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Can a resonant-mass gravitational-wave detector have wideband sensitivity?

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In principle, for an impulsive gravitational-wave signal, the signal-to-noise ratio for a resonant-mass antenna is independent of frequency if the limiting noise source is the antenna thermal noise. A gain-bandwidth restriction only arises when the coupling of the resonant antenna to the output amplifier is considered. Applying Bode's gain-bandwidth theorem we are able to derive in a general way the linear amplifier sensitivity limit for a resonant-mass detector. The practical bandwidth limits for a detector utilizing a superconducting inductive transducer are discussed. A fractional bandwidth of ~ 0.17 appears feasible with current technology. Additional bandwidth and sensitivity can be achieved with an array of detectors.

I. INTRODUCTION

It is generally recognized[1] that gravitational-wave astronomy has two broad scientific goals: (i) to verify directly the existence of gravitational radiation and (ii) to use gravitational radiation as a tool for astronomical observation. The first goal requires detectors of high sensitivity while the second goal imposes the additional requirement that the detector has sufficient bandwidth to allow detailed study of the received waveforms. This will yield unique information about the coherent bulk motions of the matter generating the radiation. In this regard gravitational waves can be an important tool for astronomical observation because "they can reveal features of their sources which no one could ever learn by electromagnetic, cosmic-ray, or neutrino studies."[2]

Since Weber's pioneering attempts to detect gravitational waves, and the subsequent program of observations with room-temperature, resonant-mass antennas,[3] there has been a steady effort to develop various kinds of improved detectors. These efforts have focused primarily on cryogenic resonant-mass detectors and free-mass laser interferometer detectors. In this paper we are concerned primarily with some issues of principle about the bandwidth and sensitivity of resonant-mass detectors.

A common form of resonant-mass detector, first developed by Weber,[4] consists of a solid right cylindrical bar with a fundamental longitudinal resonance f_0 around 1 kHz, where the signal spectral energy density is expected to be largest[1,2] A strain or motion detector monitors the dynamic strain induced in the fundamental mode by interaction with the gravitational wave. Because of the forces responsible for the bar's elasticity, the gravitational wave does work and thus deposits energy in the odd-order longitudinal modes. For a short-duration burst signal, the excitation will appear as a sudden ringing of the bar.

Because the resonant frequency of the bar is determined by the velocity of sound in the material used, which is much less than the velocity of gravitational waves, the size of a resonant-mass antenna is always much smaller than the radiation wavelength. This large mismatch accounts, in part, for the relative insensitivity of these detectors when compared with radio antennas. Long-baseline laser interferometers have been proposed as an alternative which avoids this limitation.[5,6] Although excellent progress is being made in the development of such optical-readout detectors, they have not yet reached the level of sensitivity achieved by resonant-mass detectors. The best sensitivity[7] reported to date was achieved by a 4800-kg resonant-mass antenna, operated at 4 K, with an rms noise level for pulse detection corresponding to a dimensionless antenna strain of 10^{-18}. However the bandwidth of this detector was limited to about 2 Hz.

If bar detectors are to be useful for eventually studying received waveforms both their sensitivity and bandwidth must be improved. At first it might appear that a resonant-mass antenna is particularly unsuited for use as a broadband detector. Indeed it has been stated that bar detectors "might be able to detect cosmic signals but would probably miss most of the details of any waves except those from periodic sources.[1] While this statement is correct for the present generation of bar detectors it is incorrect to assume that the high-Q, resonant nature of these detectors fundamentally limits their bandwidth. As

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will be seen below, a bandwidth restriction only arises when one considers the coupling of the resonant antenna to the output amplifier.

II. RESONANT-MASS DETECTORS

A resonant-mass detector is shown schematically in Fig. 1(a). The mechanical oscillations of the antenna are transformed into an electrical signal by a motion transducer and then amplified by an electrical amplifier. The overall sensitivity and useful bandwidth of the detector depend on both the noise characteristics of the detector and the spectral character of the signals. The principal sources of noise are antenna and transducer thermal noise and amplifier noise. We assume that the transducer and amplifier are both linear.

The amplifier noise can be characterized by a series voltage noise source with spectral density S_v, a shunt current noise source with spectral density S_i, followed by a noise-free current amplifier. The amplifier noise temperature $T_{\rm amp}$ is given by $(S_v S_i)^{1/2}/k_B$ and the noise match impedance is defined as $R_{\rm opt}=(S_v/S_i)^{1/2}$.

The antenna thermal noise is characterized by the spectral density of the Nyquist force noise applied to the antenna given by

$$S_f(\omega)=2k_B T M_a \omega_a / Q_a \ , \tag{1}$$

where T, M_a, ω_a, and Q_a are the antenna's temperature, effective mass, resonant frequency, and mechanical quality factor, respectively. The thermal noise sources associated with the transducer can be described in a similar way. Notice that $S_f(\omega)$ is a white noise source (i.e., independent of frequency). An impulsive gravitational-wave signal applied to the antenna also has a white spectrum. Thus *the signal-to-noise ratio (SNR) is independent of frequency if the limiting noise source is the antenna thermal noise.*

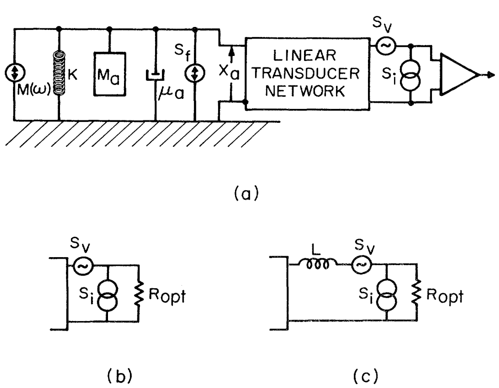

FIG. 1. (a) Electromechanical model of a single mode of a resonant-mass gravitational radiation detector. The gravitational wave signal is represented by the driving source $M(\omega)$. The signal as well as the random noise sources present result in a displacement X_a which is converted by the transducer to an electrical signal sensed by the output amplifier. (b) For the purpose of calculating the SNR, the ideal noiseless amplifier is replaced by a noiseless resistor with value $R_{\rm opt}$. (c) In the case of a SQUID amplifier there is an inductance L associated with the amplifier input.

