^c Y_{\mathscr{A}}^{\mathscr{B}}$ and Eq. (4.18). Combining (4.25), (4.24), and (4.21) we have

$$Z_{\mathscr{A}} = \left[\prod_{\mathscr{A} > \mathscr{B}}^c Z_{\mathscr{B}} \otimes_{\alpha \in \mathscr{A}} Z_{\alpha \wedge \mathscr{B}}^{-1} F((Z_{\alpha \wedge \mathscr{B}} U_{\alpha \wedge \mathscr{B}} Z_{\alpha \wedge \mathscr{B}}^{-1})_{\alpha \in \mathscr{A}})^{-1} \right] F((Z_\alpha U_\alpha Z_\alpha^{-1})_{\alpha \in \mathscr{A}}) \otimes_{\alpha \in \mathscr{A}} Z_\alpha . \quad (4.26)$$

Besides the quantities $Z_{\alpha \wedge \mathscr{B}}$, $U_{\alpha \wedge \mathscr{B}}$, Z_α, and U_α, which are known from the three-particle system, there appear the $Z_{\mathscr{B}}$

with $\mathscr{B}<\mathscr{A}$. Applying (4.26) to them, we express them by $Z_{\mathscr{C}}$ with $\mathscr{C}<\mathscr{B}$, and so on. Finally $Z_{\mathscr{A}}$ is expressed by known quantities and by $Z_{*\mathscr{A}}$. We put

$$Z_{*\mathscr{A}}\equiv\mathbb{1}\ , \tag{4.27}$$

which satisfies (4.10). Finally, we define Z_J in close analogy to (4.9) as

$$Z_J\equiv\prod_{J>\mathscr{S}}^{c} Z_{\mathscr{S}}\ . \tag{4.28}$$

Then, (4.11) and (4.12) yield the definition of $U_J=U_{1234}$.

It might be instructive to spell out these definitions: Let the representations U_i and U_{ij} be given. Then we put

$$\begin{aligned}
&Z_i\equiv\mathbb{1},\ \ Z_{ij}\equiv Z_{i/j}\equiv\mathbb{1},\ \ U_{ij/k}\equiv U_{ij}\otimes U_k,\ \ U_{i/j/k}\equiv U_i\otimes U_j\otimes U_k\ ,\\
&Z_{ij/k}\equiv F(U_{ij},U_k),\ \ \tilde U_{ij/k}\equiv Z_{ij/k}U_{ij/k}Z_{ij/k}{}^{-1}\ ,\\
&Z_{ijk}\equiv\exp(\ln Z_{ij/k}+\ln Z_{ik/j}+\ln Z_{jk/i})\ ,\\
&U_{ijk}\equiv Z_{ijk}{}^{-1}\mathrm{BT}(M(\tilde U_{ij/k})+M(\tilde U_{ik/j})+M(\tilde U_{jk/i})-2M(U_{i/j/k}),U_{i/j/k})Z_{ijk}\ ,\\
&Z_{i/j/k/l}\equiv\mathbb{1},\ \ Z_{ij/k/l}\equiv F(U_{ij},U_k,U_l)\ ,\\
&Z_{ij/kl}\equiv(\exp\{\ln[F(U_{ij},U_k,U_l)F(U_{ij},U_{k/l})^{-1}]+\ln[F(U_{kl},U_i,U_j)F(U_{kl},U_{i/j})^{-1}]\})F(U_{ij},U_{kl})\ ,\\
&Z_{ijk/l}\equiv(\exp\{\ln[Z_{ij/k/l}Z_{ij/k}{}^{-1}F(U_{ij/k},U_l)^{-1}]+\ln[Z_{jk/i/l}Z_{jk/i}{}^{-1}F(U_{jk/i},U_l)^{-1}]\\
&\qquad\qquad+\ln[Z_{ki/j/l}Z_{ki/j}{}^{-1}F(U_{ki/j},U_l)^{-1}]\})F(U_{ijk},U_l)Z_{ijk}\ ,\\
&Z_{1234}\equiv\exp(\ln Z_{1/234}+\ln Z_{2/134}+\ln Z_{3/124}+\ln Z_{4/123}+\ln Z_{12/34}+\ln Z_{13/24}+\ln Z_{14/23}\\
&\qquad\qquad-2\ln Z_{12/3/4}-2\ln Z_{13/2/4}-2\ln Z_{14/2/3}-2\ln Z_{23/1/4}-2\ln Z_{24/1/3}-2\ln Z_{34/1/2})\ ,\\
&\tilde U_{ij/kl}\equiv Z_{ij/kl}U_{ij}\otimes U_{kl}Z_{ij/kl}{}^{-1}\ ,\\
&\tilde U_{i/jkl}\equiv Z_{i/jkl}U_i\otimes U_{jkl}Z_{i/jkl}{}^{-1}\ ,\\
&\tilde U_{i/j/kl}\equiv Z_{i/j/kl}U_i\otimes U_j\otimes U_{kl}Z_{i/j/kl}{}^{-1}\ ,\\
&U_{1234}\equiv Z_{1243}{}^{-1}\mathrm{BT}(M(\tilde U_{1/234})+M(\tilde U_{2/134})+M(\tilde U_{3/124})+M(\tilde U_{4/123})+M(\tilde U_{12/34})+M(\tilde U_{13/24})\\
&\qquad\qquad+M(\tilde U_{14/23})-2M(\tilde U_{12/3/4})-2M(\tilde U_{13/2/4})-2M(\tilde U_{14/2/3})-2M(\tilde U_{23/1/4})\\
&\qquad\qquad-2M(\tilde U_{24/1/3})-2M(\tilde U_{34/1/2})+6M(\tilde U_{1/2/3/4}),U_{1/2/3/4})Z_{1234}\ .
\end{aligned} \tag{4.29}$$

