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Duality, moments, and Regge trajectories in nonrelativistic quantum mechanics

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We investigate the correctness and accuracy of some recent schemes of implementation of the QCD duality relation by means of quantum-mechanical example. Such schemes, including a recently discussed S-matrix method and the Borel-transform method, purport to determine bound states given high-energy behavior. For a power-law confining potential with no short-range component both methods work well to predict the leading Regge trajectory. Addition of short-range components makes it clear that the nature of the long-range confinement must be specified *ab initio*; then the S-matrix method gives a successful numerical approximation to the leading trajectory. The Borel-transform method is not capable of fitting the widths of states.

I. INTRODUCTION

Application[1-6] of duality principles to dispersion relations for two-point functions in field theory has been of great interest as a probe of the spectrum and of other phenomenological low-energy parameters of the theory in question. In general such principles attempt to use the high-energy behavior of the two-point function to determine the low-energy properties; the high-energy behavior is calculable in the most interesting case, quantum chromodynamics (QCD), because of asymptotic freedom. Indeed, one particular version of these principles, the so-called "QCD sum rules,"[3-6] has been extensively used in phenomenological studies of the hadronic spectrum.

Because of the phenomenological application and interest of duality, its manifestation in a nonrelativistic context has been and continues to be a useful way of assessing its validity.[7-10] In this paper we present an investigation of the validity of the duality principle in a fully calculable nonrelativistic context. We test in particular two methods of realizing the principle, one based on a recently developed proof[9] of the identity of the two-point function of the Schrödinger equation and an analog of the S matrix for confining potentials as a function of both angular momentum and energy. The second method we test is based on the use[3-6] of the Borel transformation of the two-point function. We shall refer to these two methods as the "S-matrix" and the "Borel-transform" methods, respectively. We are interested in using asymptotic behavior to determine the lowest-lying bound state, or leading trajectory, and the residue, or width, of this state.

Briefly, our conclusions are as follows.

(i) For a pure power-law confining potential—by which we mean a polynomial confining potential with a fixed known maximum power—if we know the high-energy asymptotic behavior to an accuracy determined by the maximum power, the S-matrix method reproduces the leading trajectory. The Borel-transform method does not do quite as well for a given asymptotic correction, although it does better for the full trajectory than earlier tests[9] of the S-wave bound states. More importantly, it is subject to rapid systematic improvement if higher asymptotic corrections are known.

(ii) When a short-range correction is added to the confining power-law potential all inverse powers in energy for large energy are modified. Blind application of either method with a finite number of asymptotic terms will give a completely misleading picture of the spectrum. For duality to be applied, *it is necessary to have independent knowledge, either of the confining part of the potential or equivalently of the asymptotic form of the Regge trajectory.* This result confirms earlier conclusions;[9] here we show how the problem is properly handled in the S-matrix method.

(iii) *The Borel-transform method does not allow determination of the residues of the leading trajectory,* nor even of ratios of residues along the trajectory. This is partly because all inverse powers in the asymptotic variable contribute to the residue.

In Sec. II, we discuss the basis of the two methods discussed above. Section III contains a discussion of confining power-law potentials and determination of the leading trajectory in two methods. In Sec. IV, we study the effects of short-range corrections, and in Sec. V we look at residues.

II. POTENTIALS, ASYMPTOTIC BEHAVIOR, AND BOUND STATES

We discuss in this section how asymptotic behavior is correlated with definite potentials and in turn with the bound states of those potentials. The discussion allows for a direct connection between asymptotic behavior and bound states through specification of the large-angular-

momentum region of the leading Regge trajectory, so that the tests we make will be applicable even to physical systems where nonrelativistic behavior is not directly relevant. We elaborate this important point below.

The identity[9]

$$\Pi(E,\nu)=-\hat{S}(\nu,E) \tag{2.1}$$

forms the basis for the determination of bound states from asymptotic behavior. Here

$$\Pi(E,\nu)=\sum_{k=0}^{\infty}\frac{2\nu[a_k(\nu)]^2}{E_K-E} \tag{2.2}$$

is the two-point function in the energy variable E and angular momentum variable $\nu=l+\frac{1}{2}$, with poles at the bound-state energies E_K (we suppress the dependence of E_K on ν). The residues of these poles, determining widths, are in terms of the threshold behavior of the normalized eigenfunctions $u_k(\nu,r)$ of the Schrödinger equation, namely,

$$u_k(\nu,r)\underset{r\to 0}{\to}a_n(\nu)r^{\nu+1/2}[1+(\text{terms that vanish as } r\to 0)] . \tag{2.3}$$

$\hat{S}$ is an analog to the S matrix for confining potentials,

$$\hat{S}(\nu,E)=\lim_{r\to\infty}\frac{u(-\nu,r)}{u(\nu,r)} , \tag{2.4}$$

where $u(\pm\nu,r)$ are the regular/irregular solutions to the Schrödinger equation. Here $u(-\nu,r)$ is linearly independent of $u(\nu,r)$ and finite. The $u(\pm\nu,r)$ are normalized to behave at the origin like $r^{\pm\nu+1/2}$, so that the Wronskian of the two functions is 2ν. For certain potentials there are physically relevant values of ν for which $u(-\nu,r)$ is not finite. In that case proper combinations of $u(-\nu,r)$ and $u(\nu,r)$ must be used for the irregular solution. It turns out that neither the poles nor the residues of $\hat{S}(\nu,E)$ are affected by the freedom of definition of the irregular solution, provided the value of the Wronskian is unchanged. As for the ordinary S matrix, the bound states correspond to poles of $\hat{S}$. The asymptotic behavior in E is simply obtained in terms of $\hat{S}$.

