Imbedding a Schwarzschild mass into cosmology

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We develop a method for imbedding a Schwarzschild mass into a zero-curvature universe. We work with curvature coordinates (R,T), in terms of which the metric has the form $ds^2(R,T)=A^{-1}(R,T)dR^2+R^2d\Omega^2-B(R,T)dT^2$, and coordinates (R,τ), where τ is measured by radially moving geodesic clocks. We solve the field equations for a stress-energy tensor that corresponds to a radially moving perfect geodesic fluid outside some boundary R_b. Inside R_b we take the stress-energy tensor to be composed of a perfect-fluid part and a Schwarzschild matter part. Specific examples of imbedding a mass into a de Sitter universe and a pressure-free Einstein—de Sitter universe are given, and we show how to extend our methods to general zero-curvature universes. A consequence of our results is that there will be spiralling of planetary orbits when a mass such as our Sun is imbedded in a universe. We relate our work to recent work done by Dirac with regard to his Large Numbers hypothesis.

I. INTRODUCTION

Standard cosmological theory assumes that stars and galaxies in the Universe are to be treated as a smoothed-out perfect fluid, whose particles follow geodesic trajectories. The cosmological fluid is taken to be isotropic and homogeneous. It follows from the homogeneity requirement that at any given value of a cosmological time τ the density ρ and pressure p of the cosmological fluid will have a constant value everywhere throughout the Universe. The stress-energy tensor corresponding to the fluid then serves as the source in the Einstein field equations to determine the metric of the Universe.

This standard cosmological picture is not applicable for dealing with the field in the vicinity of a star such as our Sun. In this region, the Schwarzschild field will dominate, with the cosmological field perhaps exerting some small perturbative effects. It is thus worthwhile to develop a formalism that will incorporate standard cosmology at large distances and the Schwarzschild field at small distances. At the outset, this is seen to be a basically inhomogeneous problem, because the region in the immediate vicinity of the Schwarzschild mass imbedded in the Universe will be inhomogeneous.

Various approaches to the problem can be found in the literature. McVittie[1] starts with the Schwarzschild metric expressed in isotropic coordinates (R',T'),

$$ds^2(R',T')=(1+M/2R')^4(dR'^2+R'^2d\Omega^2)-\frac{(1-M/2R')^2}{(1+M/2R')^2}dT^2\ , \tag{1.1}$$

which can be obtained from the Schwarzschild metric expressed in curvature coordinates (R,T),

$$ds^2(R,T)=\frac{dR^2}{1-2M/R}+R^2d\Omega^2-(1-2M/R)dT^2\ , \tag{1.2}$$

by means of the spatial transformation

$$R=R'(1+M/2R')^2\ ,\qquad R'=\tfrac{1}{2}[R-M\pm(R^2-2MR)^{1/2}]\ . \tag{1.3}$$

McVittie then joins the Schwarzschild metric form (1.1) smoothly onto the isotropic Robertson-Walker cosmological metric form

$$ds^2(r,t)=[1+K(r/2b)^2]^{-2}e^{2h(t)}(dr^2+r^2d\Omega^2)-dt^2 \tag{1.4a}$$

or equivalently

$$ds^2(r,t)=[1+K(r/2b)^2]^{-2}e^{2h(t)}(dx^2+dy^2+dz^2)-dt^2\ . \tag{1.4b}$$

In (1.4), the specification of the function $h(t)$ describes a particular universe, whereupon $K=+1$, 0, or -1 defines the intrinsic curvature of the three-dimensional subspace $t=\text{const}$, and the constant b is a measure of the radius of curvature of this subspace.

The end result is a metric of the form

$$ds^2(r,t)=\frac{\{1+(\mu/2r)[1+K(r/2b)^2]^{1/2}\}^4}{[1+K(r/2b)^2]^2}e^{2h(t)}(dr^2+r^2d\Omega^2)-\frac{\{1-(\mu/2r)[1+K(r/2b)^2]^{1/2}\}^2}{\{1+(\mu/2r)[1+K(r/2b)^2]^{1/2}\}^2}dt^2 \tag{1.5}$$

in which $\mu(t)$, identified with the mass of the imbedded particle, varies with time according to

$$\frac{1}{\mu}d\mu/dt=-dh/dt\ . \tag{1.6}$$

For the case of zero curvature ($K=0$), which is what we shall be interested in, (1.5) reduces to

$$ds^2(r,t)=(1+\mu/2r)^4 e^{2h(t)}(dr^2+r^2d\Omega^2) - \frac{(1-\mu/2r)^2}{(1+\mu/2r)^2}dt^2 \tag{1.7}$$

in which $\mu(t)$ is still given by (1.6).

Using a somewhat different approach from that of McVittie, Dirac[2] has obtained (1.7) for the special case of an Einstein—de Sitter (ES) universe, where the cosmological constant $\Lambda=0$ and which originally had zero pressure. For an ES universe

$$h(t)=\tfrac{2}{3}\ln(t/a), \quad a=\text{const} \tag{1.8}$$

and correspondingly, from (1.6), we obtain

$$\mu(t)=\mu_0 t^{-2/3}, \quad \mu_0=\text{const}\ . \tag{1.9}$$

The resulting metric form (1.7) describes a combined field where the pressure is not zero. The main motivation for Dirac's work is that he claims that the ES universe is in agreement with his Large Numbers hypothesis.[2,3]

There are questions that can be raised about the above approach. The spatial transformation (1.3) relating R' to R is double-valued. For each value of R there correspond two values of R', as shown in Fig. 1. The value $R=\infty$ corresponds to both $R'=\infty$ as well as $R'=0$. Also, the region $R<2M$ does not appear in the range $0\leq R'\leq\infty$. It is generally accepted that the range of space is given by $0\leq R\leq\infty$. This follows, for example, from the curvature invariants varying as R^{-n}, where n is a positive integer, and the invariant area of a sphere varying as $4\pi R^2$. Also, $R^{\mu\nu\rho\sigma}$ vanishes at $R'=0$, which corresponds to $R=\infty$, showing that spacetime there is flat. This indicates that $R'=0$ corresponds to spatial infinity. One can thus question McVitties's approach, which involves a power series in $1/R'$ to match (1.1) with (1.4), as this tacitly assumes the existence of a mass point at $R'=0$, which we now see corresponds to spatial infinity.