In general, given the form of the signal, the overall SNR is maximized by means of an optimum linear filter[8] applied to the detector's output. The optimum SNR that can be achieved is

$$S/N=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\,|M(\omega)|^2/S_n(\omega)]d\omega \ , \tag{2}$$

where $M(\omega)$ is the Fourier transform of the expected signal and $S_n(\omega)$ is the noise power spectral density. The integrand in Eq. (2) is the SNR per unit bandwidth. The signal and the noise can be referred to any point in the detector that is convenient. For an impulsive gravitational-wave signal referred to the antenna input we find that

$$|M(\omega)|^2=2\epsilon M_a \ , \tag{3}$$

where ϵ is the energy that the wave would deposit in the antenna initially at rest. A common figure of merit is the detector noise temperature for pulse detection T_d, defined so that $\rho_0=1$ for $\epsilon=k_B T_d$.

III. LINEAR AMPLIFIER SENSITIVITY LIMIT

It is illustrative to first consider Eq. (2) in the zero-temperature limit where the amplifier noise sources dominate. In this situation we can place an upper limit on the integral in (2) for the case of an arbitrary linear transducer coupling the antenna and the amplifier. In order to accomplish this, we will first express the SNR in a rather unorthodox way.

First, note that if we replace the ideal noiseless amplifier in Fig. 1(a) by a noiseless resistor, the SNR of the system is unaffected. For analytic convenience we let this resistance equal $R_{\rm opt}$. Having transformed the problem in this way we can now write the SNR in terms of the reflection coefficient $\rho(\omega)$, between $R_{\rm opt}$ and the impedance it faces. If $Z(\omega)$ is the complex impedance looking away from $R_{\rm opt}$ as shown in Fig. 1(b), then

$$\rho=\frac{R_{\rm opt}-Z}{R_{\rm opt}+Z} \ . \tag{4}$$

If we ignore dissipation in the transducer but include the antenna dissipation, the system will consist of a resistive signal source coupled by a linear lossless transmission network to a resistive load. By using reciprocity, it can be shown that the magnitude of ρ is the same when evaluated at the source end as it is at the load end. The inclusion of a transition from mechanical force to electrical voltage within the transmission system does not alter this fact. To evaluate ρ at the source side, $R_{\rm opt}$ is replaced with the value of the mechanical dissipation in the antenna and Z becomes the complex mechanical impedance seen by that dissipation.

A force with Fourier transform $F(\omega)$ applied to the antenna will dissipate power in $R_{\rm opt}$ given by

$$P_{\rm sig}(\omega)=\frac{|F(\omega)|^2 Q_a}{4\omega_a M_a}(1-|\rho|^2) \ , \tag{5}$$

where Q_a and ω_a are both evaluated before the transducer is attached to the antenna. This expression is derived from the definition of ρ at the antenna side of the network. The real part of the mechanical impedance seen by the antenna can only be attributed to $R_{\rm opt}$.

Provided that the amplifier current noise and voltage noise are uncorrelated, the amplifier noise power in $R_{\rm opt}$ can be written as

$$P_n(\omega)=R_{\rm opt}\frac{(S_V+S_I\,|Z\,|^2)}{|R_{\rm opt}+Z\,|^2} . \tag{6}$$

Using the appropriate definitions, this becomes

$$P_n(\omega)=\tfrac{1}{2}k_B T_{\rm amp}(1+|\rho\,|^2) . \tag{7}$$

The SNR can now be obtained from the ratio of signal power to noise power in $R_{\rm opt}$,

$$S/N=\frac{1}{2\pi}\frac{\epsilon Q_a}{\omega_a k_B T_{\rm amp}}\int_{-\infty}^{\infty}\frac{1-|\rho\,|^2}{1+|\rho\,|^2}d\omega . \tag{8}$$

Equation (3) has allowed us to substitute $2\epsilon M_a$ for $F(\omega)^2$ and remove it from within the integral. The gain-bandwidth theorem[9] provides the very useful limit

$$\int_{-\infty}^{\infty}\ln\frac{1}{|\rho|}d\omega\le\frac{2\pi\omega_a}{Q_a} . \tag{9}$$

Expressed in this way, the theorem will be directly useful to our present analysis, but we have obscured the fact that the application of this theorem follows from the unavoidable reactive impedance of the antenna inertial mass. Naturally, the equivalence principle tells us that this cannot be removed from the problem.

A direct power-series expansion of $\ln(1/|\rho\,|)$ in $(1-|\rho\,|^2)/(1+|\rho\,|^2)$ produces

$$\int_{-\infty}^{\infty}\frac{1-|\rho\,|^2}{1+|\rho\,|^2}d\omega\le\int_{-\infty}^{\infty}\ln\frac{1}{|\rho|}\le\frac{2\pi\omega_a}{Q_a} . \tag{10}$$

Finally, we can write

$$S/N=\frac{\epsilon}{k_B T_{\rm amp}} . \tag{11}$$

This is the same result as that first given by Giffard. The apparent difference by a factor of 2 is due to the fact that we have expressed the signal-to-noise ratio in terms of the mean square of the noise process rather than the mean square of the envelope of the noise process. The derivation given here allows for an arbitrary lossless linear transducer and rules out any improvement over Giffard's result based on more elaborate matching techniques than were included in his analysis. In fact, this derivation does not assume that either the antenna or the transducer is a resonant device.

Equation (9) involves the reflection coefficient over the entire frequency spectrum. In a practical detector, however, the response will be limited to a finite frequency range $\delta\omega$. A useful design objective might be to optimize a given transducer design so that $\ln(1/|\rho\,|)$ is a maximum over the prescribed frequency band. From Eq. (9) it is apparent that this can be achieved if $\ln(1/|\rho\,|)$ is zero outside the frequency band. In the case that $|\rho|$ is constant within the band we find that

$$|\rho\,|\ge\exp(-\pi\omega_a/\delta\omega Q_a) \tag{12}$$

over the bandwidth $\delta\omega$ and $|\rho\,|=1$ outside this band. With these restrictive conditions the limit (11) is replaced by

$$S/N\le\left[\frac{\epsilon}{k_B T_{\rm amp}}\right]\left[\frac{\delta\omega Q_a}{\pi\omega_a}\right]\tanh\left[\frac{\pi\omega_a}{\delta\omega Q_a}\right] . \tag{13}$$

If $\delta\omega\gg\pi\omega_a/Q_a$, then the linear amplifier limit of Eq. (11) is obtained. This represents a minimum bandwidth the detector must have in order to achieve the limit at zero temperature. Certainly in practice this bandwidth requirement is very easily satisfied.

