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Long-range scattering in non-Abelian gauge theory

A. Soffer
Department of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel
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We study the scattering of a quantal particle with internal degrees of freedom coupled to a long-range non-Abelian external field, acting nontrivially in its internal space. It is shown that the asymptotic states are singlets of the internal-symmetry group and the corresponding S matrix exists for singlet states (only). It is shown that the long-range interaction, although vanishing as $1/|\vec{r}|$ (for $|\vec{r}|\to\infty$), modifies the asymptotic free motion by corrections which cause precession of all degrees of freedom orthogonal to the field direction (in internal space) as t approaches infinity, thereby not admitting a well-defined color content in the asymptotic states. Remarks are made about the analogy in QCD. We explain how modified perturbation theory can be applied.

We study the scattering theory of a quantum particle with internal (non-space-time) structure interacting with a long-range potential acting nontrivially in its internal space.

We show that the long-range residual force masks the internal structure of the particle as the time approaches infinity; it is therefore possible to construct the S matrix for asymptotic singlet states only; the (color) degrees of freedom precess indefinitely, thus the asymptotic states do not admit a well-defined color content. We call this obliteration of color bleaching.

We shall describe the particle dynamics by a non-Abelian generalization of the Schrödinger equation, where the complex number field is replaced by quaternions.

The state $\psi(\vec{x})$ of the particle is given by a quaternion-valued wave function in the Hilbert space of square-integrable q-valued functions $\mathscr{H}_q$.[1,2]

The algebra of observables associated with such a particle include, apart from the position $\vec{x}$ and momentum $\vec{p}$, the operators acting in its internal space, the quaternions E_i [$\approx \mathrm{SU}(2)$] acting from the left and e_i from the right ($i=1,2,3$). The space $\mathscr{H}_q$ is linear over e_i (replacing $\mathbf{C}$) and are the (non-Abelian) phases of the state ψ, while the E_i are operators in the usual sense in $\mathscr{H}_q$.[3] It should be stressed that although we use here the quaternionic quantum-mechanics description only as a convenient tool to study non-Abelian quantum theory, this approach has an interest of its own concerning, e.g., preon models.[4]

We choose the free dynamics to be

$$U_t^0\psi = \exp\left(te_1\frac{\partial^2}{\partial\vec{x}^2}\right)\psi \tag{1}$$

(the choice of e_1 is not unique); the full dynamics is given by local gauging of the global SO(4) symmetry of the corresponding Schrödinger equation.

Since we are interested in long-range interactions we retain only the part which decays slowly; we are thus led to consider the following dynamics:

$$U_t\psi = \exp\left[t\left(e_1\frac{\partial^2}{\partial\vec{x}^2} + E_iV_{ij}e_j\right)\right]\psi \ , \tag{2}$$

where V_{ij} are real functions of $\vec{x}$; V_{0i} and V_{i0}, with $i=1,2,3$, are the only nonzero ones ($e_0=E_0=1$). The Heisenberg equations of motion for $\vec{P}(t)$ and $E_i(t)$ (under the full dynamics) are the quantum analog of a classical colored particle moving in an external gauge field.[5] Such an equation was first derived by Wong as a classical limit of SU(2) gauge theory.[6]

We study the asymptotic behavior of the operators associated with the particle in the Heisenberg picture which are then used to construct a generalized S matrix appropriate to such cases using the algebraic approach to scattering theory.

Let $\mathscr{H}_q$ denote the quaternionic Hilbert space of q-valued square-integrable functions of $\mathbf{R}^3$. This space of functions is linear over quaternions acting from the right (called a right vector module). The scalar product is given by

$$(f,g) = \int_{\mathbf{R}^3} f^*(x)g(x)d^3x \quad \forall f,g \in \mathscr{H}_q \ , \tag{3}$$

