Is the usual notion of time evolution adequate for quantum-mechanical systems? I

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Circumstances are described wherein no state at a given time (nor any definite evolution from one time to another) can be ascribed to a given physical system, but wherein the system can nonetheless be associated with definite dispersion-free values of a new sort of observable, which we call a "multiple-time" observable. The description of physical systems in terms of these new observables is discussed. It emerges as a by-product of our work that no experiment whatever (albeit that its result is certain) can be carried out on a system without disturbing the values of other measurable quantities.

I. INTRODUCTION

This work concerns prediction and retrodiction and (more importantly) combinations and superpositions of these, and arose in the context of an undertaking to make sense of the measurement process in relativistic quantum theory (wherein the distinction between prediction and retrodiction is not, in general, frame invariant; but the detailed relation of this work to *that* subject will be taken up in the paper which follows this one[1]). These considerations, it turns out, afford novel and useful lessons about the nature of time, and of the description of physical systems in terms of states, in quantum mechanics. We find that the language of the dynamical evolution of systems from one time to another through some definite succession of physical states is too narrow. Circumstances can be envisaged (and can, indeed, be prepared by suitable experiments) wherein no quantum state (and consequently no notion of evolution from one state to another) can be associated with a system and wherein the system can nonetheless be associated with definite and dispersion-free values of various combinations of different observables measured at various different times (which hereafter we shall refer to as multiple-time observables).[2] The values of these multiple-time observables constitute a description of the system which is internally complete, and which is complementary to, and incompatible with, a description in terms of states. We shall as well describe a new and very general phenomenon of measurement: No experiment whatever (albeit that its result is certain) can leave a physical system undisturbed in any measurable way; no experiment can leave all else that is certain about the system unchanged.

II. PREDICTION AND RETRODICTION AND THEIR COMBINATION AND SUPERPOSITION

Consider, to begin with, prediction and retrodiction in classical physics. Herein things are comparatively simple: The complete specification of the conditions of a closed physical system at any time (t_0, say) will serve to determine completely (via the equations of motion) the results of *any* measurements on the system carried out at *any* times, both before and after t_0. Measurements other than the one at t_0 are in a certain sense redundant. Their results are deducible; they can in principle yield no additional information about the system (nor about its past nor its future).

The quantum-mechanical situation is different in an essential way. Suppose, for example, that a particle is measured to be in the state

$$|\alpha\rangle \equiv |x_1\rangle + |x_2\rangle \tag{1}$$

(wherein $|x_1\rangle$ is the state of a particle localized at the point x_1, etc.) at t_i, and that no information concerning the results of any other measurements on the system is given. At t_1 ($t_1 > t_i$), assuming that the particle is undisturbed in the interval $t_i < t < t_1$, the measurement of an observable A of which $|\alpha\rangle$ is an eigenstate will yield a definite value, whereas a measurement of the position may yield either x_1 or x_2 (herein we assume that both A and X are constants of the motion; that, for example, there are small impenetrable boxes at x_1 and x_2 wherein the particle can be confined). Suppose, on the other hand, that we are now informed that a measurement of the position at time t_f ($t_f > t_1 > t_i$) yields the value x_1; then it will still be the case that measurements of A will yield a single definite value, but now it will also be true that any measurement at t_1 of X will with certainty yield[3] $X = x_1$. If (more generally) a system is measured at t_i to be in the state $|A=a\rangle$ and at t_f to be in the state $|B=b\rangle$, then the probability that at t_1 it will be found to be in the state $|C=c_n\rangle$ (where $|c_1\rangle, \ldots, |c_m\rangle$ form a complete basis for the Hilbert space, and assuming for simplicity that A, B, and C are all constants of the motion) is given by[4]

$$P(c_n) = \frac{|\langle a | c_n\rangle|^2 |\langle c_n | b\rangle|^2}{\sum_{i=1}^{m} |\langle a | c_i\rangle|^2 |\langle c_i | b\rangle|^2} . \tag{2}$$

Notwithstanding the fact that the state was completely determined by the measurement at t_i, then, the measurement at t_f produces additional information about the system at t_1.