It is well known[10] that the antenna thermal noise present at finite temperature imposes a more significant bandwidth requirement. This is easily seen by including the antenna noise in Eq. (8) which gives

$$S/N=\frac{\epsilon}{k_B T_{\rm amp}}\frac{Q_a}{2\pi\omega_a}\times\int_{-\infty}^{\infty}\frac{(1-|\rho\,|^2)}{(1+|\rho\,|^2)+\gamma(1-|\rho\,|^2)}d\omega , \tag{14}$$

where $\gamma=(T/T_{\rm amp})$. For an antenna at 3×10^{-3} K with a resonant frequency of 1 kHz coupled to a quantum-limited linear amplifier, $\gamma\sim10^5$. For the most sensitive cryogenic detector operated to date,[7] $\gamma\sim10^4$–10^5. If we apply Eq. (12) in this case and optimize the sensitivity over a bandwidth $\delta\omega\gg\pi\omega_a/Q_a$, then

$$S/N\le\frac{\epsilon}{k_B T_{\rm amp}}\left[1+\frac{T}{T_{\rm amp}}\frac{\pi\omega_a}{Q_a\delta\omega}\right]^{-1} . \tag{15}$$

In order for the antenna thermal noise to be neglgible the detector's bandwidth must satisfy the condition

$$\delta\omega\gg\frac{T}{T_{\rm amp}}\frac{\pi\omega_a}{Q_a} . \tag{16}$$

Before turning to the consideration of a particular transducer design, an extension of the above analysis is worth outlining. For the situation shown in Fig. 1(b) it has been assumed that there is no reactance associated with the amplifier input. This is not a limitation on the above analysis because any reactance associated with the amplifier input can be incorporated into the coupling network and not explicitly considered. We can, however, explicitly consider the effects of a parasitic amplifier reactance by suitably modifying Eq. (9).

To be specific, suppose a series inductance L is associated with the amplifier as shown in Fig. 1(c). This is, in fact, an acceptable model for a SQUID amplifier.[11] It can be easily shown (see discussion in Ref. 9) that in this case the gain-bandwidth theorem can be written as

$$\int_{-\infty}^{\infty}\ln\left|\frac{1}{\rho}\right|d\omega\le2\pi\omega_a\min\left\{\frac{R_{\rm opt}}{\omega_a L},\frac{1}{Q_a}\right\} . \tag{17}$$

If $R_{\rm opt}/\omega_a L>1/Q_a$, then the gain-bandwidth limit imposed by the amplifier is less severe than the one due to the antenna itself, and the linear amplifier sensitivity limit can be attained. In practice this inequality is easily fulfilled.

IV. AN EXAMPLE: A RESONANT TRANSDUCER DETECTOR

In the remainder of this paper we explicitly consider both the continuous source sensitivity and the pulse sensitivity and bandwidth of a resonant-mass detector coupled to a resonant transducer. Specifically we have in mind a superconducting modulated inductance transducer[12] coupled to a SQUID amplifier.[13] This type of transducer is in use both at Stanford University and the University of Maryland.[14] The results, however, are applicable to more situations than just this particular case.

The sensitivity of a resonant-mass antenna with a resonant modulated-inductance transducer has previously been discussed.[15] The detector noise temperature for pulse detection was computed numerically for conditions appropriate to the operation of the Standford 4800-kg detector. The limitations on detector bandwidth and the sensitivity to continuous sources were not considered.

The electromechanical equivalent circuit model of the complete detector is shown in Fig. 2. The electromechanical coupling is included through the force generator F_t and the voltage generator V_t. These quantities can be written as

$$F_t=(\beta M_t\omega_a{}^2L)^{1/2}I_t\ , \tag{18}$$

$$V_t=(\beta M_t\omega_a{}^2L)^{1/2}\dot{X}_t\ , \tag{19}$$

where I_t is the current flowing into the transducer, L is the inductance seen at the transducer output, M_t is the transducer mass, and β is a dimensionless coupling constant (see below). The antenna effective mass M_a is approximately one-half of the total mass for a cylindrical antenna. The compliance $K_a{}^{-1}$ and damping μ_a are fixed by the actual mechanical Q value and resonant frequency of the antenna. This also applies to the transducer parameters.

To calculate the sensitivity of the detector it is necessary to specify all the noise sources. The thermal noise associated with the antenna damping is characterized by the force noise spectral density S_f, given by Eq. (1). The transducer force-noise spectral density S_t is given by the same expression with M_t and Q_t substituted for M_a and Q_a, respectively. Finally the current noise and voltage noise of the amplifier have been included in the output electrical circuit.

The transducer in use on the gravity wave detector at Stanford[14] can be described by the model of Fig. 2. It makes use of the low mechanical and electrical losses in superconducting materials. Oscillations of the antenna

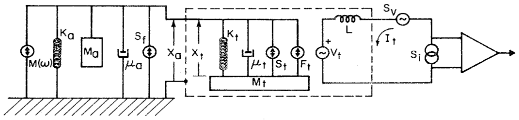

FIG. 2. Electromechanical model of a resonant, inductive transducer coupled to a resonant-mass antenna. The electromechanical transduction is included through the voltage generator V_t and the force generator F_t. These generators, as well as the noise sources, are described in the text.

end-face are coupled to the fundamental eigenmode of a superconducting diaphragm, which modulates the inductance of current-carrying superconducting pickup coils, causing an ac voltage proportional to the velocity of the diaphragm to appear at the output terminals. The output signal from the transducer is fed to the input coil of a SQUID amplifier. The inductance L in Fig. 2 includes both the input inductance of the SQUID and the inductance of the pickup coils.