Cluster separability will be shown in the general case. Nevertheless, the reader is invited to verify cluster separability for a particular $\mathscr{S}$ by direct calculation. Taking, for instance, $\mathscr{S}=12/34$, one has to show that

$$U_{ij}=U_i\otimes U_j$$

$$\text{unless } \{i,j\}=\{1,2\} \text{ or } \{3,4\} \text{ implies } U_{1234}=U_{12/34}\ .$$

Let us now describe and study the construction for an arbitrary finite number of particles. Our starting position is that described at the beginning of this section. Let us call a pair of families,

$$\Phi=((U_\alpha)_{|\alpha|\le p},(Z_{\mathscr{A}})_{|\cup\mathscr{A}|\le p}),\ \ p\le|J|\ , \tag{4.30}$$

a (J,p) *system* if every U_α is a positive representation of $\mathscr{P}$ (on H_α) and every $Z_{\mathscr{A}}$ is a unitary operator (on $H_{\cup\mathscr{A}}$) such that (4.10) is satisfied. For any partition $\mathscr{S}$ of J we easily verify that

$$\Phi^{\mathscr{S}}\equiv((U_\alpha^{\mathscr{S}})_{|\alpha|\le p},\ (Z_{\mathscr{A}}^{\mathscr{S}})_{|\cup\mathscr{A}|\le p})\ , \tag{4.31}$$

where $U_\alpha^{\mathscr{S}}\equiv U_{\alpha\wedge\mathscr{S}}\equiv\otimes_{\beta\in\alpha\wedge\mathscr{S}}U_\beta$ and $Z_{\mathscr{A}}^{\mathscr{S}}\equiv Z_{\mathscr{A}\wedge\mathscr{S}}$ is a (J,p) system too. Obviously, we have

$$(\Phi^{\mathscr{S}})^{\mathscr{S}'}=\Phi^{\mathscr{S}\wedge\mathscr{S}'}\ . \tag{4.32}$$

Thus, the commutative monoid of partitions (with $\wedge$ as the composition and $\{J\}$ as the unit element) acts on the set of (J,p) systems. Note that the action of $\mathscr{S}$ introduced after Eq. (4.13) conceptually coincides with the present one only for $|p|\le 2$. These concepts coincide generally if and only if cluster separability holds. It is convenient to associate with any (J,p) system the auxiliary families $(U_{\mathscr{A}})_{|\cup\mathscr{A}|\le p}$, $(\tilde U_{\mathscr{A}})_{|\cup\mathscr{A}|\le p}$, $(Y_{\mathscr{A}})_{|\cup\mathscr{A}|\le p}$, and $(E_{\mathscr{A}})_{|\cup\mathscr{A}|\le p}$ by

$$\begin{aligned}
&U_{\mathscr{A}}\equiv\otimes_{\alpha\in\mathscr{A}}U_\alpha,\ \ \tilde U_{\mathscr{A}}\equiv Z_{\mathscr{A}}U_{\mathscr{A}}Z_{\mathscr{A}}{}^{-1}\ ,\\
&Y_{\mathscr{A}}\equiv Z_{\mathscr{A}}\otimes_{\alpha\in\mathscr{A}}Z_\alpha{}^{-1},\ \ E_{\mathscr{A}}\equiv Y_{\mathscr{A}}F((\tilde U_\alpha)_{\alpha\in\mathscr{A}})^{-1}\ .
\end{aligned} \tag{4.33}$$