The results of the measurements at t_i and t_f do not here (as they do in the classical case) determine one another; therefore each one independently augments our knowledge of the system at t_1. This (in the present context, at least) is the meaning of the uncertainty relations: with each new measurement we learn quantitatively *more* of the system. It is in a certain sense (a limited sense, more of which will be said later) as if with each new measurement (all of which are complete, but each of which necessarily occurs at a different time) we sample another degree of freedom of the system, of which there are in principle an infinite supply.

Suppose that at t_1 we measure A (where $A\,|\alpha\rangle = a\,|\alpha\rangle$), and that at t_2 we measure X ($t_i < t_1 < t_2 < t_f$); we will with certainty find both that $A = a$ and that $X = x_1$. Suppose (Fig. 1) that we measure *either* X at t_1 *or* A at t_2; we will with certainty find if we measure X that $X = x_1$, and if we measure A that $A = a$. If, on the other hand, we measure *both* X at t_1 and A at t_2, the results of both experiments will be indefinite. This is an instance of a very general phenomenon (of which mention was made in the Introduction, and which shall assume a role of considerable importance with reference to relativistic questions): Two measurements on the same system may each individually be associated with definite dispersion-free values and may nonetheless be incompatible; for such a system *no* experiment whatever can verify *all* that we know with certainty to be true, nor can any experiment leave all that we know with certainty to be true undisturbed.

A succession of complete measurements on a system, then, defines a trajectory for that system, wherein there is in general more information about the system than is discernible within a quantum state (more, that is, that can be discovered by means of any single complete measurement); and it is natural to inquire whether (as with states) one can meaningfully speak of linear superpositions of such trajectories. It turns out that this can be done; in particular, circumstances can arise wherein it becomes nonsensical to attribute any state to a given system at a given time, and wherein nonetheless the system is associated with well-defined dispersion-free values of (for example) sums of various different observables at different times (in much the same way, say, as it is nonsensical to associate a pair of particles in an eigenstate of the observables $x_1 - x_2$ and $p_1 + p_2$, with any product of one-particle states; nontheless the particles *are* associated with well-defined dispersion-free values of $x_1 - x_2$ and $p_1 + p_2$).

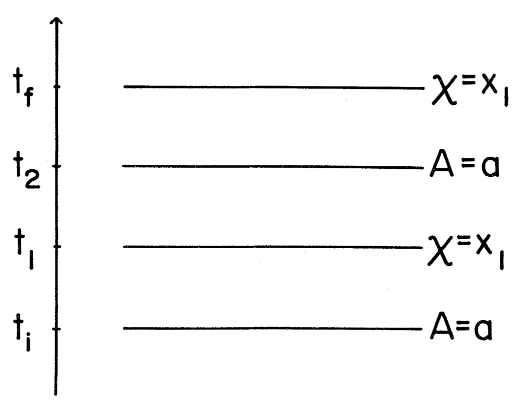

FIG. 1. A is measured at t_i and t_2, and x at t_1 and t_f.

The measurement of such (multiple-time) observables requires specialized experimental procedures.[5] Consider, to begin with, an experimental device designed to measure a single-time observable A, as follows: The device interacts with the system to be measured through a term in the Hamiltonian of the form

$$H_{\rm int} = g(t) q A \;, \tag{3}$$

wherein q is some internal variable of the apparatus, and $g(t)$ is a coupling which is nonzero only during a short interval $t_1 < t < t_f$, when the device is switched on. Then, in the Heisenberg picture,

$$\frac{\partial \pi}{\partial t} = -g(t) A \;, \tag{4}$$

where π is the momentum canonically conjugate to q, and so if we consider a short interval ($t_1 \cong t_f$) wherein A is approximately a constant, then we have

$$A = \frac{\pi(t < t_i) - \pi(t > t_f)}{\int_{t_i}^{t_f} dt\, g(t)} \tag{5}$$

and this is how the device is used to measure A.