Because of the magnetic fields produced by the stored currents, displacement of the diaphragm causes not only an induced proportional current to flow in the SQUID coil but also produces an electromagnetic force tending to return the diaphragm to its equilibrium position. This additional restoring force has the effect of raising the resonant frequency of the transducer. This is easily seen by considering the equations of motion for the system in the absence of noise or dissipation:

$$M_a(\ddot{X}_a+\omega_a{}^2X_a)=F_g-M_t(\ddot{X}_t+\ddot{X}_a)\ , \tag{20}$$

$$M_t[\ddot{X}_a+\ddot{X}_t+(K_t/m_t)X_t]=(\beta M_t\omega_a{}^2L)^{1/2}I_t\ , \tag{21}$$

$$L\dot{I}_t=-(\beta M_t\omega_a{}^2L)^{1/2}\dot{X}_t\ . \tag{22}$$

$F_g(t)$ is the signal applied to the antenna. Integrating Eq. (22) and combining with Eq. (21) gives

$$\ddot{X}_t+\omega_t{}^2X_t=-\ddot{X}_a\ , \tag{23}$$

where the transducer resonant frequency is given by[12]

$$\omega_t{}^2=\frac{K_t}{M_t}+\beta\omega_a{}^2\ . \tag{24}$$

The first term is the mechanical resonant frequency of the transducer in the absence of any magnetic restoring force. The second term is the shift due to the electromagnetic coupling. $\beta\omega_a{}^2/\omega_t{}^2$ is the ratio of electromagnetic energy stored in the transducer to the total energy in the transducer.

For maximum sensitivity the transducer is usually operated so that $\omega_t{}^2=\omega_a{}^2$. In this mode the diaphragm is made so that with the optimum magnetic restoring force present the transducer is tuned exactly to the antenna's resonant frequency. The antenna and the transducer act as a pair of coupled oscillators and the resonances split into two eigenfrequencies given by

$$\omega_{\pm}\simeq\omega_a\left[1\pm\frac{\Delta}{2}\right]\ , \tag{25}$$

where $\Delta=(M_t/M_a)^{1/2}$. It can be shown that a given amplitude of antenna motion is associated with an amplitude of transducer motion which is larger by a factor Δ^{-1}.

V. PULSE-DETECTION NOISE TEMPERATURE

Having specified the noise sources we can solve Eq. (3) for the SNR and then compute the pulse-detection noise temperature T_d. In general this can best be done numerically.[15] However, some insight can be gained by analytically considering the narrow-bandwidth case: $\delta\omega \ll \omega_a$.

The first step is to find the total detector noise referred to the antenna's input. In the limit of high Q's and $\Delta \ll 1$ we find

$$S_n(\omega)\simeq\frac{2k_BTM_a\omega_a}{Q_a}\Bigg\{\underbrace{1}_{\text{antenna noise}}+\underbrace{\frac{Q_a}{Q_t\Delta^2}\left[\left[\frac{\omega_a{}^2-\omega^2}{\omega_a{}^2}\right]^2+\frac{1}{Q_a{}^2}\right]}_{\text{transducer noise}}\Bigg\}$$

$$+\frac{k_BT_{\text{amp}}M_a\omega_a}{\Delta^2}\Bigg\{\underbrace{\beta\lambda\left[\left[\frac{\omega_a{}^2-\omega^2}{\omega_a{}^2}\right]^2+\frac{1}{Q_a{}^2}\right]}_{\text{amplifier voltage noise}}+\underbrace{\frac{1}{\beta\lambda}\left[\left[\frac{\omega_+{}^2-\omega^2}{\omega_a{}^2}\right]^2+\frac{1}{Q^2}\right]\left[\left[\frac{\omega_-{}^2-\omega^2}{\omega_a{}^2}\right]^2+\frac{1}{Q^2}\right]}_{\text{amplifier current noise}}\Bigg\}, \qquad (26)$$

where $\lambda=R_{\text{opt}}/\omega_aL$ characterizes the coupling of the transducer and the amplifier. $Q^{-1}=(Q_a{}^{1}+Q_t{}^{-1})/2$, where Q_t is the transducer Q value. In the above expression the additional approximation has been made that the amplifier voltage noise has a minimum only at ω_a. The minima can be shown to actually occur at

$$\omega\simeq\omega_a[1-\beta/2\pm(\beta^2/4+\Delta^2)^{1/2}]^{1/2}\,. \qquad (27)$$

Therefore, in order for the approximation in Eq. (26) to be strictly valid, we must require $\Delta^2 \ll \beta^2/4$.

Figure 3 shows the frequency dependence of each term that contributes to the total noise power spectrum for various values of the coupling parameter $\beta\lambda$. The value of Δ is 0.1 for all the cases shown in Figs. 3(a)—3(c). Figure 3(d) shows the SNR per unit bandwidth (for an impulse signal) for each of these cases. In general, the usable bandwidth of the system is restricted to lie between the normal-mode frequencies because of the steep increase in the amplifier current noise contribution (solid curve) outside this bandwidth. The minima of the current noise occur at the normal-mode frequencies. The contribution of the transducer thermal noise and amplifier voltage noise (dotted curve) has a minimum at the antenna frequency and increases, as shown, away from ω_a. The antenna thermal noise (dashed line) is, of course, white when referred to the antenna input.

Consider now the dependence of the noise spectrum and the SNR on $\beta\lambda$. For case (a) the value of $\beta\lambda$ has been adjusted so that the SNR as a function of frequency is maximally flat between the normal-mode frequencies. This is accomplished by requiring that $d^2S_n/d\omega^2=0$ at ω_a which leads to the condition

$$\Delta^2=\tfrac{1}{2}(\beta\lambda)^2+\frac{T}{T_{\text{amp}}Q_t}\beta\lambda+\frac{1}{Q^2}\,. \qquad (28)$$

In practice the last term can be ignored. This gives the SNR shown as the dotted curve in Fig. 3(d).