[See (4.20).] The corresponding objects of the system $\Phi^{\mathscr{S}}$ are easily seen to be given by

$$\begin{aligned}
&U_{\mathscr{A}}^{\mathscr{S}}=U_{\mathscr{A}\wedge\mathscr{S}},\ \ \tilde U_{\mathscr{A}}^{\mathscr{S}}=\tilde U_{\mathscr{A}\wedge\mathscr{S}}\ ,\\
&Y_{\mathscr{A}}^{\mathscr{S}}=Z_{\mathscr{A}\wedge\mathscr{S}}\otimes_{\alpha\in\mathscr{A}}Z_{\alpha\wedge\mathscr{S}}{}^{-1}\\
&\qquad=Y_{\mathscr{A}\wedge\mathscr{S}}\otimes_{\alpha\in\mathscr{A}}Y_{\alpha\wedge\mathscr{S}}{}^{-1}\ ,\\
&E_{\mathscr{A}}^{\mathscr{S}}=Z_{\mathscr{A}\wedge\mathscr{S}}\otimes_{\alpha\in\mathscr{A}}Z_{\alpha\wedge\mathscr{S}}{}^{-1}F((\tilde U_{\alpha\wedge\mathscr{S}})_{\alpha\in\mathscr{S}})\ .
\end{aligned} \tag{4.34}$$

If we replace $\mathscr{S}$ by any $\mathscr{B}$ satisfying $\cup\mathscr{B}\supseteq\cup\mathscr{A}$, these equations may be used to define the objects $U^{\mathscr{B}}_{\mathscr{A}}$, $\tilde{U}^{\mathscr{B}}_{\mathscr{A}}$, $Y^{\mathscr{B}}_{\mathscr{A}}$, and $E^{\mathscr{B}}_{\mathscr{A}}$. Equations (4.11), (4.28), and (4.25) can then be written in a unified form:

$$M(\tilde{U}_\alpha)=\sum_{\alpha>\mathscr{B}}^{c} M(\tilde{U}^{\mathscr{B}}_\alpha)\,,$$
$$Z_\alpha=\prod_{\alpha>\mathscr{B}}^{c} Z^{\mathscr{B}}_\alpha\,, \tag{4.35}$$
$$E_{\mathscr{A}}=\prod_{\mathscr{A}>\mathscr{B}}^{c} E^{\mathscr{B}}_{\mathscr{A}}\,.$$

Let us call an equation relating the objects of a (J,p) system Φ *partition covariant* if its validity for Φ implies its validity for $\Phi^{\mathscr{S}}$. Then, the equations (4.35) are partition covariant. Actually, partition covariance was the guiding principle for their invention. A (J,p) system satisfying (4.35) will be called *recursive* because we have seen previously that these equations enable us to construct the quantities of subsystems out of those of subsystems with a smaller number of particles. Hence, there is exactly one recursive $(J,p+1)$ system $\bar{\Phi}$ that extends Φ in the sense that $\bar{U}_{\mathscr{A}}=U_{\mathscr{A}}$ and $\bar{Z}_{\mathscr{A}}=Z_{\mathscr{A}}$ whenever $|\cup\mathscr{A}|\le p$. The defining equations for $\bar{\Phi}$ are the following: For any $\mathscr{A}$ satisfying $|\cup\mathscr{A}|=p+1$ and $|\mathscr{A}|\ge 2$ we put

$$U_{\mathscr{A}}\equiv\otimes_{\alpha\in\mathscr{A}}U_\alpha\,.$$

For each $\mathscr{A}$ such that $|\mathscr{A}|=p+1$ we put

$$Z_{\mathscr{A}}\equiv\mathbb{1}\,.$$

Now assume $Z_{\mathscr{B}}$ to be known whenever $|\mathscr{B}|>n\ge 2$. Then for $\mathscr{A}$ such that $|\mathscr{A}|=n$ we put

$$Z_{\mathscr{A}}\equiv\left[\prod_{\mathscr{A}>\mathscr{B}}^{c} Z_{\mathscr{B}}\otimes_{\alpha\in\mathscr{A}}Z_{\alpha\wedge\mathscr{B}}{}^{-1}F((Z_{\alpha\wedge\mathscr{B}}U_{\alpha\wedge\mathscr{B}}Z_{\alpha\wedge\mathscr{B}}{}^{-1})_{\alpha\in\mathscr{A}})^{-1}\right]F((Z_\alpha U_\alpha Z_\alpha{}^{-1})_{\alpha\in\mathscr{A}})\otimes_{\alpha\in\mathscr{A}}Z_\alpha\,,$$