Now consider two such devices, which interact with the system through the Hamiltonian

$$H_{\rm int} = g_1(t) q_1 A + q_2(t) q_2 B \;. \tag{6}$$

Our intent here is to design some combinations of these devices which will collectively measure $A + B$ *without* measuring either A or B individually. This can be accomplished as follows: Imagine that at some time prior to t_{i_1} and t_{i_2} we prepare the two devices in an initial state with the properties

$$\pi_1 + \pi_2 = 0, \quad q_1 - q_2 = 0 \;. \tag{7}$$

Then we allow the devices to interact with the sytem. When the interaction is over, the devices will have measured A (at $t_{i_1} \cong t_{f_1}$) *plus* B (at $t_{i_2} \cong t_{f_2}$), that is,

$$A(t_{f_1}) + B(t_{f_2}) = \frac{-[\pi_1(t_f) + \pi_2(t_f)]}{\int dt\, g(t)} \tag{8}$$

(wherein we have assumed that

$$\left. \int_{t_{i_1}}^{t_{f_i}} dt\, g_1(t) = \int_{t_{i_2}}^{t_{f_2}} dt\, g_2(t) \equiv \int dt\, g(t) \right) .$$

On the other hand, they will *not* have measured A or B or $A - B$. In order to measure, say, A, we need to know $\pi_1(t_{i_1}) - \pi_1(t_{f_1})$; however, π_1 does not commute with

q_1-q_2, and hence is not well defined in the state (7) in which the devices were initially prepared. No measurement of A has occurred; that is, no information about A can be discerned from the devices. Similarly,

$$[\pi_2,q_1-q_2]\neq 0,\ \ [\pi_1-\pi_2,q_2-q_2]\neq 0 \tag{9}$$

so no measurement of B nor of $A-B$ has occurred either.

Suppose that (by the procedure we have just described) a measurement of the two-time observable

$$\sigma_z(t_i)+\sigma_x(t_f)\equiv\sigma_{zx}(t_i,t_f) \tag{10}$$

is carried out on a spin-$\frac{1}{2}$ particle whereby it emerges, say, that $\sigma_{zx}(t_i,t_f)=0$ (see Fig. 2). For times $t_i\leq t\leq t_f$, the particle is in *no* particular quantum state (that is, it can be associated with *no* definite values of any complete set of observables; it is, strictly speaking in a mixture of states, in what Everett and Wheeler[6] call a relative state: its state is correlated to that of the apparatus). Nevertheless it can be said with certainty of this system that any measurement of, say, $\sigma_{zx}(t_1,t_2)$ (where $t_i<t_1<t_2<t_f$) will yield $\sigma_{zx}(t_1,t_2)=0$ (and the same can be said of each of an infinite family of observables of the general form σ_{zx}, within this interval).

It ought to be born in mind that multiple-time measurements of the sort we have just considered are *not* collections of one-time measurements [such as we have treated in Eq. (2) and what surrounds it], but something altogther different. A measurement of $\sigma_{x,z}(t_1,t_2)$ entails neither a measurement of σ_x at t_1 nor one of σ_z at t_2; indeed, it is [in the manner of Eq. (9)] incompatible with the latter.

Consider now, more generally, how probabilities associated with given experimental results are to be calculated under such circumstances. The system is not in any state; if, then, we are to make calculations involving states we must enlarge our dynamical system to the point where this language makes sense: we must, that is, include the measuring apparatus in that system. States do not afford here a language (such as exists, via the collapse of the wave function, for single-time observables) wherein we need only refer to the system itself in describing the results of experiments carried out on it. Another language exists, however; that of paths (in the manner of Feynman), and *this* (in the present circumstances) is more powerful; this affords us what we are seeking.