For case (b) the value of $\beta\lambda$ has been increased by a factor of 3. This has reduced the overall contribution of the current noise and increased the transducer and voltage noise contributions. The SNR now peaks at the antenna resonant frequency and the usable bandwidth is now controlled primarily by the voltage noise. Increasing $\beta\lambda$ further will narrow the bandwidth but not substantially increase the SNR at the antenna frequency because eventually the antenna thermal noise becomes the limiting noise source [provided $\beta^2/4 \gg \Delta^2$; see discussion of Eq. (27)]. Increasing $\beta\lambda$ above the value given by Eq. (28) will not improve the overall pulse sensitivity and eventually will lead to loss of sensitivity. This statement assumes that Δ is held fixed and that the antenna temperature is nonzero. In practice if a larger than optimal value of $\beta\lambda$ were achieved the bandwidth Δ of the system could be increased by increasing the transducer mass to satisfy Eq. (28).

The effect of decreasing $\beta\lambda$ below the value given by Eq. (28) is illustrated in Fig. 3(c). The voltage noise is reduced but the current noise is increased. This leads to the SNR shown as the solid curve in Fig. 3(d). The SNR peaks near the normal-mode frequencies. By decreasing $\beta\lambda$ further the SNR at the normal-mode frequencies will increase until the antenna and transducer thermal noise dominates. The overall bandwidth will decrease and the pulse sensitivity will be reduced.

The optimization of the detector for pulse detection can be achieved by obtaining a flat SNR over the widest possible bandwidth. At zero temperature the linear amplifier

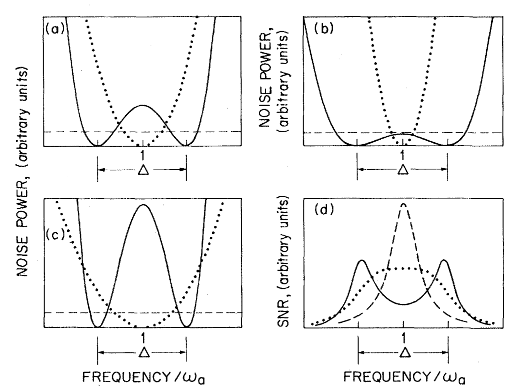

FIG. 3. Contributions to the detector noise power spectrum referred to the antenna input. For cases (a), (b), and (c) Δ is 0.10. The contribution from the antenna thermal noise (dashed line), the amplifier current noise (solid curve), and the amplifier voltage noise and transducer thermal noise (dotted curve) are shown for each case. For the approximation discussed in the text, the amplifier voltage noise and the transducer thermal noise have the same frequency dependence. In all these cases T/T_{amp} is 10^6 and the Q values are 10^8. For case (a) $\beta\lambda$ is equal to the value given by Eq. (28), while for cases (b) and (c) $\beta\lambda$ is 3 times and $\frac{1}{3}$ times the value for case (a), respectively. The signal-to-noise ratio for each case is shown in (d): (dotted curve) case (a), (dashed curve) case (b), and (solid curve) case (c).

limit can easily be attained under far less restrictive conditions that do not necessarily dictate a flat SNR [see Eq. (13)]. At finite temperature, consideration of the wide-band antenna thermal noise leads to the conclusion stated above.

Using the bandwidth condition of Eq. (28) and Eqs. (2), (3) and (26), and letting $Q_t = Q_a$, the detector noise temperature for pulse detection is given approximately by

$$T_d \simeq \frac{\pi}{4} T_{\text{amp}} \left\{ \frac{5T}{\Delta T_{\text{amp}} Q_a} + \left[\left[\frac{T}{\Delta T_{\text{amp}} Q_a} \right]^2 + 2 \right]^{1/2} \right\} . \tag{29}$$

In the limit $T/Q_a\Delta \gg T_{\text{amp}}$ the thermal noise dominates and the detector noise temperature is given by

$$T_d \sim \frac{3}{2} \frac{\pi T}{\Delta Q_a} . \tag{30}$$

In this regime the detector noise temperature improves as the detection bandwidth is widened. In the opposite limit $T/Q_a\Delta \ll T_{\text{amp}}$ the linear amplifier limit is achieved (approximately) and the pulse sensitivity is independent of bandwidth.

VI. CONTINUOUS SOURCE SENSITIVITY

The detection of a narrowband continuous source places different requirements on the detector. If the frequency of the source is known the best strategy is well known to be to tune the antenna so that its resonant frequency coincides with the source frequency. This is only necessary to overcome the amplifier noise. Since resonant-mass detectors are usually operated under conditions for which $T_{\text{amp}} \ll T$, it is possible to obtain sensitivity that is limited by the antenna thermal noise over a bandwidth large compared to ω_a/Q_a.

The SNR for detection of a continuous source at frequency ω_g with dimensionless strain amplitude h_c is given by

$$S/N = h_c^{\,2} \frac{M_a^{\,2} l^2 \omega_g^{\,4} \tau}{8 S_n(\omega_g)} , \tag{31}$$

where τ is the observation time, l is the effective length of the antenna, and $S_n(\omega_g)$ is the noise power spectral density, referred to the input, evaluated at $\omega = \omega_g$. If the antenna resonant frequency coincides with the source frequency then the minimum detectable wave strain for unity SNR is given by

$$h_c^{\,2} \simeq \frac{4k_B T}{\omega_a \tau Q} \frac{1}{\frac{1}{2} M_a \omega_a^{\,2} l^2} \times \left[1 + \frac{1}{\Delta^2 Q^2} + \frac{1}{2} \frac{T_{\text{amp}}}{T} \left[\frac{\beta\lambda}{\Delta^2 Q} + \frac{2}{\beta\lambda Q} + \frac{\Delta^2 Q}{\beta\lambda} \right] \right] . \tag{32}$$

Since $T_{\text{amp}}/T \ll 1$ there is no need to minimize the contribution of the amplifier noise by excessively restricting the bandwidth Δ. For $\Delta \gg Q^{-1}$ and $\beta\lambda \gg Q^{-1}$ and imposing the bandwidth condition of Eq. (28), the minimum detectable wave strain becomes