where the objects on the RHS either belong to Φ or are $Z_{\mathscr{B}}$'s with $|\mathscr{B}|>n$. Finally, we know all $Z_{\mathscr{A}}$'s such that $|\mathscr{A}|\ge 2$. For each α such that $|\alpha|=p+1$ we put

$$Z_\alpha\equiv\prod_{\alpha>\mathscr{B}}^{c} Z_{\mathscr{B}}$$

and

$$U_\alpha\equiv Z_\alpha{}^{-1}\mathrm{BT}\left[\sum_{\alpha>\mathscr{B}}^{c} Z_{\mathscr{B}}M(U_{\mathscr{B}})Z_{\mathscr{B}}{}^{-1},U_{*\alpha}\right]Z_\alpha\,.$$

In order to show that this procedure defines in fact a $(J,p+1)$ system, we have to prove

$$Z_{\mathscr{A}}U_{\mathscr{A}}Z_{\mathscr{A}}{}^{-1}\sim U_{*\mathscr{A}}\,.$$

With the help of the statement including (4.22), this is easily done by induction with respect to $|\mathscr{A}|$ in the same direction used in the construction of $Z_{\mathscr{A}}$. Let the construction start with a $(J,2)$ system Φ_2. Then the objects of the $(J,|J|)$ system arising by successive extension of Φ_2 depend as functions on Φ_2:

$$U_{\mathscr{A}}=U_{\mathscr{A}}(\Phi_2),\;\; Z_{\mathscr{A}}=Z_{\mathscr{A}}(\Phi_2)\,.$$

Cluster separability means for all partitions $\mathscr{S}$

$$U^{\mathscr{S}}_{\mathscr{A}}=U_{\mathscr{A}\wedge\mathscr{S}}=U_{\mathscr{A}}(\Phi^{\mathscr{S}}_2),\;\; Z^{\mathscr{S}}_{\mathscr{A}}=Z_{\mathscr{A}\wedge\mathscr{S}}=Z_{\mathscr{A}}(\Phi^{\mathscr{S}}_2)\,.$$

This is in fact satisfied because in constructing the $(J,|J|)$ system we used only partition-covariant equations, so that the process of extending a (J,p) system commutes with the action of partitions:

$$\overline{\Phi^{\mathscr{S}}}=\bar{\Phi}^{\mathscr{S}}\,.$$

Finally, one easily shows that the $(J\,|J|)$ system satisfies $U_{\mathscr{A}}(g)=U_{*\mathscr{A}}(g)$ for all $g\in\mathscr{G}$.

Concluding this section, let us reflect once more on what we have done. The representation U_J that results from the construction described above determines the Hamiltonian P^0_J of the system J via (2.1). Since the individual particle observables $\vec{X}_j$, $\vec{P}_j$, and $\vec{S}_j$ [see (2.4)] that are determined by the free representation $\otimes_{j\in J}U_j$ form an irreducible set of operators, P^0_J is a function of these operators (more precisely, any bounded function of P^0_J belongs to the von Neumann algebra generated by these operators). Therefore, we are within the conventional scheme of Hamiltonian quantum mechanics where the Hamiltonian is a Hermitian operator (on a Hilbert space with positive-definite metric) that is a function of a specified finite set of dynamical variables. The function representing the Hamiltonian is rather complicated, so that only approximations of it may actually be written. The principles of relativity by themselves would not enforce such a complicated structure, but cluster separability does.

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APPENDIX A: THE POINCARÉ GROUP

In order to avoid the use of projective representations in describing relativistic invariance, we always work with the quantum-mechanical Poincaré group in the sense of Wigner. This group, denoted by $\mathscr{P}$, is the topological space $R^4\times\mathrm{SL}(2,C)$ together with the law of multiplication

$$(a,A)(a',A')=(a+Aa',AA')\,,$$

where the matrix group SL(2,C) acts (by definition) on four-vectors as follows:

$$(Ax)^\mu=\tfrac{1}{2}\,\mathrm{tr}\sigma_\mu Ax^\nu\sigma_\nu A^+,\;\; \mu\in\{0,1,2,3\}$$

where σ_0 is the unit element in SL(2,C) and $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ is the triplet of Pauli matrices. For any