Suppose that a spin-$\frac{1}{2}$ particle is known to be in the state $|\sigma_x=\frac{1}{2}\rangle\equiv|\frac{1}{2}\rangle_x$ at time t_i, and in the state $|\frac{1}{2}\rangle_z$ at t_f, and that in addition $\sigma_{zx}(t_1,t_4)=0$. We wish to inquire about the probability that a measurement of, say $\sigma_{xz}(t_2,t_3)(t_i<t_1<t_2<t_3<t_4<t_f)$ will yield $\sigma_{xz}(t_2,t_3)=0$ (see Fig. 3). We are instructed by the formalism to sum the functional $\exp\{iS[\text{path}]\}$ over all paths satisfying the constraints above, to take the absolute square of this sum (which gives the probability that, given the circumstances at t_0, all that is stipulated about the future above will come to pass), and then to divide this probability by the sum of those of all possible outcomes of the measurement at (t_2,t_3). Neglecting the interaction of the measuring apparatus with the system (of which we shall speak presently), the sum over all paths in the present case reduces to a sum over four amplitudes, corresponding to the four separate trajectories which can contribute to the result $\sigma_{xz}(t_2,t_3)=0$. These are

$$\begin{aligned}
&{}_x\langle\tfrac{1}{2}|-\tfrac{1}{2}\rangle_{z\,z}\langle-\tfrac{1}{2}|-\tfrac{1}{2}\rangle_{x\,x}\langle-\tfrac{1}{2}|\tfrac{1}{2}\rangle_{z\,z}\langle\tfrac{1}{2}|\tfrac{1}{2}\rangle_{x\,x}\langle\tfrac{1}{2}|\tfrac{1}{2}\rangle_z\,,\\
&{}_x\langle\tfrac{1}{2}|\tfrac{1}{2}\rangle_{z\,z}\langle\tfrac{1}{2}|-\tfrac{1}{2}\rangle_{x\,x}\langle-\tfrac{1}{2}|\tfrac{1}{2}\rangle_{z\,z}\langle\tfrac{1}{2}|-\tfrac{1}{2}\rangle_{x\,x}\langle-\tfrac{1}{2}|\tfrac{1}{2}\rangle_z\,,\\
&{}_x\langle\tfrac{1}{2}|\tfrac{1}{2}\rangle_{z\,z}\langle\tfrac{1}{2}|\tfrac{1}{2}\rangle_{x\,x}\langle\tfrac{1}{2}|-\tfrac{1}{2}\rangle_{z\,z}\langle-\tfrac{1}{2}|-\tfrac{1}{2}\rangle_{x\,x}\langle-\tfrac{1}{2}|\tfrac{1}{2}\rangle_z\,,\\
&{}_x\langle\tfrac{1}{2}|-\tfrac{1}{2}\rangle_{z\,z}\langle-\tfrac{1}{2}|\tfrac{1}{2}\rangle_{x\,x}\langle\tfrac{1}{2}|-\tfrac{1}{2}\rangle_{z\,z}\langle-\tfrac{1}{2}|\tfrac{1}{2}\rangle_{x\,x}\langle\tfrac{1}{2}|\tfrac{1}{2}\rangle_z\,.
\end{aligned} \tag{11}$$

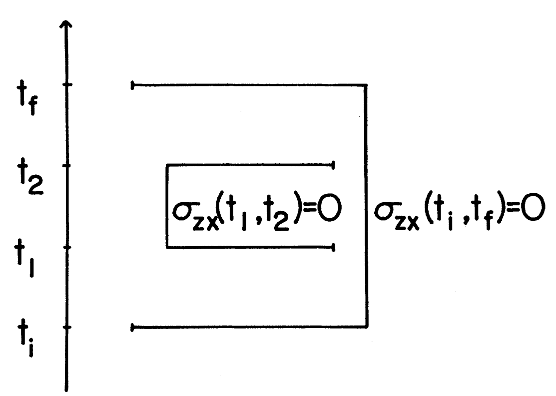

FIG. 2. Two-time measurements are carried out at (t_i,t_f) and (t_1,t_2).