Equation (2.2) is a spectral representation for the two-point function, analogous to the Källén-Lehman representation for the vacuum expectation value of the time-ordered product of two currents used for relativistic application.[3–5]

Although QCD is surely not describable by a local potential, the present study has direct relevance to QCD because in that theory too there is a confining interaction and a perturbative short-range modification, because the duality principle discussed here is analogous to those used in QCD sum rules, and also because effective potentials have been so useful in quarkonium phenomenology. Furthermore, in the narrow width approximation, the various two-point functions for arbitrary spin in QCD are structurally no different than the two-point functions for a Schrödinger equation with some equivalent, although possibly spin and energy dependent, potential.[11] Our work here focuses on local potentials as a test of two different methods of extracting information about the spectrum from high-energy behavior, but the QCD application should firmly be kept in mind.

The relation between the confining potential and the asymptotic behavior is as follows: At large E, $\hat{S}(\nu,E)$ takes the form

$$\hat{S}(\nu,E)=(-E)^{\nu}\sum_{s=0}\frac{\alpha_s(\nu)}{E^s},\quad \alpha_0(\nu)=1 . \tag{2.5}$$

$\alpha_1(\nu)$ may be taken equal to zero by redefinition of E. [In practical application Eq. (2.5) will be cut off at some $s_{\max}$.] Consider now the confining potential to be a pure (even) power:

$$V(r)=g_{2N-2}r^{2N-2}=\bar{g}_{2N-2}(r/r_0)^{2N-2} , \tag{2.6}$$

where $\bar{g}$ has dimensions of inverse length squared. Keeping $\bar{g}$ finite allows us to study the infinite square well as one limit. This parametrization is convenient because the first nonzero α_s for such a potential occurs at $s=N$: for potentials as in Eq. (2.6),

$$\hat{S}(\nu,E)=(-E)^{\nu}\left[1-g_{2N-2}\frac{4^{N-1}\Gamma^2(N)\Gamma(\nu+N)}{\Gamma(2N)\Gamma(\nu-N+1)}\frac{1}{E^N}+\cdots\right] . \tag{2.7}$$

This result may be shown for integer values of N using the methods of Ref. 11. Since the Green's function and hence $\hat{S}$ is a smooth function of N for $N>0$ the result derived for integer N may be continued to include noninteger N. Alternatively, the Green's-function method,[8,10] in which the confining potential is a perturbation on free particle motion, will also give Eq. (2.7) even for noninteger values of N.

The next correction to $\hat{S}$ for the given potential (2.6) occurs at the level E^{-2N}. Its coefficient is of course completely determined by the parameters of $V(r)$.

If $V(r)$ were a finite polynomial of maximum power r^{2N-2}, then this *maximum* power would first manifest itself at level E^{-N}; lower powers in r would already have appeared in $\hat{S}(\nu,E)$ as lower powers of E^{-1}. There is no new information in powers of E^{-1} beyond E^{-N}.

We now turn to a discussion of how the two methods discussed in the Introduction can be used to specify the ground states given the asymptotic behavior. The S-matrix method makes direct use of Eqs. (2.1) and (2.2) by "reading" the potential which produces the asymptotic behavior in (2.5). More precisely, suppose the α_s are specified up to $s_{\max}$. Then one would first check that the coefficients are not in the specified ratios which would imply a confining force is of lower power than corresponds to the highest power of E^{-1}. [An r^2 confining potential first appears in $O(1/E^2)$ but would also appear in $O(1/E^4),\ldots$. If $s_{\max}=4$ then by looking at the coefficients of E^{-s}, $s<s_{\max}$, one could decide if an r^4 confinement or an r^2 confinement were implied.] Let us suppose that $s_{\max}$ implies a confinement $\sim r^{2s_{\max}-2}$. Then the S-matrix method simply "reads" this potential[12] and writes its bound states using the Schrödinger equation. To the extent that $s_{\max}$ is not large enough to reflect the "true" confining potential the result will be increasingly wrong as

ν increases. In physical application to QCD, where it has been shown[13] that asymptotic trajectories are linear, this is unlikely to be the case, since such trajectories correspond to nonrelativistic harmonic confinement $\sim r^2$, i.e., $s_{\max}=2$.

Consider linear confinement with $s_{\max}=2$. Suppose that perturbative calculation at large E yielded an E^{-3} term in addition to E^{-2}, not in accord with linear trajectories. We would conclude that such a term came from a short-range correction to the long-range confining potential and is part of a necessarily infinite series to be handled with perturbation theory as described in Ref. 9. We return to this question in Sec. IV.

Let us now summarize the Borel-transform method: As ordinarily used the Borel-transform method[3-6] uses no information beyond that contained in the asymptotic form (2.5). In Ref. 3 it was noted that if the sign of E in Eq. (2.2) is changed, then the factor $(E_0+E)^{-1}$ exceeds $(E_k+E)^{-1}$, $k\neq 0$. To make this factor dominate, a large number of derivatives in E suffices,[14] that is, for large n,

$$M_n\equiv(-1)^n\frac{1}{\Gamma(n+1)}\left[\frac{d}{dE}\right]^n\Pi(-E,\nu)$$
$$=\sum_k\frac{2\nu[a_k(\nu)]^2}{(E_k+E)^{n+1}}\approx\frac{2\nu[a_0(\nu)]^2}{(E_0+E)^{n+1}} \tag{2.8}$$

and in particular

$$R_n\equiv\frac{M_n}{M_{n+1}}\rightarrow(E_0+E)\ . \tag{2.9}$$

While this is rigorously true for infinite n and E, the choice of, say, using M_{n-1}/M_n rather than (2.9), or of other possibilities at finite n has been investigated for S waves in Ref. 8, and we shall not deal with such questions here. We treat the problem in the now standard way, namely, to search for a minimum in R_n as a function of n and evaluate R_n at that $n_{\min}$. As we shall see, for power-type confinement or for potentials which can be expanded in powers of r, $n_{\min}$ occurs at a value proportional to E, namely, $n_{\min}=E/\epsilon$. The result expressed in (2.8) then becomes equivalent to Borel transformation of $\Pi(-E,\nu)$:

$$M_{n_{\min}}=\frac{1}{E^{n+1}}\sum\frac{2\nu[a_k(\nu)]^2}{(1+E_k/E)^{n_{\min}+1}}$$
$$\rightarrow\frac{1}{E^{n+1}}\sum 2\nu[a_k(\nu)]^2e^{-E_k/\epsilon}\ . \tag{2.10}$$