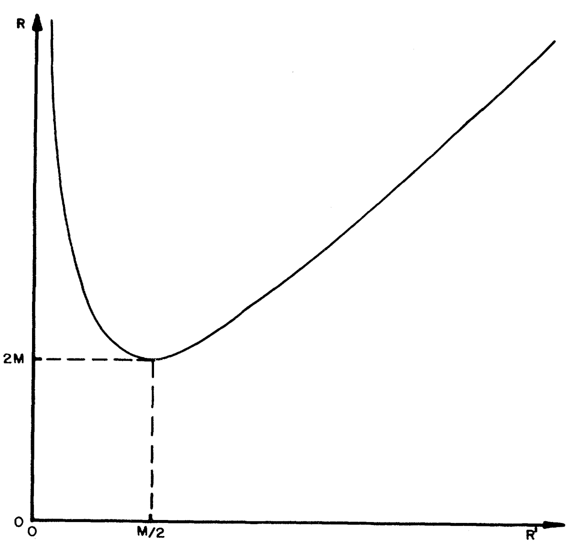

FIG. 1. The relationship between the isotropic radial coordinate R' used in the metric form (1.1) and the curvature radial coordinate R used in the metric form (1.2), as given by the transformation (1.3). It is seen that for each value $R>2M$ there are two values of R', and that the region $R<2M$ is not included in the range $0\leq R'\leq\infty$. In particular, the value $R'=0$ corresponds to $R=\infty$, where $R^{\mu\nu\rho\sigma}=0$, so that there spacetime is flat. This indicates that a point $R'=0$ corresponds to spatial infinity.

In addition to the spatial coordinate, the time coordinate in (1.1) or (1.2) is fundamentally different in origin from the time coordinate used in (1.4). In (1.1) and (1.2), T is a curvature time coordinate, which is related to times measured by clocks located at fixed values of R' or R. As such, these clocks do not follow geodesic trajectories. On the other hand, t in (1.4) is measured by clocks that are moving along geodesic trajectories, and thus are moving relative to fixed R' or R points.

Also, the above approach does not give results that would be expected when a Schwarzschild mass is imbedded in a de Sitter universe. For a de Sitter universe, $K=0$, and $h(t)$ in (1.4) is given by[4]

$$h(t)=t/R_0, \quad R_0=(3/\Lambda)^{1/2}\ . \tag{1.10}$$

One then finds from (1.6) that the mass $\mu(t)$ to be used in (1.7) is

$$\mu(t)=\mu_0 e^{-t/R_0}, \quad \mu_0=\text{const}\ . \tag{1.11}$$

It is well known, however, that the most general solution to the Einstein vacuum field equations with cosmological constant has the curvature form

$$ds^2(R,T)=\frac{dR^2}{1-(R/R_0)^2-2M/R}+R^2d(\Omega)^2 - [1-(R/R_0)^2-2M/R]dT^2 \tag{1.12}$$

in which M is a constant. For $\Lambda=0$, (1.12) is the Schwarzschild metric, while for $M=0$, (1.12) is the de Sitter metric. It therefore seems reasonable to regard (1.12) as representing the field of a Schwarzschild mass M imbedded in a de Sitter universe. This interpretation of (1.12), though, results in a field different from (1.7) with (1.10) and (1.11).

Einstein and Straus have taken a different approach to the imbedding problem.[5,6] Inside a cosmological fluid with zero pressure and zero cosmological constant, they cut out a spherical vacuum region, in which they place a Schwarzschild mass M, as shown in Fig. 2. They then work out the relationships for the vacuum Schwarzschild metric to join on smoothly to the cosmological metric form (1.4) at some radius $R_u(T)$. Working with the metric form (1.4b), Einstein and Straus showed that a solution exists for the problem they posed, but were not able to give an explicit expression for the corresponding metric form. Subsequently, Schücking[7] obtained an explicit form for the metric of Einstein and Straus by working in curvature coordinates.

In this paper, we develop an approach to the problem of imbedding a Schwarzschild mass into a given cosmology that is different from the ones described above. In our approach, we extend the methods of our paper immediately

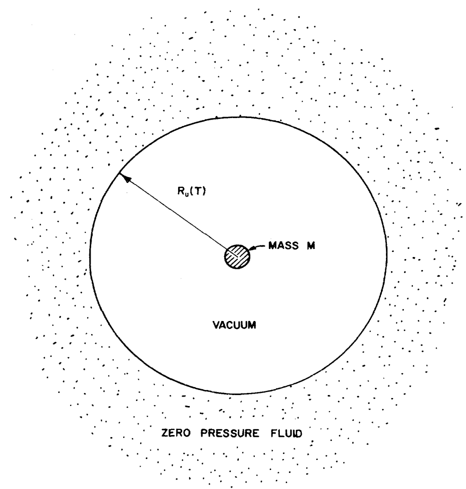

FIG. 2. The imbedding model used by Einstein and Straus, and Schücking. Inside the expanding radius $R_u(T)$ there is a vacuum region with the imbedded (constant) Schwarzschild mass M. Outside $R_u(T)$ there is a pressure-free cosmological fluid. The metrics in each of the two regions are matched on smoothly at $R_u(T)$.

preceding this one,[8] in which we have developed cosmological theory in terms of curvature coordinates (R,T) and coordinates (R,τ), where τ is measured by geodesic clocks fixed in the cosmological fluid that serves as the source of the universe.

In Sec. II, we develop the Einstein field equations corresponding to a stress-energy tensor where the matter of the imbedded Schwarzschild mass is located within some bounding curvature radius R_b (see Fig. 3). Outside R_b, there is a radially moving perfect fluid with stress-energy tensor F^μ_ν, whose particles follow geodesic trajectories. Inside R_b, we assume that the stress-energy density component τ^4_4 can be written as

$$\tau^4_4 = M^4_4 + F^4_4 = M^4_4 - \rho \ .$$

Here M^4_4 is associated with the density of the imbedded Schwarzschild matter, and $\rho = -F^4_4$ is the density of the fluid, which is taken to extend from inside R_b to outside R_b.