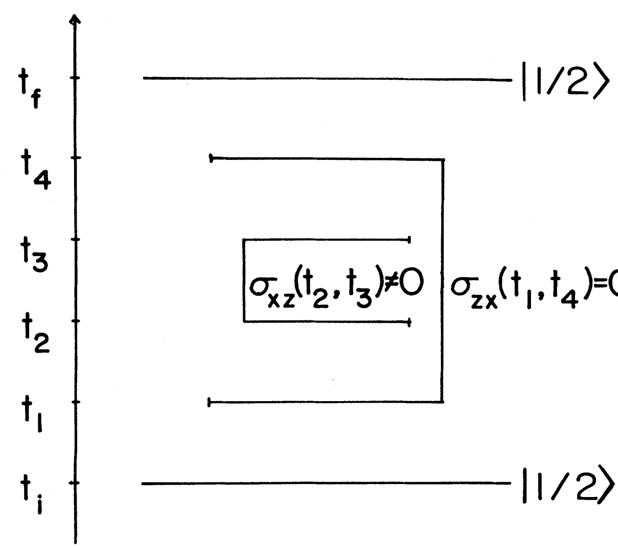

FIG. 3. σ_x is measured at t_i, σ_z at t_f, and two-time measurements are carried out at (t_1,t_4) and (t_2,t_3).

The interaction with the measuring apparatus is, on each of these four trajectories, precisely the same, since each trajectory interacts throughout with an identical q [in the manner of Eqs. (6) and (7), wherein q_1 and q_2 are taken to be constants of the motion] and the two-time variable of the system with which q interacts is identical in each case (each q, that is, interacts by the end of the process with $\sigma_x+\sigma_z=0$). The consequences of the interaction with the apparatus are therefore the same on each trajectory; the interaction merely produces an overall phase in the final amplitude (albeit an uncertain phase); it can, then, for our purposes, be entirely ignored.

The correct amplitude (up to a phase), then, is the sum of the four amplitudes, which, as it happens (since the first and third are negative and the second and fourth are positive), vanishes. The same result can be obtained within the language of states, by incorporating the measuring apparatus into the picture (which is a very cumbersome procedure); here, however, the language [in the manner of Eq. (11)] refers *exclusively to* the spin system itself.

So it emerges even in the nonrelativistic theory that the formalism of paths is broader and more powerful, in this sense, at least, than that of states. In such circumstances as we have considered here, no state at a given time, (nor, therefore, any notion of evolution from one time to another) can be associated with the system in question. Furthermore, a description of the system in terms of a density matrix (as this, too, is a description *at a given time)* will predict merely that the one-time observables of the system are uncertain. Nonetheless, the path formalism applies to such circumstances in a very natural way, and leads to a description of such systems in terms of *dispersion-free* observables: multiple-time observables. Such a description can be obtained as well within the conventional formalism, if the described system is enlarged to include the measuring apparatus. But what we see here (and it is precisely here that the path language can do more than the languages of states or statistical density operators) is that such an enlargement is unnecessary, that the multiple-time properties are intrinsic properties of the system itself, and that a language exists wherein they can be described without reference to the measuring apparatus, albeit this is impossible within the language of states or density matrices. It may be that some generalized state language is possible, a language of multiple-time states which are eigenstates of multiple-time observables, wherein one might accomplish the same thing; we should like, in the future, to study this question more deeply.

And here again it emerges that no experiment whatsoever can be constructed (albeit its result is certain) which leaves all else that is with certainty the case unaltered. Given (as above), for example, that $\sigma_{zx}(t_i,t_f)=0$, it follows both that any measurement of $\sigma_{zx}(t_1,t_2)$ will with certainty yield zero, and that any measurement of $\sigma_{zx}(t_3,t_4)$, $(t_i<t_1<t_2)$, and $(t_3<t_4<t_f)$, will certainly yield zero, and nonetheless if *both* measurements are carried out on the same system, the results of both will be uncertain.