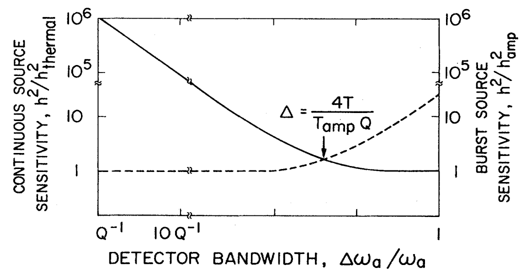

FIG. 4. Detector sensitivity to burst sources and continuous sources as a function of detector bandwidth. It is assumed that the frequency of the continuous source is within the bandwidth $\Delta\omega_a$ and that the spectral density of the burst source is constant over this bandwidth. For a fractional bandwidth equal to $4T/QT_{\text{amp}}$ the sensitivity is nearly optimal for both kinds of sources.

$$h_c^{\,2} \geq \frac{4k_B T}{\omega_a \tau Q} \frac{1}{\frac{1}{2} M_a \omega_a^{\,2} l^2} \left\{ \frac{5}{4} + \frac{1}{4} \left[1 + 2 \left[\frac{T_{\text{amp}} Q \Delta}{T} \right]^2 \right]^{1/2} \right\} . \tag{33}$$

The prefactor in the above expression is the strain sensitivity if only antenna thermal noise is present. The factor in brackets approaches $\frac{3}{2}$ as $T_{\text{amp}} \to 0$. This is a consequence of imposing Eq. (28) which forces $\beta\lambda \to 0$ for fixed Δ in such a way that the amplifier current noise at ω_a approaches half the antenna thermal noise. It is of course not necessary to impose Eq. (28). By restricting the bandwidth Δ and maintaining sufficient coupling such that $\Delta^2/\beta\lambda \ll 2T/T_{\text{amp}}Q$ the thermal sensitivity limit can be achieved. However, imposing Eq. (28) ensures that the SNR will be nearly constant over the bandwidth Δ.

Figure 4 illustrates the sensitivity to both burst sources and continuous sources as a function of bandwidth. The burst sensitivity can be expressed in terms of the minimum detectable dimensionless wave strain of the signal pulse h_b. For the uniform bandwidth condition of Eq. (28) it is straightforward to show that the sensitivities are related by

$$h_c^{\,2} = h_b^{\,2} \left[\frac{\Delta}{2\omega_a \tau} \right] . \tag{34}$$

For a bandwidth $\Delta \sim 4T/QT_{\text{amp}}$ the sensitivity is nearly optimal for both kinds of sources. If $T = 50$ mK, $Q = 10^8$ and $T_{\text{amp}} = 5\times 10^{-8}$ K then a 4800-kg detector resonant at 1 kHz has a continuous source sensitivity at the noise level of $5\times 10^{-26}/\text{day}^{1/2}$ in a bandwidth of 40 Hz. The pulse-detection noise temperature is within a factor of 3 of the amplifier noise temperature for these conditions.

VII. PRACTICAL CONSIDERATIONS

It is clear that the most important factors affecting detector sensitivity and bandwidth are Q, antenna temperature, amplifier noise temperature, and the coupling parameters β and λ. If the temperature is low enough the pulse-detection noise temperature will approach $T_{\rm amp}$ and the fractional bandwidth Δ will be approximately $\beta\lambda$. It is therefore desirable to achieve the largest values possible for both β and λ.

Recall that β is the ratio of electromagnetic energy stored in the transducer to the total energy in the transducer. In principle, this ratio can be unity. In the modulated-inductance transducer in use at Stanford it is limited by practical considerations. To discuss these limitations it is convenient to express β as

$$\beta=\frac{M_r}{M_t}\ , \tag{35}$$

where M_t is the transducer mass and M_r is the mass that can be resonated at the antenna frequency by the electromagnetic contribution to the spring constant. For a particular transducer design M_r does not depend on M_t. In the Stanford transducer it is determined by the strength of the stored magnetic field, the spacing of the pickup coils, etc. A design objective is to maximize M_r. In the present Stanford transducer M_r is about 10 g. It appears that this value can be increased by at least a factor of 3 to 4 by optimizing the present design.

The 4800-kg detector at Stanford has been operated at about 4 K. Because of electrical losses in the transducer the value of Q_t was limited to about 2×10^5. For these reasons the detector has not yet achieved the linear amplifier sensitivity limit and the optimum transducer mass determined from Eq. (28) is governed primarily by the second term.

By operating the detector at lower temperature and achieving much higher Q values, the limiting factor in determining Δ will be the first term of Eq. (28). Under these conditions the fractional bandwidth can be written as

$$\Delta=(M_t/M_a)^{1/2}\simeq(M_r\lambda/M_a)^{1/3}\ . \tag{36}$$

If $M_r=10$ g and $\lambda=1$, then for the Stanford detector, $\Delta f\sim15$ Hz. Clearly it is desirable to increase M_r as much as possible.

The parameter $\lambda=R_{\rm opt}/\omega_a L$ depends both on the noise characteristics of the amplifier and the output impedance of the transducer as seen by the amplifier. In the case of a properly optimized dc SQUID amplifier[16] the amplifier noise match impedance $R_{\rm opt}$ is approximately given by

$$R_{\rm opt}\sim\epsilon\alpha^2\omega L_i\ , \tag{37}$$

where L_i is the inductance of the SQUID input coil, ω is the signal frequency, α^2 is the input coil coupling coefficient, and ϵ is a complicated factor that depends on the detailed SQUID parameters. Numerically ϵ is of order unity. If the input coil is tightly coupled to the SQUID, then α^2 can also approach unity. Typically values of $\alpha^2\simeq0.5$ to 0.8 have been achieved.[16]

The inductance L is the sum of L_i and the inductance of the pair of superconducting transducer pickup coils L_t. For a flat pair of circular coils of area A wound with wire of diameter d, the inductance of each pickup coil is

$$L(x)\simeq L_0\pm\frac{A\mu_0}{d^2}X\ , \tag{38}$$

where L_0 is the inductance of each coil when the diaphragm is in its equilibrium position, X is the displacement of the diaphragm from equilibrium. $L_t=L_0/2$ since the pickup coils are connected in parallel.