Now use Eqs. (2.1) and (2.5). We have

$$(-1)^n\left[\frac{d}{dE}\right]^n\hat{S}(\nu,-E)=(-1)^n\sum_{s=0}^{s_{\max}}\frac{\Gamma(\nu-s+1)}{\Gamma(\nu-s-n+1)}\alpha_s(\nu)(-1)^sE^{\nu-n-s}$$
$$=(-1)^n\frac{\Gamma(\nu+1)}{\Gamma(\nu-n+1)}E^{\nu-n}\sum_{s=0}^{s_{\max}}\frac{\alpha_s(\nu)(-1)^s}{E^s}\frac{(\nu-n)\cdots(\nu-n-s+1)}{\nu\cdots(\nu-s+1)}\ . \tag{2.11}$$

Thus

$$R_n=E\left[\frac{n+1}{n-\nu}\right]\left[1+\sum_{s=2}^{s_{\max}}\frac{\alpha_s(\nu)(-1)^s}{E^s}\frac{(\nu-n)\cdots(\nu-n-s+1)}{\nu\cdots(\nu-s+1)}\right]\Bigg/\left[1+\sum_{s=2}^{s_{\max}}\frac{\alpha_s(\nu)(-1)^s}{E^s}\frac{(\nu-n-1)\cdots(\nu-n-s)}{\nu\cdots(\nu-s+1)}\right]\ . \tag{2.12}$$

One now finds the minimum of Eq. (2.12) as a function of $n,n_{\min}$, notes for large E that $n_{\min}=E/\epsilon(\nu,s_{\max})$, and substitutes this value back into R_n, which has the form

$$R_{n_{\min}}=E+E_0+O\left[\frac{I}{E}\right]\ .$$

The constant term is E_0. We save specific evaluation for Sec. III, but reemphasize that in its usual form this method approximates bound states with no assumption on the nature of the confinement. This fact proves to be a failure of the method. The point is that, as in the $\hat{S}$-matrix method, knowledge of the nature of the confining potential (as distinguished from the short-range corrections) is necessary for finding the leading trajectory, and the Borel-transform method does not take that into account. This limitation of the uncritical use of the Borel-transform method, as applied to a calculation of the effective potential was independently recognized by the authors of the first references in Ref. 10. The correct way to deal with short-range corrections in the S-matrix method has been indicated by us in Ref. 9 and will be made more explicit in Sec. IV of this paper.

III. TEST OF THE BOREL-TRANSFORM METHOD FOR POWER-LAW CONFINING POTENTIALS

For a numerical test we consider a single power as in Eq. (2.6). The asymptotic behavior in E of $\hat{S}(\nu,E)$ is then truncated with E^{-N}, and is given by Eq. (2.7). For such potentials it is uninteresting to test the S-matrix method, which simply reads back the corresponding confining potential. To test the Borel-transform method we compute R_n as in Eq. (2.12):

$$R_n = E\frac{(n+1)}{(n-\nu)}\left[1+\frac{(-1)^N\alpha_N N(\nu-n-1)\cdots(\nu-n-N+1)}{E^N\nu\cdots(\nu-N+1)}\right]$$
$$= E\left[\frac{n+1}{n-\nu}\right]\left[1-\frac{(-1)^N g_{2N-2}4^{N-1}\Gamma(N+1)\Gamma(N)\Gamma(\nu+N)}{E^N\Gamma(2N)\Gamma(\nu+1)}(\nu-n-1)\cdots(\nu-n-N+1)\right] . \quad (3.1)$$

Anticipating that the minimum in R_n occurs at $n_{\min}\sim E$, we keep the largest power of n in the $O(g_{2N-2})$ term of (3.1) and the first correction in $(n+1)/(n-\nu)$. Then

$$R_n \approx E\left[1+\frac{1+\nu}{n}\right]\left[1+\frac{g_{2N-2}4^{N-1}\Gamma(N+1)\Gamma(N)\Gamma(\nu+N)}{E^N\Gamma(2N)\Gamma(\nu+1)}n^{N-1}\right] , \quad (3.2)$$

and minimizing with respect to n, we find

$$n_{\min}\approx E/\epsilon , \quad (3.3)$$

where

$$1/\epsilon(\nu,N) \equiv \left[\frac{\Gamma(2N)\Gamma(\nu+2)}{(N-1)4^{N-1}\Gamma(N+1)\Gamma(N)\Gamma(\nu+N)g_{2N-2}}\right]^{1/N} . \quad (3.4)$$

Inserting $n_{\min}$ back into Eq. (3.2), we find for the leading trajectory

$$E_0=\epsilon(\nu,N)\frac{N}{N-1}(\nu+1) . \quad (3.5)$$

Let us first compare the result to the harmonic oscillator $N=2$. Here we find

$$E_0=2\sqrt{4/3}\sqrt{g_2}\,(1+\nu) , \quad (3.6)$$

which should be compared to the exact answer

$$E_0^{\text{exact}}=2\sqrt{g_2}\,(1+\nu) . \quad (3.7)$$

The approximation gets the linearity right and slightly overshoots the slope. We return to the harmonic oscillator case later to study systematic improvement of the method.