As specific examples, we show in Sec. III that when our approach is applied to imbedding in a de Sitter universe, we obtain the expected metric form (1.12). In Sec. IV, we describe how to imbed a Schwarzschild mass into an Einstein-de Sitter universe to obtain a field where the cosmological fluid has zero pressure outside the boundary R_b of the Schwarzschild mass. In Sec. V, we show how to generalize the procedures of Secs. III and IV to imbed a Schwarzschild mass into a general zero-curvature cosmology.

In Sec. VI, we show that if g_{44} depends on time, there will be a spiralling of planetary orbits around a Schwarzschild mass imbedded in a universe. Relationships between the work presented here and Dirac's work on his Large Numbers hypothesis are discussed in Sec. VII.

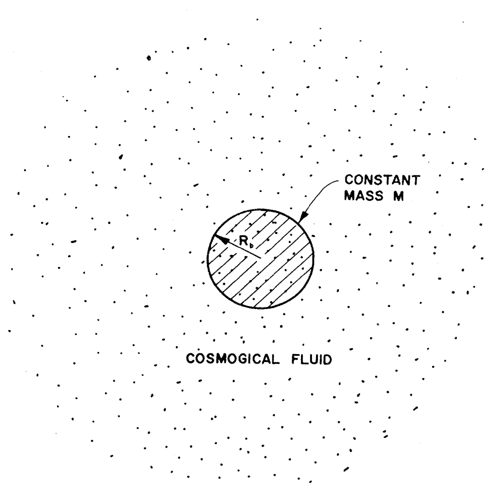

FIG. 3. The imbedding model used in this paper. A constant Schwarzschild mass of radius R_b is immersed in a perfect fluid whose particles follow radial geodesics. Outside R_b, the stress-energy tensor corresponds to that of a perfect fluid: $\tau^\mu_\nu = F^\mu_\nu$. Inside R_b, we take $\tau^4_4 = F^4_4 + M^4_4$, where $M^4_4 = \text{const}$ corresponds to the imbedded Schwarzschild matter.

II. THE FIELD EQUATIONS FOR AN IMBEDDED MASS

For convenience, we will restate relevant expressions from our previous paper.[8] A completely general curvature metric form for a spherically symmetric field is

$$ds^2(R,T) = \frac{dR^2}{A(R,T)} + R^2 d\Omega^2 - A(R,T) f^2(R,T) dT^2 \ . \tag{2.1}$$

The time coordinate T in (2.1) is associated with times recorded by clocks located at fixed $R = \text{const}$ points. These clocks do not move along geodesics. If we change the time coordinate from T to a new time coordinate τ measured by a congruence of radially moving geodesic clocks, the metric form (2.1) takes the nondiagonal form

$$ds^2(R,\tau) = k^{-2}[dR - m(k^2 - A)^{1/2} d\tau]^2 + R^2 d\Omega^2 - d\tau^2 \ , \tag{2.2}$$

where k is a parameter related to the energy per unit mass of the geodesic clocks measuring τ, and $m = +1$ or -1 if, respectively, the clocks are moving in the sense of increasing or decreasing R. The transformation between the two time coordinates is defined by

$$\tau_{,R} = -mA^{-1}(k^2 - A)^{1/2} \ , \tag{2.3a}$$

$$\tau_{,T} = kf \ . \tag{2.3b}$$

The trajectories of the geodesic clocks measuring τ are given in (R,T) coordinates by

$$dR/d\tau = m(k^2 - A)^{1/2}, \quad dT/d\tau = k/fA \tag{2.4a}$$

and in (R,τ) coordinates by

$$dR/d\tau = m(k^2 - A)^{1/2}, \quad d\tau/d\tau = 1 \ . \tag{2.4b}$$

From the field equations, with cosmological constant Λ, we obtain

$$\frac{\partial}{\partial R}[R(1-A)]=-8\pi R^2\tau_4^4+R^2\Lambda , \tag{2.5}$$

$$\frac{\partial}{\partial \tau}[R(1-A)]=8\pi R^2\tau_4^1 , \tag{2.6}$$

$$\frac{\partial}{\partial R}[R(1-A)]+m(k^2-A)^{-1/2}\frac{\partial}{\partial \tau}[R(1-A)]+2m(RA/k)(k^2-A)^{-1/2}[\partial k/\partial\tau+m(k^2-A)^{1/2}\partial k/\partial R]$$
$$=-8\pi R^2\tau_1^1+R^2\Lambda , \tag{2.7}$$

where τ_ν^μ is the stress-energy tensor in (R,τ) coordinates.

The above expressions hold for arbitrary τ_ν^μ. We now look at the form we would expect for the stress-energy tensor describing a Schwarzschild mass imbedded in a given cosmology.

It seems reasonable to break the spacetime region into two parts. We will take the matter of the imbedded Schwarschild mass to lie within the sphere $R=R_b$. Outside R_b, we will take the stress-energy tensor to correspond to a cosmological perfect fluid with stress-energy tensor F_ν^μ related to the fluid density ρ and pressure p by

$$F_\nu^\mu=(\rho+p)V^\mu V_\nu+p\sigma_\nu^\mu,\quad V^\mu=dx^\mu/d\tau , \tag{2.8}$$

whose particles move along radial geodesics. Further, the time τ used in the metric form (2.2) will be taken as the time recorded by clocks that remain coincident with the fluid particles. Substitution of (2.4b) into (2.8) gives, for $R>R_b$,

$$\tau_4^4=F_4^4=-\rho , \tag{2.9a}$$

$$\tau_1^1=F_1^1=p , \tag{2.9b}$$

$$\tau_4^1=F_4^1=-m(k^2-A)^{1/2}(\rho+p) , \tag{2.9c}$$

$$\tau_2^2=\tau_3^3=F_2^2=F_3^3=p . \tag{2.9d}$$

We then obtain from (2.7) (Ref. 8)

$$dk/d\tau=0 , \tag{2.10}$$

which shows that the energy parameter k will be a constant along each streamline of the cosmological fluid where $R>R_b$.