The coupling parameter $M_r\lambda$ can be expressed in terms of these inductances as

$$M_r\lambda=\frac{R_{\rm opt}}{\omega_a}\left[\frac{I_0}{\omega_a}\left[\frac{dL}{dX}\right]\right]^2\Big/(L_i+L_t)^2\ , \tag{39}$$

where I_0 is the persistent dc current stored in the transducer coils. Using Eq. (37), this becomes

$$M_r\lambda=\frac{\epsilon\alpha^2L_i}{{\omega_a}^2}\left[\frac{I_0dL/dX}{L_i+L_t}\right]^2 . \tag{40}$$

Assuming that $\epsilon\alpha^2$ is independent of L_i, this quantity is optimized if $L_i=L_t$. If we approximate the transducer inductance by

$$L_t\simeq\frac{A\mu_0X_0}{2d^2}\ , \tag{41}$$

where X_0 is the equilibrium diaphragm coil spacing, then Eq. (40) can be rewritten as

$$(M_r\lambda)_{\rm opt}\simeq\frac{\epsilon\alpha^2}{{\omega_a}^2}\frac{A\mu_0}{2X_0}H^2\ , \tag{42}$$

where I_0 has been expressed in terms of the magnetic field intensity H between the coil and diaphragm. μ_0 is the permeability of free space. The maximum value of H is limited by the lower critical field of the superconductor. It is desirable to use a material with a large critical field and to achieve very close spacing with a large area coil. Of course the coil characteristics must be selected so that $L_t\simeq L_i$. For example, with $A=5\times10^{-3}$ m^2, $X_0=10^{-5}$ m, $\omega_a=5\times10^3$ sec^{-1}, $\epsilon\alpha^2=1$, and $H=9\times10^4$ amp/m^2 (the lower critical field of niobium), we find $(M_r\lambda)\simeq100$ g. With this value the Stanford detector would have a bandwidth $\Delta f\sim30$ Hz. With a SQUID amplifier operating at the quantum limit ($T_{\rm amp}\sim5\times10^{-8}$ K) and $Q=10^8$, the pulse-detection noise temperature T_d will approach $T_{\rm amp}$ if $T<150$ mK.

Recall that Q is determined by both the antenna Q and the transducer Q. It has been experimentally observed[12] that the Q value of the modulated inductance transducer in use at Stanford varies with β approximately as

$$\frac{1}{Q_t}=\frac{1}{Q_m}+\frac{\beta}{Q_e}\ , \tag{43}$$

where Q_m is the mechanical Q value with no current stored. The relationship indicates that a damping mechanism exists that is equivalent to a fixed loss in the electrical circuit. This is characterized by a Q value of Q_e. Typically we have found $Q_e\sim3\times10^4$ although in one prototype transducer $Q_e\gtrsim10^6$. In order to achieve a high overall Q value a large value of Q_e/β is required. For example, if $\Delta f\sim30$ Hz, then $Q_e\gtrsim4\times10^6$ is required to have $Q\sim10^8$.

VIII. MULTIMODE TRANSDUCER

Richard has suggested[17] the use of a mechanically resonant intermediate mass between the antenna and the transducer as a way of effectively increasing β. Richard has considered the case where the intermediate mass M_i is equal to $(M_a M_t)^{1/2}$. If this two-mode transducer is operated in a resonant mode with the stored magnetic field adjusted to tune the resonant frequency of the transducer mass to the uncoupled antenna resonant frequency and if the intermediate mass spring constant is selected so that $k_i = M_i \omega_a{}^2$, then it is easily shown that the three normal-mode frequencies are given by

$$\omega_1 = \omega_a \, , \quad \omega_\pm = \omega_a \left[1 \pm \frac{\Delta_2}{2} \right] , \tag{44}$$

where $\Delta_2 = (4M_t/M_a)^{1/4}$. Recall that for the single-mode transducer the analogous quantity is given by $(M_t/M_a)^{1/2}$. Thus for a given transducer mass the maximum available bandwidth $\Delta_2\omega_a$ for the two-mode transducer case is larger by $(4M_a/M_t)^{1/4}$ than the bandwidth for the single-mode transducer case. Richard has previously emphasized this benefit.[17]

The sensitivity analysis in this case can be carried out in the same manner as the analysis for the single-mode transducer case. Accurate results are best obtained numerically, but for $\Delta_2 \ll 1$ approximate analytic results can be obtained. The results are only summarized here.

If the temperature is low enough and the Q's are sufficiently high then the detector noise temperature T_d will be limited by the amplifier noise. Again, the usable bandwidth of the system is restricted to lie between the normal-mode frequencies $\omega_\pm$ because of the increase of the amplifier current noise contribution outside this bandwidth.

Figure 5 shows the frequency dependence of the noise spectrum and the SNR for the case when $\beta\lambda$ satisfies Eq. (28). Δ_2 is equal to 0.1 and all of the Q values are 10^8. Applying condition (28) gives the SNR shown as the solid curve in Fig. 5(b). This leads to approximately uniform ripple of the SNR over the bandwidth $\Delta_2\omega_a$. The dotted curve shows the SNR when $\beta\lambda$ is adjusted to give $d^2S_n/d\omega^2$ equal to zero at the antenna resonant frequency. This condition is identical to Eq. (28) provided that Δ^2 is replaced by $\Delta_2{}^2/2$. As can be seen from Fig. 5(b) these two choices lead to essentially the same result for T_d. In the amplifier limit the fractional bandwidth is given by

$$\Delta_2 = (4M_t/M_a)^{1/4} \simeq (4M_r\lambda/M_a)^{1/5} \, . \tag{45}$$

This expression should be compared with Eq. (36) for the single-mode transducer case. For $M_r\lambda = 100$ g and $M_a = 2.4 \times 10^6$ g this gives a bandwidth $\Delta f \sim 150$ Hz for the Stanford detector. With a SQUID amplifier operating at the quantum limit and $Q = 10^8$, the pulse detection noise temperature T_d will approach $T_{\rm amp}$ if $T \leq 750$ mK. To achieve a high overall Q and the larger bandwidth with the two-mode transducer places a more severe requirement on Q_e than does the single-mode transducer.