For S waves ($\nu=\frac{1}{2}$), a case previously studied,[8] Eq. (3.5) reduces to

$$E_0(\nu=\tfrac{1}{2})=\frac{3}{2}\frac{N}{N-1}\left[\frac{2(g_{2N-2})N(N-1)\Gamma(N)}{3}\right]^{1/N} .$$

As it stands, and as it was remarked in Ref. 8, this approximation, quite good for the harmonic oscillator, gets progressively worse as we approach the square well, $N\to\infty$, when

$$E_0(\nu=\tfrac{1}{2})\approx\frac{3}{2}\frac{N}{e}\to\infty .$$

For the full trajectory, however, the method does far better. If we first let ν become large, we find

$$E_0 \underset{\nu\to\infty}{\to} \nu^{2-2/N}(g_{2N-2})^{1/N}\frac{4N}{N-1}\left[\frac{N(N-1)\Gamma^2(N)}{4\Gamma(2N)}\right]^{1/N} . \quad (3.8)$$

This result has the correct ν dependence. If we now let N become large,

$$E_0 \underset{\text{large } N}{\to} \nu^{2-2/N}(g_{2N-2})1/N \underset{N\to\infty}{\to} \left[\frac{\nu}{r_0}\right]^2 . \quad (3.9)$$

This should be compared to the exact square-well trajectory, determined by the zeros of $J_\nu(kr_0)$, $k^2=E$. For large ν, the leading zero is at ν, i.e., $E_0^{\text{exact}}=(\nu/r_0)^2$. The approximation becomes *exact* in this limit.

It is instructive to compare Eq. (3.8) with the large-ν classical trajectory, determined by minimal circular orbits,

$$E_0^{\text{cl}}=\frac{\nu^2}{r^2}+V(r)\,|_{r=r_{\min}} , \quad (3.10)$$

where $r_{\min}$ is determined by minimization of $V(r)+\nu^2/r^2$,

$$r_{\min}=\left[\frac{\nu^2}{g_{2N-2}(N-1)}\right]^{1/(2N)} , \quad (3.11)$$

so that

$$E_0^{\text{cl}}=\nu^{2-2/N}[(g_{2N-2})^{1/N}]N(N-1)^{1/N-1} . \quad (3.12)$$

The (correct) ν dependences are the same for Eqs. (3.8) and (3.12). In Fig. 1 we plot the coefficients of

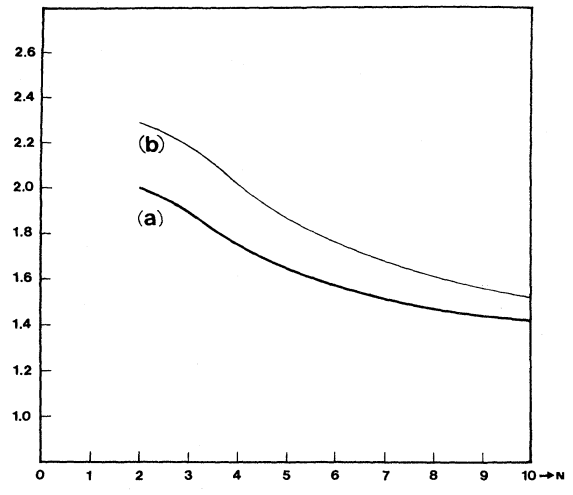

FIG. 1. Comparison of two large-ν approximations for the leading Regge trajectory E_0 for $V(r)=gr^{2N-2}$, with ν dependence and g dependence removed. Curve (a) corresponds to the classical trajectory (circular orbits) (Eq. (3.12), while curve (b) is determined by the Borel-summation method (Eq. 3.8). Both curves approach the asymptote 1 for $N\to\infty$ (the square well).

$\nu^{2-2/N}(g_{2N-2})^{1/N}$ as a function of N.

One further test is useful, namely, an assessment of the effect of carrying more terms in $\hat{S}$. As we shall see, the importance of this test is magnified when we consider short-range corrections in Sec. IV. The work of Beavis, Chu, and Kaus[11] can be used to extract the second correction for the harmonic oscillator, namely,

$$\alpha_1(\nu)=0,\quad \alpha_2(\nu)=-\tfrac{2}{3}\nu(\nu^2-1)g_2,\quad \alpha_3(\nu)=0\ ,$$
$$\alpha_4(\nu)-\tfrac{4}{90}\nu(\nu^2-1)(\nu-2)(\nu-3)(5\nu+7){g_2}^2\ . \tag{3.13}$$

Application of the Borel-transform method to $\hat{S}$ truncated now at E^{-4} rather than E^{-2} gives us

$$n_{\min}=E(\tfrac{15}{16})^{1/4}{g_2}^{-1/2}$$

rather than the earlier value

$$n_{\min}=E(\tfrac{3}{4})^{1/2}{g_2}^{-1/2}\ ,$$

and a value for E_0,

$$E_0=\left[\tfrac{2}{3}(\tfrac{16}{14})^{1/4}+\tfrac{4}{3}(\tfrac{15}{16})^{1/4}\right]\sqrt{g_2}\,(1+\nu)$$
$$=1.99\sqrt{g_2}\,(1+\nu)\ . \tag{3.14}$$

The coefficient 1.99 can be compared to the leading-approximation result 2.3 and the exact coefficient 2. Evidently the Borel-transform method is susceptible to systematic improvement for the class of potentials we have discussed. Many of these results and others on the Borel-transform method may be found in the first paper in Ref. 10.

It is interesting to compare these results to an early implementation of the duality principle, the α expansion. This method was applied[2] to the harmonic oscillator and "zeroth-order" and "first-order" corrections were used to search for the ground state starting from the large-E behavior of the two-point function. The slopes of the leading trajectory in these two approximations were found to be, respectively, 2.33, $\sqrt{g_2}$ and 2.06 $\sqrt{g_2}$, compared with the corresponding numbers 2.30 $\sqrt{g_2}$ and 1.99 $\sqrt{g_2}$ for the Borel-transform method.