Following the procedure developed in our previous paper,[8] we obtain in the region $R>R_b$ from (2.5), (2.6), and (2.7), respectively,

$$A(R,\tau)=1+(8\pi/R)\int_0^{R_b}\tau_4^4R^2dR-(8\pi/R)\int_{R_b}^{R}\rho R^2dR$$
$$-(\Lambda/3)R^2 , \tag{2.11}$$

$$\tau_4^1=-(1/R^2)\int_0^{R_b}(\partial\tau_4^4/\partial\tau)R^2dR$$
$$+(1/R^2)\int_{R_b}^{R}(\partial\rho/\partial\tau)R^2dR , \tag{2.12}$$

$$p+\rho-(m/R^2)(k^2-A)^{-1/2}\left[\int_0^{R_b}(\partial\tau_4^4/\partial\tau)R^2dR\right.$$
$$\left.-\int_{R_b}^{R}(\partial\rho/\partial\tau)R^2dR\right]=0 , \tag{2.13}$$

where the integrals are taken over a $\tau=$const surface (which is different from a $T=$const surface).

What remains now is the description of the stress-energy tensor component τ_4^4 in the region $R\le R_b$. One possible approach is to assume that τ_4^4 for $R\le R_b$ is composed of two parts. One part is $F_4^4=-\rho$ of the cosmological fluid given in (2.9a). The other part will be a stress-energy tensor component M_4^4 associated with the matter of the imbedded Schwarzschild mass. Thus, we assume we can write τ_4^4 in the region $R\le R_b$ as

$$\tau_4^4=F_4^4+M_4^4=-\rho+M_4^4 . \tag{2.14}$$

We now make a further assumption that M_4^4 is independent of the time τ (or time T), so that $\partial M_4^4/\partial\tau=0$. (It is a straightforward matter to generalize the following equations if the assumption of time independence of M_4^4 is dropped.) We define the (constant) Schwarzschild mass M inside R_b by

$$M=-4\pi\int_0^{R_b}M_4^4R^2dR . \tag{2.15}$$

Equations (2.11), (2.12), and (2.13) then become, respectively,

$$A(R,\tau)=1-2M/R-(8\pi/R)\int_0^R\rho R^2dR-(\Lambda/3)R^2 , \tag{2.16}$$

$$\tau_4^1=(1/R^2)\int_0^R(\partial\rho/\partial\tau)R^2dR , \tag{2.17}$$

$$p+\rho+(m/R^2)(k^2-A)^{-1/2}\int_0^R(\partial\rho/\partial\tau)R^2dR=0 . \tag{2.18a}$$

In our previous paper,[8] we found that the condition of homogeneity relative to the cosmological time τ required that the cosmological fluid density ρ depend only on τ and not on the spatial coordinate R. Now, however, with an imbedded Schwarzschild mass M, we do not have a homogeneous field. Thus, we are not justified in assuming ρ and $\partial\rho/\partial\tau$ are independent of R so that they may be brought outside the spatial integrals in the above expressions.

In our previous paper,[8] we also found that the homogeneity condition required setting the energy parameter $k=1$. Even though we here are not dealing with a homogeneous situation, the following reasoning shows that we should still set $k=1$ for the fluid particles. It seems reasonable to assume that as R tends toward zero inside the imbedded mass, spacetime should become flat (condition of elementary flatness). This is equivalent to requiring $A\to1$ as $R\to0$. If we had $k<1$, (2.4) shows that at some sufficiently small value of R, we would have a turning radius where $dR/d\tau=0$, indicating a hole in the cosmolgical fluid, which hardly seems like a good description of a cosmological fluid. On the other hand, if $k>1$, (2.9c) shows that τ_4^1 would tend toward a nonzero value as $R\to 0$, indicating that $R=0$ is a source or sink of cosmological fluid, depending on the sign of m. This also seems like an unsatisfactory description of a cosmological fluid. Thus we are led to the value $k=1$ for the energy parameter of the cosmological fluid particles, giving $\tau_4^1\to 0$ as $R\to 0$. (The following equations can easily be generalized if $k\neq 1$.)

Setting $k=1$ in (2.18a), we get

$$p+\rho+(m/R^2)(1-A)^{-1/2}\int_0^R (\partial\rho/\partial\tau)R^2 dR=0 \qquad (2.18b)$$

and the metric form (2.2) becomes

$$ds^2(R,\tau)=[dR-m(1-A)^{1/2}d\tau]^2 + R^2 d\Omega^2 - d\tau^2 \; . \qquad (2.19)$$

In the subspace $\tau=\text{const}$, (2.19) becomes

$$ds^2(R,\tau=\text{const})=dR^2+R^2 d\Omega^2 \; , \qquad (2.20)$$

which is seen to be flat. Thus, just as in our previous paper,[8] R is the spatial coordinate that explicitly exhibits the flatness of the $\tau=\text{const}$ subspace. Also, R has the physical significance that it measures proper distance between τ-simultaneous events.

III. A SCHWARZSCHILD MASS IMBEDDED IN A DE SITTER UNIVERSE

A de Sitter universe corresponds to a vacuum cosmology, i.e., a universe in which there is no cosmological fluid. Setting $\rho=0$ in (2.18a) and (2.17) we find $p=0$ and $\tau_4^1=0$, which are to be expected. Setting $R_0=(3/\Lambda)^{1/2}$, we find from (2.16)

$$A(R,\tau)=1-2M/R-(R/R_0)^2 \qquad (3.1)$$

so that the metric form (2.2) is

$$ds^2(R,\tau)=k^{-2}\{dR-m[(R/R_0)^2+2M/R+k^2-1]^{1/2}d\tau\}^2+R^2 d\Omega^2-d\tau^2 \; . \qquad (3.2)$$

As we have discussed in our previous paper,[8] we are not justified in setting $k=1$ in a universe where there is no cosmological fluid. The equations (2.3) defining the transformation that relates the times τ and T become

$$\tau_{,R}=-m\frac{[2M/R+(R/R_0)^2+k^2-1]^{1/2}}{1-2M/R-(R/R_0)^2} \; , \qquad (3.3a)$$

$$\tau_{,T}=k \qquad (3.3b)$$

resulting in the metric form (3.2) taking the curvature form

$$ds^2(R,T)=\frac{dR^2}{1-2M/R-(R/R_0)^2}+R^2 d\Omega^2-[1-2M/R-(R/R_0)^2]dT^2 \; . \qquad (3.4)$$