These considerations stand at present at their extreme beginnings; let us in closing indicate along what lines they may be expected to develop. The trajectories in (11), according to the familiar manner of Feynman paths, all begin and end with the same single-time states ($|\frac{1}{2}\rangle_x$ and $|\frac{1}{2}\rangle_z$, respectively). There is in this formalism, then, some vestige of the notion of one-time states, and this (in the present case, at least) can be done away with, as follows. It turns out (in this instance) that as long as the initial x spin and the final z spin have the same sign [as they do in (11)], $\sigma_{xz}(t_2,t_3)$ cannot vanish. If they have opposite signs, on the other hand, it can. If it is found, then, that $\sigma_{zx}(t_1,t_4)=0$ and $\sigma_{xz}(t_2,t_3)=0$, it *must* be that either $\sigma_x(t_i)=+\frac{1}{2}$ and $\sigma_z(t_f)=-\frac{1}{2}$ *or* $\sigma_x(t_i)=-\frac{1}{2}$ and $\sigma_z(t_f)=+\frac{1}{2}$ and this (it can easily be seen) is true entirely irrespective of what experiments may or may not occur within the interval $t_2<t<t_3$ [in (11), that interval is free of any such experiments, for the sake of simplicity]. Consider the amplitude for such a process (wherein we leave the experiments in the innermost interval unspecified):

$$ {}_x\langle \pm\tfrac{1}{2} | -\tfrac{1}{2}\rangle_z \, [{}_z\langle -\tfrac{1}{2} | \cdots | \tfrac{1}{2}\rangle_x] \, {}_x\langle \tfrac{1}{2} | \mp\tfrac{1}{2}\rangle_z \tag{12a} $$

$$ +{}_x\langle \pm\tfrac{1}{2} | \tfrac{1}{2}\rangle_z \, [{}_z\langle \tfrac{1}{2} | \cdots | -\tfrac{1}{2}\rangle_x] \, {}_x\langle -\tfrac{1}{2} | \mp\tfrac{1}{2}\rangle_z \, . \tag{12b} $$

(12) consists of two main terms [(a) and (b) which themselves each consist of some unspecified, but necessarily equal, number of terms] corresponding to the two ways whereby it can come to pass that $\sigma_{zx}(t_1,t_4)=0$. The crucial feature of (12) is that whatever our initial state is $|+\frac{1}{2}\rangle_x$ *or* $|-\frac{1}{2}\rangle_x$, there is a relative negative sign between (12a) and (12b). Given then that $\sigma_{xz}(t_2,t_3)=0$ and $\sigma_{zx}(t_1,t_4)=0$, we require no information whatsoever concerning the one-time states at t_i and t_f in order to evaluate any amplitude of the form (12), since, in any case, everything outside the square brackets serves merely to impose a relative minus sign between (12a) and (12b). Up to a phase (and a normalization factor), then, (12) can be replaced by

$$ \begin{aligned} &{}_z\langle -\tfrac{1}{2} | \cdots | \tfrac{1}{2}\rangle_x \\ &-{}_z\langle \tfrac{1}{2} | \cdots | -\tfrac{1}{2}\rangle_x \end{aligned} \tag{13} $$

wherein two trajectories are superimposed from beginning to end, and interfere in a new and unfamiliar way, without even having their initial or final states in common, and with a definite relative phase. The behavior of such systems, then, can be defined entirely in terms of the results of multiple-time experiments (they need not be two-time experiments; they may be 3-time or 4-time or N-time experiments) and these, it is hoped, can constitute a rich and entirely complementary formalism to that of states and of ordinary time evolution.

ACKNOWLEDGMENTS

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[1]Y. Aharonov and D. Z. Albert, following paper, Phys. Rev. D 29, 228 (1984).

[2]Combinations of this type are of course not unknown in quantum field theory. The averaging of observables over finite regions of space-time is a common practice there; but, as the reader will discover, we shall be thinking of such combinations here for different reasons, and in a different way.

[3]That this cannot be known to an observer until after t_f is irrelevant; the fact remains that such an observer, albeit after t_f, can say with certainty that *any* experiment of t_1, whether or not he is aware of one having been carried out, must have yielded $X = x_1$.

[4]Y. Aharonov *et al.*, Phys. Rev. 134, B1410 (1969).

[5]A detailed consideration of procedures roughly of this type appears in Y. Aharonov and D. Albert, Phys. Rev. D 24, 359 (1981).

[6]H. Everett, Rev. Mod. Phys. 29, 454 (1957); J. Wheeler, *ibid.* 29, 463 (1957).