IV. EFFECTS OF SHORT-RANGE POTENTIALS

Short-range contributions to the confining interaction present us with an entirely different set of problems. Such components can affect the large-E asymptotic form of the two-point function in ways which mimic long-range forces if only a finite number of terms in E^{-1} are known. Physically, however, short-range forces should have no effect on the large ν portion of the trajectory. This difficulty is already evident in the work of Ref. 9, where it is pointed out that information on the nature of the long-range force must be known or assumed, to distinguish the short-range corrections. A scheme to deal with the short-range forces is developed there.

Our purpose in this section is twofold: first, to show how straightforward application of either the S-matrix or the Borel-transform method when there is a short-range component is wrong. The key observation is already contained in Ref. 11: the large-E form of $\hat{S}$ in inverse powers of E coming from a short-range potential corresponds to the short-distance expansion of the potential in powers of r. If the only information available from perturbative calculation is a finite number of terms in E^{-1}, then there is no way to tell if the series corresponds to a short-range or a long-range potential. A simple example which illustrates this point is provided by the function $1/(E+\mu^2)$, associated with short-range potentials, which has large-E expansion

$$\frac{1}{E}\left[1-\frac{\mu^2}{E^2}+\frac{\mu^4}{E^4}-\cdots\right].$$

Use of only a few terms would be quite misleading, since these suggest a power law $V\sim r^2, r^6\cdots$.

Our second purpose is to test numerically the perturbative scheme developed in Ref. 9 for finding the shifts in the bound-state poles due to a short-range interaction.

We consider for these purposes the potential

$$V(r)=gr^2-\lambda(1-e^{-\mu^2r^2})\ . \tag{4.1}$$

The expansion of $\hat{S}(\nu,-E)$ is to $O(E^{-3})$

$$\hat{S}(\nu,-e)=(+E)^{\nu}\left[1+\frac{\alpha_2}{E^2}-\frac{\alpha_3}{E^3}+\cdots\right], \tag{4.2}$$

where[11]

$$\alpha_2=-\tfrac{2}{3}\nu(\nu^2-1)(g-\lambda\mu^2)\ , \tag{4.3}$$

$$\alpha_3=-\tfrac{4}{15}\nu(\nu^2-1)(\nu^2-4)\lambda\mu^4\ . \tag{4.4}$$

We first apply the Borel-transform method uncritically to this case. If we cut the series in Eq. (4.2) with the E^{-2} term, we "read" an harmonic oscillator with a shifted spring constant, $g\to g-\lambda\mu^2$. If we include the E^{-3} term, we "read" a mixed r^2, r^4 potential and find from Eq. (2.12) that

$$R_n=E\left[\frac{n+1}{n-\nu}\right]\left[1+\frac{2\alpha_2(\nu-n-1)}{E^2\nu(\nu-1)}\right.$$
$$\left.-\frac{3\alpha_3(\nu-n-1)(\nu-n-2)}{E^3\nu(\nu-1)(\nu-2)}\right]. \tag{4.5}$$

As discussed in Sec. II, we anticipate $n_{\min}=E/\epsilon$ and keep the leading terms in E while searching for $n_{\min}$. We find ϵ to be determined by the roots of the cubic equation

$$\epsilon^3-\tfrac{4}{3}(g-\lambda\mu^2)-\tfrac{8}{5}\lambda\mu^4(\nu+2)=0\ , \tag{4.6}$$

and E_0 to be given by

$$E_0\approx(1+\nu)\epsilon+\frac{4}{3}\frac{(g-\lambda\mu^2)(\nu+1)}{\epsilon}$$
$$+\frac{4}{5}\lambda\mu^4\frac{(\nu+1)(\nu+2)}{\epsilon^2}\ . \tag{4.7}$$

Before studying these equations in detail, several comments are in order. First, for sufficiently large ν we see that the only real root of Eq. (4.6) is

$$\epsilon\underset{\nu\to\infty}{\to}[\tfrac{8}{5}\lambda\mu^4(\nu+2)]^{1/3}\ , \tag{4.8}$$

which only involves the parameters of the short-range potential. The interpretation of this result is simple: by keeping only up to $O(E^{-3})$ in $\hat{S}(\nu,E)$, the Borel-transform method "reads" a confining potential r^4, with coefficient $\lambda\mu^4$, corresponding to the $O(r^4)$ expansion of the potential (4.1). This will lead to a misleading trajectory. We show the numerics below; analytically we observe that for large ν,

$$E_0 \sim \nu^{4/3} , \tag{4.9}$$

a trajectory rising more slowly [$\nu \sim (E_0)^{3/4}$] than linearly and characteristic of r^4 confinement.

Second, the Borel-transform method requires the right *sign* to be applicable: if the short-range component of Eq. (4.1) were positive rather than negative, we would expect the physics to change very little. But in the expansion (4.2) the sign of α_3 would change, the large-ν root (4.8) would change sign, and so would E_0. This anomaly occurs because the method reads an *anticonfining* r^4 term.