$$p\in V_+\equiv\{p\in R^4\,|\,p^0>0,\ p\cdot p=p_0{}^2-\vec{p}\,^2>0\}$$

we define the positive Hermitian matrix $A(p)$ by

$$A(p)\equiv\frac{(m+p^0)\sigma_0+\vec{p}\cdot\vec{\sigma}}{[2m(m+p^0)]^{1/2}}=\exp(\chi\vec{n}\cdot\vec{\sigma}/2)\ ,$$

where $m\equiv(p\cdot p)^{1/2}$, $\vec{n}\equiv\vec{p}/|\vec{p}|$, and $\chi=\text{arctanh}\,|\vec{p}|/p^0=\text{arcsinh}\,|\vec{p}|/m$. The essential properties of $A(p)$ are

$$A(p)^{-1}p=(m,\vec{0}),\quad A(p)^{-1}=A(p^0,-\vec{p})\ ,$$
$$A(p)=A(w),\quad \text{where } w\equiv p/m\ ,$$
$$A(p)^{-1}x=(w\cdot x,\vec{x}-x^0\vec{w}+(1+w^0)^{-1}\vec{w}\cdot\vec{x}\vec{w})\ .$$

For all $p\in V_+$ and $A\in\text{SL}(2,C)$ the Wigner rotation $R(p,A)\in\text{SU}(2)$ is defined by

$$R(p,A)\equiv A(p)^{-1}AA(A^{-1}p)\ .$$

For all $A,A'\in\text{SL}(2,C)$, $B\in\text{SU}(2)$, and $p,q\in V_+$ we have

$$\begin{aligned}&R(p,A)R(A^{-1}p,A')=R(p,AA')\ ,\\&R(p,A)^{-1}=R(A^{-1}p,A^{-1})\ ,\\&R(p,AB)=R(p,A)B\ ,\\&R(p,1)=R(p,A(p))=1\ ,\\&R(p,B)=B\ ,\\&R(A^{-1}p,A(A^{-1}q))=R(p,A)^{-1}R(p,A(q))R(q,A)\ .\end{aligned}\tag{A1}$$

APPENDIX B: CLUSTER SUMS AND CLUSTER PRODUCTS

In this section, $\mathscr{P}(J)$ denotes the set of all partitions of the finite set J, and $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{F}$ denote elements of $\mathscr{P}(J)$. In statements that contain $\mathscr{A},\mathscr{B},\ldots$ freely, a quantor $\forall\mathscr{A},\mathscr{B},\ldots\in\mathscr{P}(J)$ is understood.

Let f be a function on $\mathscr{P}(J)$ with values in an arbitrary Abelian group. Then we define the "algebraic-cluster sum" $\sum^{\text{ac}}$ by

$$\begin{aligned}\sum_{\mathscr{A}>\mathscr{B}}^{\text{ac}} f(\mathscr{B})&\equiv\sum_{\mathscr{A}>\mathscr{B}}-g(\mathscr{A},\mathscr{B})f(\mathscr{B})\\&\equiv\sum\{-g(\mathscr{A},\mathscr{B})f(\mathscr{B})\,|\,\mathscr{B}\in\mathscr{P}(J),\mathscr{A}>\mathscr{B}\}\ ,\end{aligned}\tag{B1}$$

where

$$\begin{aligned}&g(\mathscr{A},\mathscr{B})\equiv\Pi_{\alpha\in\mathscr{A}}\varphi(\,|\,\alpha\wedge\mathscr{B}\,|\,)\ ,\\&\varphi(n)\equiv(-1)^{n-1}(n-1)!\ .\end{aligned}\tag{B2}$$

For instance, $g(1234/5,12/3/4/5)=\varphi(3)\varphi(1)=2$. The coefficients g are fixed by (B2) just in such a way that

$$\sum_{\mathscr{A}>\mathscr{B}}^{\text{ac}} f(\mathscr{B}\wedge\mathscr{C})=f(\mathscr{A}\wedge\mathscr{C})\ \text{ unless } \mathscr{C}\geq\mathscr{A}\ .\tag{B3}$$

This property implies the partition covariance of Eqs. (4.35) and, therefore, plays a decisive role in our construction. That (B2) implies (B3) and vice versa is shown by the following.

Theorem[10] *[Möbius transform on $\mathscr{P}(J)$].* Let g: $\mathscr{P}(J)\times\mathscr{P}(J)\to Z$ ($\equiv$ set of integers) be a function such that

$$g(\mathscr{A},\mathscr{A})=1 \text{ and } g(\mathscr{A},\mathscr{B})=0\ \text{ unless } \mathscr{A}\geq\mathscr{B}\ .$$

Then the following statements are equivalent.