If $\Lambda=0$, (3.4) is the Schwarzschild metric, while if $M=0$, (3.4) is the metric for a de Sitter universe. Hence the diagonal metric form (3.4) can be regarded as describing a Schwarzschild mass imbedded in a de Sitter universe, i.e., a de Sitter/Schwarzschild (S/S) field described in (R,T) coordinates, where T is related to times measured by clocks located at fixed $R=\text{const}$ points. The nondiagonal metric form (3.2) describes the same dS/S field in (R,τ) coordinates, where τ is measured by clocks moving along radial geodesics in the S/S field. Since there is no cosmological fluid to tie the geodesic clocks to, the energy parameter k is not determined by the field equations, and depends upon the type of geodesic-clock reference system we choose to use to measure τ. The choice of reference system will determine the form that the transformation defined by (3.3) will take. Examples of different types of geodesic-clock reference systems for the de Sitter universe are given in Ref. 4.

IV. A SCHWARZSCHILD MASS IMBEDDED IN AN EINSTEIN–DE SITTER UNIVERSE

An Einstein-de Sitter (ES) universe is characterized by a zero cosmological constant $(\Lambda=0)$ and a cosmological fluid that has zero pressure. As we have shown in our previous paper,[8] an ES universe can be described either in (R,τ) coordinates by the metric form

$$ds^2(R,\tau)=[dR-m(2R/3\tau)d\tau]^2 + R^2 d\Omega^2 - d\tau^2 \qquad (4.1)$$

or by curvature coordinates (R,T) as

$$ds^2(R,T)=\frac{dR^2}{1-(2R/3\tau)^2}+R^2 d\Omega^2 - \frac{dT^2}{[1-(2R/3\tau)^2][1+\frac{1}{2}(2R/3\tau)^2]} \; , \qquad (4.2)$$

in which $\tau(R,T)$ is given as an implicit function of the coordinates (R,T) by the transformation

$$T=\tau[1+\tfrac{1}{2}(2R/3\tau)^2]^{3/2} \tag{4.3}$$

that relates the two time coordinates τ and T. The density ρ of the cosmological fluid of the ES universe varies with the time τ according to

$$\rho=\frac{1}{6\pi\tau^2} \; . \tag{4.4}$$

Dirac claims that the ES universe is in agreement with his Large Numbers hypothesis.[2,3] As described in the Introduction, Dirac and McVittie have developed an approach, different from ours, for imbedding a Schwarzschild mass into the ES universe to produce an Einstein—de Sitter/Schwarzschild (ES/S) field.

One way of proceeding from our constitutive equations (2.16), (2.17), and (2.18b) to describe a Schwarzschild mass imbedded in an ES/S field is to assume that the imbedded mass has no effect on the cosmological fluid. In this case, we would assume a density given by (4.4) independent of R, so that from (2.16) we obtain

$$A(R,\tau)=1-(2R/3\tau)^2-2M/R \; . \tag{4.5}$$

Thus, this assumption results in simply adding a $-2M/R$ term to the $A(R,\tau)$ term in the original ES metric form (4.1) to produce an ES/S field. The pressure now will not be zero. We find from (2.18b).

$$p=-(6\pi\tau^2)^{-1}\{1-[1+(2M/R)(3\tau/2R)^2]^{-1/2}\} \tag{4.6}$$

so that the pressure is negative, going over to zero at large distances away from the Schwarzschild mass centered at $R=0$.

Objections can be raised about the approach just described. In essence, what has been done is to make a guess as to the form that the metric coefficients should have when a Schwarzschild mass is imbedded into an ES universe. The expression for the pressure is then taken to be whatever follows from the field equations. This approach seems contradictory to the spirit of the field equations, which imply that we should be the ones who specify the structure of the source, with the field equations determining the form of the metric coefficients.

Therefore, we approach a combined ES/S field in the following manner. The primary characteristic of an ES universe is that the cosmological fluid has zero pressure. We will assume that this feature holds also in a combined ES/S field where $R>R_b$. Correspondingly, we set $p=0$ in (2.18b) to get

$$\rho+(m/R^2)(1-A)^{-1/2}\int_0^R(\partial\rho/\partial\tau)R^2dR=0 \; . \tag{4.7}$$

Substitution of $A(R,\tau)$ from (2.16) then gives

$$\left[(-m/R^2\rho)\int_0^R(\partial\rho/\partial\tau)R^2dR\right]^2 =(8\pi/R)\int_0^R\rho R^2dR+2M/R \; , \tag{4.8}$$

which is an integral-differential equation that determines the density $\rho(R,\tau)$ that generates the ES/S field with zero pressure. If we set

$$Z(R,\tau)=8\pi\int_0^R\rho R^2dR \; , \tag{4.9}$$

Eq. (4.8) becomes

$$R^{1/2}\partial Z/\partial\tau+m(Z+2M)^{1/2}\partial Z/\partial R=0 \; . \tag{4.10}$$

From (2.16) we can then write

$$A=1-(2M+Z)/R \tag{4.11}$$

and the metric form (2.19) becomes

$$ds^2(R,\tau)=\{dR-m[(2M+Z)/R]^{1/2}d\tau\}^2 +R^2d\Omega^2-d\tau^2 \; . \tag{4.12}$$

Changing the time coordinate from the geodesic time τ to the curvature time T requires solving (2.3a), which is

$$\tau_{,R}=-m\frac{[(2M+Z)/R]^{1/2}}{1-(2M+Z)/R} \tag{4.13}$$

under the conditions that for small R we should have $\tau_{,T}\rightarrow 1$, while for large R the relationship (4.3) should hold. The metric form for an ES/S universe in curvature coordinates is then

$$ds^2(R,T)=\frac{dR^2}{1-(2M+Z)/R}+R^2d\Omega^2 -[1-(2M+Z)/R](\tau_{,T})^2dT^2 \; , \tag{4.14}$$

where $\tau_{,T}$ is determined from the solution $\tau(R,T)$ to (4.13).