Third, let us discuss the choice of roots of Eq. (4.6) in order to make numerical fits. Knowledge of the large-ν behavior leads one in this case to pick that real root which reduces to the single real root (4.8) in the large-ν limit. Define the two parameters

$$x = +\tfrac{4}{5}\lambda\mu^4(\nu+2), \quad y = -\tfrac{4}{9}(g-\lambda\mu^2) . \tag{4.10}$$

Then for fixed $\lambda\mu^2$ and sufficiently large μ^2 and/or ν, y^3+x^2 is positive and there is only the single real root

$$\epsilon = [x+(y^3+x^2)]^{1/3}+[x-(y^3+x^2)^{1/2}]^{1/3} . \tag{4.11}$$

As μ^2 and/or ν decreases, y^3+x^2 goes negative and the proper root is

$$\epsilon = 2[x^2+|y^3+x^2|]^{1/6} \times \cos\left\{\tfrac{1}{3}\arctan\left[\frac{(|y^3+x^2|)^{1/2}}{x}\right]\right\} . \tag{4.12}$$

Next we discuss the S-matrix method in this context. From the above comments it is clear that knowledge of the true nature of the confining force is necessary. As a given of the method, we take such knowledge, not only of the form of the asymptotic slope but of the coefficient of that slope. There are claims that this information is indeed provided in QCD.[13] In that case we can treat the effects of the short-range contribution as a perturbation. The short-range contribution is determined as far as it can be determined by the large-E behavior of $\hat{S}$.

To illustrate this, we consider again $\hat{S}$ as in Eq. (4.2) and add the independent knowledge that the trajectory behaves at large ν like $2\sqrt{g}\,\nu$. In that case we assume $\hat{S}$ results from a potential $V=gr^2+\delta v$. δV can be any short-range form with two parameters which are fit from

$$\bar{\alpha}_2 = \alpha_2 + \tfrac{2}{3}\nu(\nu^2-1)g$$

and α_3. We could choose a form such as in Eq. (4.1) but another form such as $\delta V = \lambda e^{-\mu^2 r^2}$ or even a nonexponential Padé form would be equally good. We now treat this δV as a perturbation to $O(\lambda)$ on the $\hat{S}$ which follows from gr^2 alone. To be specific, write as in Ref. 9

$$\hat{S}(\nu,E) = \lim_{R\to\infty} \frac{D(-\nu,k,R)}{D(\nu,k,R)} , \tag{4.13}$$

where $D(\pm\nu,k,r)$ are the two solutions to the Schrödinger equation with the Wronskian $W(+\nu,-\nu) = -2\nu$:

$$D(\nu,k,R) = D_0(\nu,k,R)[1+I_1(\nu,k,R)] + D_0(-\nu,k,R)I_2(\nu,k,R) , \tag{4.14}$$

$$I_1 = \frac{1}{2\nu}\int_0^R D_0(-\nu,k,r)D_0(\nu,k,r)\delta V\,dr , \tag{4.15}$$

$$I_2 = -\frac{1}{2\nu}\int_0^R D_0^2(\nu,k,r)\delta V\,dr . \tag{4.16}$$

The zeros of $\lim_{R\to\infty} D_0(\nu,k,R)$ determine the bound-state poles of the unperturbed problem just as the zeros of $\lim_{R\to\infty} D(\nu,k,R)$ determine the poles of the perturbed problem.

It is in fact simpler to look for the zeros of $\hat{S}(-\nu,E)$. We have for the unperturbed problem, near the state E_{n0},

$$\hat{S}_0(-\nu,E) = \lim_{R\to\infty}\frac{D_0(\nu,k,R)}{D_0(-\nu,k,R)} \approx \gamma_n(E-E_{n0}) . \tag{4.17}$$

Then, expanding in δV and $(E-E_{n0})$, we find

$$\hat{S}(-\nu,E) = \gamma_n(E-E_{n0}) + \lim_{R\to\infty} I_2(\nu,k_{n0},R) ,$$

or the perturbed ground-state energies

$$E_n = E_{n0} - \lim_{R\to\infty} I_2(\nu,k_{n0},R)/\gamma_n . \tag{4.18}$$

If we assume the short-range potential to be of the form in Eq. (4.1), the integral I_2 appearing in Eq. (4.18) is easily evaluated using harmonic-oscillator wave functions. This gives for the perturbed top trajectory

$$E = E_0 + \lambda\left[\left[\frac{1}{1+\mu^2/\sqrt{g}}\right]^{\nu+1} - 1\right] , \qquad E_0 = 2\sqrt{g}\,(\nu+1) . \tag{4.19}$$

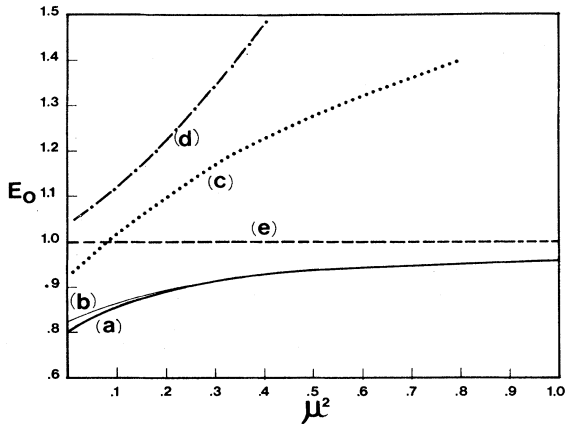

FIG. 2. The ground-state energy E_0 for the potential Eq. (4.1). What is plotted is E_0 vs μ^2, with $g=\frac{1}{9}$ and $\lambda\mu^2=0.04$. (a) Exact solution; (b) S-matrix method solution; (c) Borel-summation solution; (d) α-expansion (Ref. 2) solution; (e) the ground-state energy for the "unperturbed" problem, $\lambda=0$.