(1) For any function f defined on $\mathscr{P}(J)$ with values in an Abelian group one has

$$\sum_{\mathscr{A}\geq\mathscr{B}} g(\mathscr{A},\mathscr{B})f(\mathscr{B}\wedge\mathscr{C})=0\ \text{ unless } \mathscr{C}\geq\mathscr{A}\ .\tag{B4}$$

(2) The relation (B4) is valid for some function f as in (1) having the property that $\sum_{\mathscr{A}}c_{\mathscr{A}}f(\mathscr{A})=0$, $c_{\mathscr{A}}\in Z$, implies $c_{\mathscr{A}}=0$.

(3) $\sum\{g(\mathscr{A},\mathscr{B})\,|\,\mathscr{B}\in\mathscr{P}(J),\mathscr{B}\wedge\mathscr{C}=\mathscr{D}\}=0$, if $\mathscr{A}>\mathscr{C}\geq\mathscr{D}$.

(4) $\sum_{\mathscr{A}\geq\mathscr{B}\geq\mathscr{C}}g(\mathscr{A},\mathscr{B})=\delta_{\mathscr{A}\mathscr{C}}$.

(5) The $\mathscr{P}(J)\times\mathscr{P}(J)$ matrix A,

$$A_{\mathscr{A}\mathscr{B}}\equiv\begin{cases}1 & \text{for } \mathscr{A}\geq\mathscr{B},\\0 & \text{otherwise,}\end{cases}$$

is invertible and we have $(A^{-1})_{\mathscr{A}\mathscr{B}}=g(\mathscr{A},\mathscr{B})$.

(6) For $\mathscr{A}\geq\mathscr{B}$, $g(\mathscr{A},\mathscr{B})$ is given by (B2).

This theorem shows that any transformation of the form

$$f\to\tilde{f},\quad \tilde{f}(\mathscr{A})\equiv\sum_{\mathscr{A}\geq\mathscr{B}}f(\mathscr{B})=\sum_{\mathscr{B}}A_{\mathscr{A}\mathscr{B}}f(\mathscr{B})$$

may be inverted by

$$f(\mathscr{A})=\sum_{\mathscr{A}\geq\mathscr{B}}g(\mathscr{A},\mathscr{B})\tilde{f}(\mathscr{B})\ .$$

This justifies the name Möbius transform. It should be noted that (B4) is a transcription of (B3), using $g(\mathscr{A},\mathscr{A})=1$. Before we come to the proof of this theorem, let us consider a motivation for (B1). To a quantity being described by a function as in (1) one wishes to associate a function f^c such that $f^c(\mathscr{A})$ is intuitively the $\mathscr{A}$-connected part of this quantity. As an example we consider $f(\mathscr{A})\equiv$ sum over all graphs (for instance, in the free Abelian group generated by all graphs) with vertices in J and lines that do not join points belonging to different $\mathscr{A}$ clusters. Then $f^c(\mathscr{A})$ should be the sum over all those graphs for which any two points i,j of every $\mathscr{A}$ cluster α can be connected by lines of the graph. Let us now consider the action of the monoid $\mathscr{P}(J)$ on functions:

$$(f^{\mathscr{S}})(\mathscr{A})\equiv f(\mathscr{A}\wedge\mathscr{S}),\quad (f^{\mathscr{S}})^{\mathscr{S}'}=f^{\mathscr{S}\wedge\mathscr{S}'}\ .$$

Then an obvious condition on the c operation is

$$(f^{\mathscr{S}})^c(\mathscr{A})=\begin{cases}(f^c)^{\mathscr{S}}(\mathscr{A}) & \text{if } \mathscr{S}\geq\mathscr{A},\\0 & \text{otherwise,}\end{cases}$$

which means that the connectivity of $f^{\mathscr{S}}$ is never higher than $\mathscr{S}$. If we further assume that the c operation is of the form

$$f^c(\mathscr{A})=\sum_{\mathscr{A}\geq\mathscr{B}} h(\mathscr{A},\mathscr{B})f(\mathscr{B})\ ,$$

where the coefficients are integers satisfying $h(\mathscr{A},\mathscr{A})=1$, then the theorem tells us

$$f(\mathscr{A})=f^c(\mathscr{A})+\sum_{\mathscr{A}>\mathscr{B}}^{ac} f(\mathscr{B})$$

and

$$f(\mathscr{A})=\sum_{\mathscr{A}\geq\mathscr{B}} f^c(\mathscr{B})\ \text{(cluster expansion)}\ .$$

Proof of the theorem.