V. IMBEDDING A SCHWARZSCHILD MASS INTO A GENERAL UNIVERSE

The above descriptions of combined S/S and ES/S fields suggest the following approach for imbedding a Schwarzschild mass into a general zero-curvature universe to obtain a combined general universe/Schwarzschild (GU/S) field. Substitute (2.16) into (2.18b) to get

$$\left[(-m/R^2)(\rho+p)^{-1}\int_0^R(\partial\rho/\partial\tau)R^2dR\right]^2 =(8\pi/R)\int_0^R\rho R^2dR+2M/R+(\Lambda/3)R^2 \; , \tag{5.1}$$

which gives a relationship between ρ and P. Assume that for $R>R_b$, p and ρ satisfy the same equation of state as in the original universe before the Schwarzschild mass was imbedded. With this equation of state $p=p(\rho)$, solve (5.1) to get $\rho(R,\tau)$. With $Z(R,\tau)$ defined by (4.9), the metric form is then determined from (2.19) as

$$ds^2(R,\tau)=\{dR-m[(2M+Z)/R+(\Lambda/3)R^2]^{1/2}d\tau\}^2 +R^2d\Omega^2-d\tau^2 \tag{5.2}$$

with the flow of energy across an $R=$const surface being given from (2.17) as

$$8\pi\tau_4^1=(1/R^2)\partial Z/\partial\tau \; . \tag{5.3}$$

To convert from (R,τ) to (R,T) coordinates, we use the time-coordinate transformation obtained by solving (2.3a):

$$\tau_{,R}=-m\frac{[(2M+Z)/R+(\Lambda/3)R^2]^{1/2}}{1-(2M+Z)/R-(\Lambda/3)R^2} \tag{5.4}$$

subject to the conditions that for small R we should have $\tau_{,T} \to 1$, while for large R the relationship between τ and T should be what it was before the Schwarzschild mass was imbedded into the Universe. The nondiagonal metric form (5.2) will then assume the diagonal-curvature form

$$ds^2(R,T) = \frac{dR^2}{1-(2M+Z)/R-(\Lambda/3)R^2} + R^2 d\Omega^2 - [1-(2M+Z)/R-(\Lambda/3)R^2](\tau_{,T})^2 dT^2 \tag{5.5}$$

in which $\tau_{,T}$ is determined from the solution $\tau(R,T)$ to (5.4).

VI. THE SPIRALLING OF PLANETARY ORBITS

Now that we have developed a formalism for imbedding a Schwarzschild field into a general zero-curvature universe, we here investigate how planetary orbits around a Schwarzschild gravitating mass M are affected by the inclusion of a surrounding universe. We will work in curvature coordinates (R,T). In the metric form (5.5) we set

$$A(R,T) = 1-(2M+Z)/R-(\Lambda/3)R^2 , \tag{6.1a}$$

$$B(R,T) = [1-(2M+Z)/R-(\Lambda/3)R^2](\tau_{,T})^2 . \tag{6.1b}$$

Let $V^\mu = dx^\mu/ds$, where s is the proper time along a particle's trajectory. Without loss of generality we take $\theta = 90°$ and $V^2 = d\theta/ds = 0$. The geodesic equations given by

$$dV^\mu/ds + \Gamma^\mu_{\rho\sigma} V^\rho V^\sigma = 0, \quad g_{\rho\sigma} V^\rho V^\sigma = -1 \tag{6.2}$$

are

$$dV^4/ds - [A_{,T}/(2A^2B)](V^1)^2 + [B_{,R}/B]V^1V^4 + [B_{,T}/2B](V^4)^2 = 0 , \tag{6.3}$$

$$dV^1/ds - [A_{,R}/2A](V^1)^2 - [A_{,T}/A]V^1V^4 + [AB_{,R}/2](V^4)^2 - AR(V^3)^2 = 0 , \tag{6.4}$$

$$dV^3/ds + (2/R)V^1V^3 = 0 , \tag{6.5a}$$

$$(1/A)(V^1)^2 + R^2(V^3)^2 - B(V^4)^2 = -1 . \tag{6.6}$$

Since $V^1 = dR/ds$, we can write (6.5a) in the alternate form

$$R^2(dV^3/ds) + V^3 2R(dR/ds) = \frac{d}{ds}(R^2V^3) = 0 \tag{6.5b}$$

so that

$$R^2V^3 = R^2(d\phi/ds) = L ,$$

$$L = \text{const} = \text{angular momentum per unit mass} . \tag{6.7}$$

Planetary orbits around a Schwarzschild mass *in vacuo* do not spiral inwards or outwards. However, in a cosmological situation where a Schwarzschild mass is surrounded by, or imbedded in, the cosmological fluid of a nonstatic universe, spiralling of planetary orbits will occur. To show this, we will assume circular motion, i.e.,

$$R = \text{constant}, \quad V^1 = 0 \tag{6.8}$$

and find that a contradiction arises, showing that spiralling must occur.

With $dR/dT = 0$, a partial derivative with respect to T is the same as a total derivative with respect to T. Using

$$dV^4/dS = (dV^4/dT)(dT/ds) = V^4(dV^4/dT) ,$$

we can then rewrite (6.3) as

$$2BV^4(dV^4/dT) + B_{,T}(V^4)^2 = (d/dT)[B(V^4)^2] = 0 \tag{6.9}$$

so that

$$B(V^4)^2 = \text{constant} . \tag{6.10}$$

The value of this constant is obtained by combining (6.7) with (6.6) to get

$$B(V^4)^2 = 1 + (L/R)^2 . \tag{6.11}$$

Using (6.11) in the remaining geodesic equation (6.4), we obtain

$$[\ln(B)]_{,R} = \frac{2/R}{1+(R/L)^2} . \tag{6.12}$$

If R=const, the right-hand side of (6.12) will be a constant, while if $B = -g_{44}$ depends upon T, the left-hand side of (6.12) will not be a constant. Since we get a contradiction, the starting assumption (6.8), that R=const, can not hold. Therefore, with the exception of the static S/S field for which B is independent of T, there will be spiralling of planetary orbits for GU/S fields.