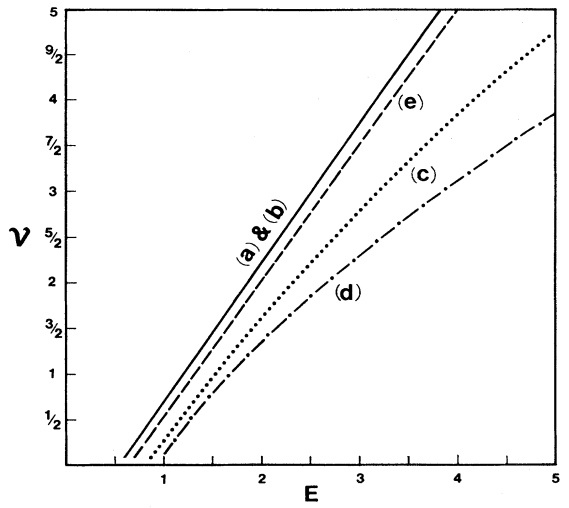

FIG. 3. The top Regge trajectory for the potential Eq. (4.1). What is plotted is ν vs E, with $\lambda\mu^2=0.04$ and $\lambda=\mu^2=0.02$. (a) and (b) (indistinguishable on this scale), the exact and S-matrix method results; (c) Borel-summation result; (d) α-expansion solution; (e) The ground-state trajectory for the "unperturbed" problem, $\lambda=0$.

For numerical comparison of the various methods discussed above, we first concentrate on S waves ($\nu=\frac{1}{2}$). We take the following parameters: $g=\frac{1}{9}$ [convenient because it gives $E_0(\nu=\frac{1}{2})=1$], $\lambda\mu^2=0.04$, and μ^2 varying over the range 0 to 1. Figure 2 shows four curves: (i) the exact result according to numerical solution of the Schrödinger equation, (ii) the S-matrix method result with the short-range term treated as a perturbation, (iii) the Borel-transform result, and for completeness (iv) the α-expansion result, as taken from Ref. 2. Note that as μ^2 increases for fixed $\lambda\mu^2$ the δV term becomes shorter range, and the Borel-transform method "misreads" the confinement to greater degree.

Figure 3 shows the top Regge trajectory for the exact solution, the S-matrix method, and the Borel-transform method. The curvature in the Borel-transform trajectory ($\nu\sim E^{3/4}$) is clearly visible.

The significance of the observations above for phenomenology should be clear.

V. RESIDUES

Because the residue of the pole at $E=E_k$, $2\nu[a_k(\nu)]^2$, is proportional to the width of the state, it is interesting to ask if it is possible to extract this quantity from methods discussed above. The S-matrix method "reads" the potential and so computes the residue directly. Of more interest is the question of whether the residue can be obtained using the asymptotic behavior alone. In particular, Eq. (2.8) suggests that

$$2\nu[a_0(\nu)]^2=(E_0+E)^{n+1}M_n \ . \tag{5.1}$$

As we shall see by example, this is inadequate, for the following two reasons. First, higher-order corrections in E^{-1} to the asymptotic form of $\hat{S}$ for long-range confining potentials provide more accurate approximations to the bound states. However each one of these corrections make roughly equal or even increasing contributions to the residues. Thus *all* orders in E^{-1} are required to give the residues. Second, overall functions of ν which multiply the form (2.5) will be important; these factors, which help ensure the positivity of the residues, would have to be included in perturbative calculation.

Let us apply Eq. (5.1) in a more general form. We have from Eq. (2.11)

$$2\nu[a_0(\nu)]^2=-\frac{(E_0+E)^{n+1}E^{\nu-n}}{\Gamma(n+1)} \times\sum_{s=0}^{s_{\max}}(-1)^s\frac{\alpha_s(\nu)\Gamma(n+s-\nu)}{\Gamma(s-\nu)E^s}\Bigg|_{n=n_{\min}} . \tag{5.2}$$

Using $n_{\min}=E/\epsilon(\nu)$, we have

$$2\nu[a_0(\nu)]^2=-[n\epsilon(\nu)]^{\nu+1}\left[1+\frac{E_0}{\epsilon n}\right]^{n+1}\frac{1}{\Gamma(n+1)} \times\sum_{s=0}^{s_{\max}}(-1)^s\frac{\alpha_s(\nu)\Gamma(n+s-\nu)}{\Gamma(s-\nu)E^s} . \tag{5.3}$$

We can also use for large n

$$\frac{\Gamma(n+s-\nu)}{\Gamma(n+1)E^s}\approx n^{-\nu-1}\left[1+O\left[\frac{1}{n^2}\right]\right][\epsilon(\nu)]^{-s} . \tag{5.4}$$

Thus

$$2\nu[a_0(\nu)]^2\approx-[\epsilon(\nu)]^{\nu+1}e^{E_0/\epsilon(\nu)} \times\sum_{s=0}^{s_{\max}}(-1)^s\frac{\alpha_s(\nu)}{\Gamma(s-\nu)[\epsilon(\nu)]^s} . \tag{5.5}$$

$\epsilon(\nu)$ generally will contain powers of the coupling to compensate those in $\alpha_s(\nu)$. So unless the numerical coefficient in $\epsilon(\nu)$ changes significantly with $s_{\max}$, and it will not, each term in the sum of Eq. (5.5) will be important.

The harmonic oscillator provides a useful laboratory to test these questions. For $V(r)=gr^2$, the exact value of the residue is

$$2\nu[a_0(\nu)]^2\,|_{\text{exact}}=4\nu g^{(\nu+1)/2}\frac{1}{\Gamma(\nu+1)} . \tag{5.6}$$

The exact $\hat{S}$ is given by[9]

$$\hat{S}(\nu,E)=g^{\nu/2}\frac{\Gamma\left[-\frac{\eta}{2}+\frac{1}{2}+\frac{\nu}{2}\right]}{\Gamma\left[-\frac{\eta}{2}+\frac{1}{2}-\frac{\nu}{2}\right]}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} , \tag{5.7}$$

where

$$\eta\equiv E/2(\sqrt{g}) .$$

The series expression Eq. (2.5) comes from the large-E expansion of the ratio of the two E-dependent Γ functions in Eq. (5.7). The factor $\Gamma(1-\nu)/\Gamma(1+\nu)$, which changes sign as ν passes through positive integers and is crucial for the positivity of the residues (5.6), will be multiplicative in Eq. (2.5) and thus not generally available in perturbative

calculation. This pinpoints one problem in the calculation of residues. Note that for small ν this factor is $O(1)$.