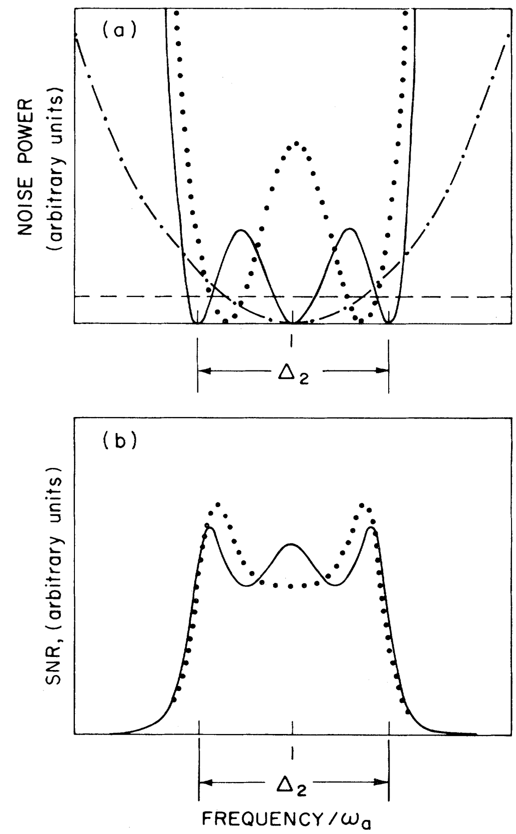

FIG. 5. (a) Contributions to the detector noise power spectrum referred to the antenna input for a detector with a two-mode transducer: (dashed line) antenna thermal noise, (dot-dashed curve) intermediate transducer mass thermal noise, (solid curve) amplifier current noise and (dotted curve) final transducer mass thermal noise and amplifier voltage noise. Δ_2 is 0.1 and $\beta\lambda$ satisfies Eq. (28). All of the Q values are 10^8. (b) Signal-to-noise ratio: the solid curve corresponds to the conditions of (a) while the dotted curve is the result of adjusting $\beta\lambda$ so that the SNR is flat at the antenna resonant frequency.

This is because β is larger in the two-mode transducer for the same value of magnetic spring constant. This is offset to some extent by the wider bandwidth that is obtained.

In either case for the transducer thermal noise to be negligible compared to the amplifier voltage noise requires that

$$(\beta\lambda) \gg \frac{2T}{T_{\rm amp} Q_t} \simeq \frac{2T}{T_{\rm amp}} \left[\frac{1}{Q_m} + \frac{\beta}{Q_e} \right] . \tag{46}$$

This implies that λ must satisfy

$$\lambda \gg \frac{2T}{T_{\rm amp}} \frac{1}{Q_e} \, . \tag{47}$$

If $\lambda \lesssim 1$, $T = 50$ mK and $T_{\rm amp} = 5 \times 10^{-8}$ K, then (47) requires that $Q_e \gg 2 \times 10^6$. This condition may be very difficult to satisfy in practice. Lower temperature operation is a possible remedy.

IX. MULTITRANSDUCER READOUT

As we have noted above, to achieve amplifier-limited detector sensitivity to impulsive signals and to obtain wideband sensitivity, it is desirable to maximize the quantity $\beta\lambda$. The quantity β is maximized, for either the single-mode or two-mode transducer, by maximizing the magnetic spring constant. The impedance matching parameter λ is maximized by proper design and coupling to the SQUID amplifier. Ideally $\lambda \sim 1$ could be achieved with a SQUID amplifier, but it would be difficult to obtain $\beta \sim 1$ with a single-mode or two-mode transducer. In principle larger values of β could be achieved with an N-mode transducer ($N \geq 3$) but the design and construction of such a system may be difficult.

Given a particular transducer design, one way to increase the detector bandwidth is by coupling N transducers in parallel to the antenna and summing their outputs. It is straightforward to show that if each transducer is tuned to the same resonant frequency, then the SNR is maximized if all the transducers have the same mass. Under these conditions the results previously obtained for the single-transducer case apply, provided that M_t and R_{opt} are replaced by NM_t and NR_{opt}, respectively. In the amplifier limit for the single-mode transducer the fractional bandwidth is now given by

$$\Delta \simeq (NM_r\lambda / M_a)^{1/3} . \tag{48}$$

However, in order to reach the amplifier limit the condition (47) for one transducer must still be satisfied. A similar result holds for the two-mode transducer case.

We note here that the same result is not obtained in the case of a single transducer coupled to N SQUID amplifiers, either in series or in parallel; the only effect this has is to change the value of L_i and R_{opt}. Such a scheme may be useful in obtaining the condition $L_i = L_t$ but the inductance matching can probably be more easily satisfied by proper design of the transducer coils.

X. SUMMARY AND DISCUSSION

The ultimate sensitivity of a resonant-mass gravitational wave detector using a linear transducer-amplifier readout is limited by the amplifier noise. Using the gain-bandwidth theorem[9] we have shown that the amplifier limit (11) is a general limit that applies to the use of an arbitrary linear transducer.

To actually achieve the amplifier limit for the detection of impulsive signals requires that the amplifier noise contribution dominate the thermal noise over a bandwidth $\delta\omega > (T\pi\omega_a / T_{\text{amp}}Q)$, Obviously a high-$Q$, low-temperature detector is desirable. For a detector resonant at 1 kHz, operated at 10 mK with an overall Q value of 10^7 and readout with a quantum-limited linear amplifier, this requires a bandwidth greater than 60 Hz in order to achieve the amplifier limit. Thus a relatively wideband detector is not only desirable for studying the received waveforms, but is necessary for attaining the linear amplifier limit of detector sensitivity. This will, of course, remain true whether a resonant transducer is used or not.

With the technology presently available using superconducting inductive resonant transducers, a fractional bandwidth of ~ 0.17 is feasible for a 4800-kg antenna. This is certainly not a fundamental limit and alternative technologies may offer improvements, but this bandwidth is already adequate for achieving the linear amplifier limit with detectors now under construction

Further increases in sensitivity and bandwidth can be achieved with an array of independent resonant-mass detectors. A bandwidth of several kilohertz can be gotten by appropriate selection of the size of the individual antennas in the array. In addition, an array of independent detectors can offer a significant advantage over a single wideband detector in identifying and eliminating nonthermal events that are internally generated in the detector.

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