Now that we know a planetary orbit will spiral if g_{44} depends on T, let us analyze a planet's motion taking the spiraling into account. To do this, I have followed a method developed by Dirac,[2] but using curvature coordinates instead of the isotropic coordinates employed by Dirac.

Define the velocity v of the planet by

$$Bv^2 = g_{ab}(dx^a/dT)(dx^b/dT) \quad (a,b=1,2,3) . \tag{6.13}$$

Following Dirac, we assume nearly circular orbits, and neglect dR/dT in the geodesic equations where appropriate. The results of the calculations, which are shown in the Appendix, are that for nearly circular orbits the following three relations follow from the geodesic equations:

$$(dR/dT)[\ln(B)]_{,R} + 2v(dv/dT)(1-v^2)^{-1} = 0 , \tag{6.14}$$

$$(L/R)^2 = v^2(1-v^2)^{-1} , \tag{6.15}$$

$$[\ln(B)]_{,R} = 2v^2/R , \tag{6.16}$$

where only two of the three equations are independent.

Eliminating v between (6.15) and (6.16), we obtain (6.12). Thus, once a particular universe is chosen, so that $B(R,T)$ is specified, (6.12) gives R implicitly as a function of T. It is seen from (6.12) that R will be constant only for a S/S field, where B is independent of T.

The magnitude of the orbital spiralling can be estimated from a Newtonian calculation. A flow of cosmological fluid across a sphere R=const means that the attractive mass inside the orbit of a planet will vary with time, causing the planet to spiral. From Newton's second law we have

$$GM(t)/R^2=v^2/R \; , \tag{6.17}$$

where $M(t)$ is the mass inside the planet's orbit. From $Rv=\text{const}$ from conservation of angular momentum, we obtain

$$v\, dR/dt+R\, dv/dt=0 \; , \tag{6.18}$$

which, when combined with (6.17), yields

$$dR/dt=-(R/M)dM/dt \; . \tag{6.19}$$

The mass $M(t)$ consists of the constant central mass $M_\odot$ plus the mass of the cosmological fluid with density $\rho(t)$:

$$M(t)=M_\odot+\tfrac{4}{3}\pi R^3\rho(t) \; . \tag{6.20}$$

Substituting this into (6.19) we obtain

$$dR/dt=-\tfrac{4}{3}\pi(R^4/M_\odot)d\rho/dt \; , \tag{6.21}$$

where we have used $M(t)\approx M_\odot$.

There are two equivalent ways we can proceed from this point. The density variation inside a sphere is

$$d\rho/dt=(\tfrac{4}{3}\pi R^3)^{-1}dM/dt \; , \tag{6.22}$$

which can be related to the flux ρv by

$$dM/dt=-4\pi R^2\rho v \tag{6.23}$$

to get

$$d\rho/dt=-3\rho v/R \; . \tag{6.24}$$

From cosmological considerations we know $v=HR$, where H is Hubble's constant, so that (6.24) can be written as

$$d\rho/dt=-3H\rho \; . \tag{6.25}$$

Substitution of (6.25) into (6.21) gives

$$dR/dt=(8\pi R^4H\rho)/(2M_\odot) \; . \tag{6.26a}$$

Alternatively, we can substitute the density for an ES universe

$$\rho=(6\pi Gt^2)^{-1} \tag{6.27}$$

into (6.21) to get

$$dR/dt=(8\pi R^4\rho)/(3M_\odot t) \; . \tag{6.26b}$$

Since $H=2/3t$ for an ES universe, (6.26a) and (6.26b) are equal to each other.

Taking the age of the Universe as $t=3\times10^{10}$ yr, the cosmological density as $\rho=10^{-29}\,\text{g/cm}^3$, we find that in our solar system ($M_\odot=2\times10^{30}$ kg) the orbit of Saturn ($R=1.4\times10^{12}$ m) will change by

$$dR/dt\,|_{\,\text{Saturn}}=6\times10^{-18}\ \text{m/yr} \; , \tag{6.28}$$

which is much too small to be measured. If we assume that the above analysis holds also for a galaxy, we have for the spiralling of a star at the edge of the Andromeda galaxy $M31$ ($R=25$ kpc, $M=4\times10^{11}M_\odot$)

$$dR/dt\,|_{\,\text{galactic star}}=1\,100\ \text{km/yr} \; , \tag{6.29}$$

which may possibly be measurable. Since $dR/dt\propto t^{-3}$ for an ES universe, the spiralling effect may have been appreciable at the formation of galaxies in the early universe.

VII. DISCUSSION

Einstein and Straus find with their approach that there is no planetary spiralling. The planetary motion behaves as if there is no influence by an expanding universe.[6] This is to be expected, since with their approach of treating a hole cut out of the Universe, in which there is a Schwarzschild mass M, they are dealing in the vacuum region around M with a usual Schwarzschild field, where we know spiralling does not exist.

McVittie also finds that his approach does not result in planetary spiralling. An observer using the metric form (1.5) or (1.7) will find that the orbit of a planet remains fixed.[1]

Dirac, who works with McVitties's metric form (1.7) specialized to an ES universe by means of (1.8) and (1.9), also finds, just as in a Schwarzschild field, a planet will move with a constant velocity at a constant distance from the Sun. However, Dirac concludes that there will be planetary spiralling by going outside the bounds of general relativity to atomic theory. Proceeding from his Large Numbers hypothesis (LNh), Dirac argues that there are two times of consequence in nature[2]:

(1) An ephemeris time which governs the motions of planets and galaxies. Dirac has identified ephemeris time with the time t used in the metric form (1.7).

(2) An atomic time t_A which governs atomic processes. It is atomic time that Dirac refers to when he says that the gravitational constant varies with time.