Let us now apply Eq. (5.5) with the factor $[\Gamma(1-\nu)]/[\Gamma(1+\nu)]$ included. Truncate the series at $s=2$, and recall that then we have Eq. (3.7) for E_0, and $\epsilon(\nu)=\sqrt{4g/3}$, independent of ν for the harmonic oscillator. This will give us

$$2\nu[a_0(\nu)]^2=-e^{2(1-\nu)}\left[\frac{4g}{3}\right]^{(\nu+1)/2}\frac{1}{\Gamma(\nu+1)}\frac{\nu(\nu-1)}{2}\,. \tag{5.8}$$

Here the g dependence of α_2 cancels the g dependence of ϵ^2, so the α_2 term contribution to the sum is $-\frac{1}{2}(\nu+1)$ times as big as the α_0 term. The contributions of higher-order terms would contribute *growing* powers of ν. Comparing Eq. (5.8) to (5.6) we see the ν dependence of the approximation to be thoroughly incorrect.

It is interesting to try to include the entire series in α_s for the harmonic oscillator to see if satisfactory results for the residues are thereby achieved. For the ratio of E-dependent Γ functions of Eq. (5.7) for $\hat{S}(-E)$ we have the general expansion[15]

$$\frac{\Gamma\left[\frac{E}{4\sqrt{g}}+\frac{1+\nu}{2}\right]}{\Gamma\left[\frac{E}{4\sqrt{g}}+\frac{1-\nu}{2}\right]}=\left[\frac{E}{4\sqrt{g}}\right]^{\nu}\sum_{s=0}^{\infty}\frac{\Gamma(\nu+1)}{\Gamma(\nu-s+1)}q_s\left[\frac{1+\nu}{2},\frac{1-\nu}{2}\right]\left[\frac{4\sqrt{g}}{E}\right]^{s}, \tag{5.9}$$

where q_s is determined by

$$e^{-at}(1-e^{-t})^{b-a-1}=\sum_{s=0}^{\infty}(-1)^s q_s(a,b)t^{s+b-a-1},\quad |t|<2\pi\,. \tag{5.10}$$

Comparison with Eq. (2.5) yields

$$\alpha_s=(-1)^s\frac{\Gamma(\nu+1)}{\Gamma(\nu-s+1)}q_s\left[\frac{1+\nu}{2},\frac{1-\nu}{2}\right](4\sqrt{g})^s \tag{5.11}$$

and a factor $(4\sqrt{g})^{-\nu}$ multiplying (2.5). We next need to evaluate the full sum, $s_{\max}\to\infty$, in Eq. (5.5). This sum is of the form

$$\begin{aligned}\sum_s(-1)^s\frac{\alpha_s}{\Gamma(s-\nu)\epsilon^s}&=\sum_s\frac{\Gamma(\nu+1)}{\Gamma(s-\nu)\Gamma(\nu-s+1)}q_s\left[\frac{1+\nu}{2},\frac{1-\nu}{2}\right]\left[\frac{4\sqrt{g}}{\epsilon}\right]^s\\&=-\frac{\sin\pi\nu}{\pi}\Gamma(\nu+1)\sum_s(-1)^s q_s\left[\frac{1+\nu}{2},\frac{1-\nu}{2}\right]\left[\frac{4\sqrt{g}}{\epsilon}\right]^s\\&=-\frac{\sin\pi\nu}{\pi}\Gamma(1+\nu)\left[\frac{4\sqrt{g}}{\epsilon}\right]^{\nu}\exp\left[-\frac{(1+\nu)}{2}\frac{4\sqrt{g}}{\epsilon}\right]\left[1-\exp\left[-\frac{4\sqrt{g}}{\epsilon}\right]\right]^{-\nu-1}.\end{aligned} \tag{5.12}$$

Insert this result into (5.5), together with the multiplicative factors

$$(\sqrt{g})^{\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}$$

to find

$$2\nu[a_0(\nu)]^2=4\nu\frac{1}{\Gamma(1+\nu)}[\epsilon(\nu)]^{\nu+1}\left[\frac{\sqrt{g}}{\epsilon}\right]^{\nu}\exp\left\{[E_0-2(1+\nu)\sqrt{g}\,]/\epsilon\right\}\left[1-\exp\left[-\frac{4\sqrt{g}}{\epsilon}\right]\right]^{-\nu-1}, \tag{5.13}$$

which must be compared to the exact result (5.6). When we calculated in the Borel-transform method to $O(1/E^2)$, we found $\epsilon=\sqrt{4/3}\sqrt{g}$; to $O(1/E^4)$, $\epsilon=(\frac{16}{15})^{1/4}\sqrt{g}$. Evidently as we improve the accuracy of our approximation ϵ rapidly approaches $\sqrt{g}$. Similarly, E_0 approaches $2(1+\nu)\sqrt{g}$, the exact answer. Thus (5.13) approaches the exact answer to exponential accuracy.

While it may be comforting to know that the answer can be reached for a known long-range potential, we should recall that an infinite number of terms in E^{-s} are never available in practice and that the residues can therefore not be calculated this way. It is an accident that for ν of $O(1)$, the truncated result (5.8) is near the exact form. For example, for $\nu=\frac{1}{2}$ (S waves), (5.8) *without* the external factor $\Gamma(1-\nu)/\Gamma(1+\nu)$ is $g^{3/4}\times 1.76$, while the exact answer is $g^{3/4}\times 2.26$. Fortuitous agreement of lower-order calculation[6] at low ν is thus possible.

Of course there may be other handles on the residues, as, for example, in Refs. 16 and 17. We hope to discuss these questions in the future.

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