$(1)\Longleftrightarrow(2)\Longleftrightarrow(3)$: This results easily from the equation

$$\sum_{\mathscr{A}\geq\mathscr{B}} g(\mathscr{A},\mathscr{B})f(\mathscr{B}\wedge\mathscr{C}) = \sum_{\mathscr{C}\geq\mathscr{D}} f(\mathscr{D}) \sum_{\substack{\mathscr{A}\geq\mathscr{B}\\ \mathscr{B}\wedge\mathscr{C}=\mathscr{D}}} g(\mathscr{A},\mathscr{B})\ \text{ if } \mathscr{A}>\mathscr{C}\ .$$

$(3)\Longrightarrow(4)$: Put $\mathscr{C}=\mathscr{D}$ in (3).

$(4)\Longleftrightarrow(5)$: Defining a $\mathscr{P}(J)\times\mathscr{P}(J)$ matrix B by $B_{\mathscr{A}\mathscr{B}}\equiv g(\mathscr{A},\mathscr{B})$, we see that (4) is equivalent to $BA=1$. For the invertability of A see, for instance, Theorem 3 in Ref. 7.

$(5)\Longleftrightarrow(6)$: See Theorem 3.1 in Ref. 8.

$(5)\Longrightarrow(1)$: This results from the equation

$$\begin{aligned}\sum_{\mathscr{A}\geq\mathscr{B}} g(\mathscr{A},\mathscr{B})f(\mathscr{B}\wedge\mathscr{C})
&=\sum_{\mathscr{F}}\sum_{\mathscr{A}\geq\mathscr{B}} g(\mathscr{A},\mathscr{B})\delta_{\mathscr{B}\wedge\mathscr{C},\mathscr{F}}f(\mathscr{F})\\
&=\sum_{\mathscr{F},\mathscr{D}}\sum_{\mathscr{A}\geq\mathscr{B}} g(\mathscr{A},\mathscr{B})A_{\mathscr{B}\wedge\mathscr{C},\mathscr{D}}g(\mathscr{D},\mathscr{F})f(\mathscr{F})\\
&=\sum_{\mathscr{F},\mathscr{D}}\sum_{\mathscr{A}\geq\mathscr{B}} g(\mathscr{A},\mathscr{B})A_{\mathscr{B},\mathscr{D}}A_{\mathscr{C}\mathscr{D}}g(\mathscr{D},\mathscr{F})f(\mathscr{F})\\
&=\sum_{\mathscr{F},\mathscr{D}} \delta_{\mathscr{A}\mathscr{D}}A_{\mathscr{C},\mathscr{D}}g(\mathscr{D},\mathscr{F})f(\mathscr{F})\\
&=\sum_{\mathscr{F}} A_{\mathscr{C},\mathscr{A}}g(\mathscr{A},\mathscr{F})f(\mathscr{F})=0\end{aligned}$$

unless $\mathscr{C}\geq\mathscr{A}$. This argument is taken from Eq. (6.20) in Ref. 9. The proof in Ref. 10 makes use of the free Abelian group generated by the graphs in J.

Now let $(W_{\mathscr{A}})_{\mathscr{A}\in\mathscr{P}(J)}$ be a family of unitary operators on a Hilbert space H. We define $\ln W_{\mathscr{B}}=i\arg W_{\mathscr{B}}$ in the sense of the functional calculus of normal operators, where

$$\arg\text{: } \{\omega\in C\mid |\omega|=1\}\to(-\pi,+\pi],\ \ e^{i\varphi}\to\varphi \quad (-\pi<\varphi\leq\pi)\ .$$

Since the operators $\arg W_{\mathscr{B}}$ are bounded ($\|\arg W_{\mathscr{B}}\|\leq\pi$) and self-adjoint, the definition

$$\prod_{\mathscr{A}>\mathscr{B}}^{c} W_{\mathscr{B}}\equiv\exp\left[i\sum_{\mathscr{A}>\mathscr{B}}^{ac}\arg W_{\mathscr{B}}\right]$$

makes sense and yields a unitary operator. A bounded operator that commutes with all $W_{\mathscr{B}}$, $\mathscr{B}<\mathscr{A}$, obviously commutes also with $\prod^{c}_{\mathscr{A}>\mathscr{B}} W_{\mathscr{B}}$. Further the construction commutes with unitary transformations:

$$U\prod_{\mathscr{A}>\mathscr{B}}^{c} W_{\mathscr{B}}U^{-1}=\prod_{\mathscr{A}>\mathscr{B}}^{c} UW_{\mathscr{B}}U^{-1}\ .$$