From his LNh, Dirac finds that atomic and ephemeris time are related by[2]

$$t=\tfrac{1}{2}t_A{}^2 \; . \tag{7.1}$$

In turn, distances measured in atomic units are different from distances measured in ephemeris units. In this manner, Dirac concludes from his ES/S metric form (1.7) with (1.8) and (1.9) that, in terms of atomic units, the distance r_A of a planet from the Sun will vary according to[2]

$$r_A\propto t_A{}^{-1} \; . \tag{7.2}$$

In contrast with the above results, we have showed that planetary spiralling is predicted from the usual general relativity theory. The reason for our result of planetary spiralling is straightforward. The cosmological fluid, which is the source of a particular universe, will be moving across a sphere $R=\text{const}$. Hence the amount of matter inside a sphere $R=\text{const}$ will be changing with time, either decreasing for an expanding universe, or increasing for a contracting universe. Therefore, a planet at the radius R will experience a force that varies with time T or τ, resulting in the planet's curvature radial coordinate R changing with time. As we see with (2.20), R measures the proper distance in the subspace $\tau=\text{const}$, so that, in this sense, a variation of R indicates a variation of a planet's proper distance from the Sun.

To close, I would like to offer a speculation. Dirac has hypothesized the existence of two different times in nature, an atomic time and an ephemeris time. In the development of our work, we have seen the natural existence of two different times, a curvature time T and a cosmological time τ. Perhaps the times T and τ might be related, in some fashion, with the two times hypothesized by Dirac. This would have the effect of merging, in a natural way, Dirac's LNh, with its associated atomic constants, with Einstein's theory of general relativity.

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APPENDIX

I will here redo Dirac's calculations in the section titled "Planetary orbits" in his paper of Ref. 2. For ease of comparison, I will use his same notation of letting z^μ denote the planet's coordinates as a function of z^4, with an overdot denoting d/dz^4, so that $\dot z^4=1$. The difference between what Dirac has done and what is being done here is that Dirac worked with an isotropic metric form, while I will be using the curvature metric form

$$ds^2(R,T)=\frac{dR^2}{A(R,T)}+R^2d\Omega^2-B(R,T)dT^2 \ . \tag{A1}$$

Let s denote proper time measured along the world line of the planet. The geodesic equations are

$$\ddot z^\mu+\Gamma^\mu_{\rho\sigma}\dot z^\rho\dot z^\sigma-\dot z^\mu\ddot s/\dot s=0 \ . \tag{A2}$$

Define the velocity v of the planet by

$$-g_{44}v^2=Bv^2=g_{ab}\dot z^a\dot z^b \quad (a,b=1,2,3) \ . \tag{A3}$$

Then

$$\dot s^2=g_{\rho\sigma}\dot z^\rho\dot z^\sigma=B(1-v^2) \tag{A4}$$

from which

$$2\dot s\ddot s=(B_{,4}+B_{,a}\dot z^a)(1-v^2)-2Bv\dot v \ . \tag{A5}$$

Apply (A2) with $\mu=4$ to get

$$\begin{aligned}\Gamma^4_{\rho\sigma}\dot z^\rho\dot z^\sigma&=\ddot s/\dot s\\&=(1/2B)(B_{,4}+B_{,a}\dot z^a)-v\dot v/(1-v^2) \ ,\end{aligned} \tag{A6}$$

where we have used (A4) and (A5). From the definition of $\Gamma^4_{\rho\sigma}$ we have

$$\Gamma^4_{\rho\sigma}\dot z^\rho\dot z^\sigma=B^{-1}(\tfrac12 B_{,4}+B_{,a}\dot z^a+\tfrac12 g_{ab,4}\dot z^a\dot z^b) \ . \tag{A7}$$

But for the metric form (A1), only g_{11} in the g_{ab} depends on z^4, and B is independent of z^2 and z^3, so that (A7) becomes

$$\Gamma^4_{\rho\sigma}\dot z^\rho\dot z^\sigma=B^{-1}[\tfrac12 B_{,4}+B_{,1}\dot z^1+\tfrac12 g_{11,4}(\dot z^1)^2] \ . \tag{A8}$$

Equating (A8) with (A6), we get

$$\tfrac12 B_{,1}\dot z^1+\tfrac12 g_{11,4}(\dot z^1)^2+Bv\dot v/(1-v^2)=0 \tag{A9}$$

or, with $g_{11}=A^{-1}$,

$$\tfrac12 B_{,1}\dot z^1-A^{-2}A_{,4}(\dot z^1)^2+Bv\dot v/(1-v^2)=0 \ . \tag{A10}$$

For the case of an orbit that is nearly circular except for cosmological effects on the radius coordinate $z^1=R$, $\dot z^1$ is small, and we may neglect its square in (A10) to get

$$B^{-1}B_{,1}\dot z^1+2v\dot v/(1-v^2)=\dot z^1[\ln(B)]_{,1}+2v\dot v/(1-v^2) \ . \tag{A11}$$

This is Eq. (6.14).

Similarly, (A3) reduces to (since $\dot z^2=0$ for our case)

$$Bv^2=R^2(\dot z^3)^2 \ . \tag{A12}$$

From (6.7) we have

$$\begin{aligned}V^3&=dz^3/ds=(dz^3/dT)(dT/ds)\\&=\dot z^3(dT/ds)=\dot z^3V^4=L/R^2 \ .\end{aligned} \tag{A13}$$

Since $g_{\rho\sigma}V^\rho V^\sigma=-1$, we have from (A3)

$$B(V^4)^2=(1-v^2)^{-1} \ . \tag{A14}$$

Substituting (A14) and (A13) into (A12), we obtain

$$(L/R)^2=v^2(1-v^2)^{-1} \ . \tag{A15}$$

This is Eq. (6.15).

Applying (A2) with $\mu=1$ and neglecting $\dot z^1$, we obtain, using (A13) and (A14),

$$\begin{aligned}\Gamma^1_{\rho\sigma}\dot z^\rho\dot z^\sigma&=\Gamma^1_{44}+\Gamma^1_{33}(\dot z^3)^2\\&=B_{,1}A/2-(AR)[L^2B(1-v^2)R^{-4}]=0 \ .\end{aligned} \tag{A16}$$

Rearranging (A16) and using (A15), we obtain

$$[\ln(B)]_{,1}=2v^2/R \ . \tag{A17}$$

This is Eq. (6.16).

Since (A2) with $\mu=4,1$ and (A3) constitute only two independent equations, it follows that there will be only two independent equations among (A11), (A15), and (A17).

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