We have for $\mathscr{A}\wedge\mathscr{C}<\mathscr{A}$ (i.e., unless $\mathscr{C}\geq\mathscr{A}$)

$$\begin{aligned}\prod_{\mathscr{A}>\mathscr{B}}^{c} W_{\mathscr{B}\wedge\mathscr{C}}&=\exp\left[i\sum_{\mathscr{A}>\mathscr{B}}^{ac}\arg W_{\mathscr{B}\wedge\mathscr{C}}\right]\\
&=\exp(i\arg W_{\mathscr{A}\wedge\mathscr{C}})\\
&=W_{\mathscr{A}\wedge\mathscr{C}}\ .\end{aligned}\tag{B5}$$

Instead of the function arg and its inverse $\exp(i\,\cdot)$, Coester and Polyzou[9] use the bijective function

$$\text{Caley: } \{\omega\in C\mid |\omega|=1\}\to(-\infty,+\infty],\ \ \omega\to\tan(\tfrac{1}{2}\arg\omega)\ [=i(1-\omega)(1+\omega)^{-1}\ \text{ if } \omega\neq-1]$$

and its inverse. Then, however, $\sum^{ac}_{\mathscr{A}>\mathscr{B}}\text{Caley}(W_{\mathscr{B}})$ is a sum of unbounded operators, which is not necessarily self-adjoint, so that the unitarity of $\prod^{c}_{\mathscr{A}>\mathscr{B}}W_{\mathscr{B}}$ is not obvious. If we would define the sum (4.11) of mass operators via (B1) [as it was done, for the sake of simplicity, in (4.29)] we would have a similar problem. This difficulty can be avoided by regularizing. Comments on this method are given in Sec. I. We choose a large cutoff mass m_c (e.g., $m_c=10^6\max\{m_j\mid j\in J\}$, a small positive auxiliary mass m_0 (e.g., $m_0=10^{-6}\min\{m_j\mid j\in J\}$), and a bijective function

$$\lambda\text{: } (0,\infty)\to(0,m_c)$$

such that $\lambda(x)\approx x$ for $x\ll m_c$. For instance,

$$\lambda(x)=\frac{2}{\pi}m_c\arctan\frac{\pi x}{2m_c}\ .$$

Further, we choose a function

$$\mu\text{: } (-\infty,+\infty)\to(0,\infty)\ \text{ such that } \mu(\lambda(x))=x$$

for all $x\in(0,\infty)$. For instance,

$$\mu(y)=\begin{cases}|y| & \text{for } |y|\geq m_c,\\ \lambda^{-1}(|y|) & \text{for } 0<|y|<m_c,\\ m_0 & \text{for } y=0.\end{cases}$$

Now let $(A_{\mathscr{A}})_{\mathscr{A}\in\mathscr{P}(J)}$ be a family of strictly positive operators on H. Then we put

$$\sum_{\mathscr{A}>\mathscr{B}}^{c} A_{\mathscr{B}} \equiv \mu\left[\sum_{\mathscr{A}>\mathscr{B}}^{\mathrm{ac}} \lambda(A_{\mathscr{B}})\right].$$

We easily see that this is again a strictly positive operator. For any unitary operator U in H we have

$$U \sum_{\mathscr{A}>\mathscr{B}}^{c} A_{\mathscr{B}} U^{-1} = \sum_{\mathscr{A}>\mathscr{B}}^{c} U A_{\mathscr{B}} U^{-1}.$$

Further, $\sum_{\mathscr{A}>\mathscr{B}}^{c} A_{\mathscr{B}}$ commutes with all those bounded operators that commute with all $A_{\mathscr{B}}$'s. Finally, we have for $\mathscr{A}\wedge\mathscr{C}<\mathscr{A}$

$$\sum_{\mathscr{A}>\mathscr{B}}^{c} A_{\mathscr{B}\wedge\mathscr{C}} = \mu\left[\sum_{\mathscr{A}>\mathscr{B}}^{\mathrm{ac}} \lambda(A_{\mathscr{B}\wedge\mathscr{C}})\right] = \mu(\lambda(A_{\mathscr{A}\wedge\mathscr{C}})) = A_{\mathscr{A}\wedge\mathscr{C}}. \quad \text{(B6)}$$

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