where the asterisk is the conjugation of quaternions. The norm of $f\in\mathscr{H}_q$ is given by $(f,f)^{1/2}$. An operator A in H_q is called q linear if $A(fq)=(Af)q \forall q\in Q$ (the quaternions). So the E_i are q linear but the e_i are not. $\psi\in\mathscr{H}_q$ can be represented as a four-component (real) function that we associate with a quantal particle with non-Abelian phase degrees of freedom, which is naturally coupled to an SO(4) gauge field $A_\mu^a(x)$. When ψ is viewed as a q-valued function the gauge field can be described as multiplication from left and right by ($\vec{x}$-dependent) quaternions (hence it is not, in general, q linear). A complete description of a freely moving particle [up to the right SU(2) phase] can be given by the algebra of operators generated by $\mathscr{A}_0 \equiv \{\vec{P}, E_i | i=0,1,2,3\}$ (with $\vec{P} = -E_1\vec{\nabla}$) that commutes with the free dynamics.

In the algebraic approach to scattering theory[7] we seek the asymptotic motion of these operators under the full dynamics (by construction $\mathscr{A}_0$ is expected to be asymptotically time independent if the full dynamics approaches the free one):

$$\mathscr{A}_0 \in A \to A(t) = \exp\{t[e_1(-\vec{\nabla}^2) + E_i V_{ij} e_j]\} A \exp\{-t[e_1(-\vec{\nabla}^2) + E_i V_{ij} e_j]\} P_M \quad . \tag{4}$$

P_M denotes the projection on those states which are to be identified with scattering states (typically the continuous spectral part of the full dynamics).

If the functions V_{ij} decay fast enough as $|\vec{x}| \to \infty$ (faster than $1/|\vec{x}|$) the usual wave operators exist and the asymptotic motion is given by the free one, i.e.,

$$\| U_t \psi - U_t^0 \phi \| \to 0 \text{ as } |t| \to \infty, \quad \forall \psi \in \mathscr{H}_q \quad . \tag{5}$$

In this case the asymptotic algebra is given by the asymptotic limits of $\mathscr{A}_0$:

$$\{\vec{P}^{\pm}, E_i^{\pm}\} = \Omega_{\pm} \{\vec{P}, E_i\} \Omega^*_{\pm} \quad , \tag{6}$$

where $\Omega_{\pm}$ are the usual wave operators.

In the presence of a long-range potential, in the Abelian case, the asymptotic algebras exist, though the usual wave operators do not; this is true since the only effect of the long-range part is to modify the phase shifts in an uncontrollable way, leaving the momentum an asymptotic observable.[7] In our case the analog of the phase shifts belongs to a non-Abelian algebra. If they become unmeasurable it is expected to be reflected in the asymptotic behavior of the $E_i(t)$. In fact, as we shall show, the limits do not exist, leaving the asymptotic particle state bleached from its original internal degrees of freedom.

If we can show that $\vec{P}^{\pm}$ exists we can still associate the asymptotic states with a new (structureless) object. We may also be able to construct S: $\vec{P}^+ \to \vec{P}^-$, the generalized S matrix; in this way we get a scattering theory for the case where the asymptotic particle is qualitatively different from the (nonasymptotic) original one.

We shall construct a simple model generalizing the Coulomb interaction to the non-Abelian case which allows one to achieve the generalized scattering theory described above.

A direct way of showing that bleaching occurs is to prove that limits defining $E_i^{\pm}$ do not exist. Since the free part of the dynamics commutes with all of the E_i we need the long-range part to depend on some E_i (namely, the long-range part acts nontrivially in the internal space of the particle). Let us restrict our attention to the simplest case satisfying the above demands:

$$E_i V_{ij} e_j = \alpha E_1 V(\vec{x}) \quad , \tag{7}$$

where α is real and $V(\vec{x}) = 1/|\vec{x}|$. Motivated by the Coulomb problem we deduce the following asymptotic motion:[7]

$$U_D(t) = \exp\left[-e_1 t(-\vec{\nabla}^2) - \frac{\alpha}{\sqrt{H_0}} E_1 \ln|4H_0 t|\right] \quad . \tag{8}$$

The explicit dependence of the modified part on E_1 which does not commute with E_2 and E_3 is the essential new feature, which is, of course, expected to hold for more general long-range interactions. One can prove in a similar way as in the Abelian case that the following limits exist:

$$\operatorname*{s-lim}_{t\to\pm\infty} U_{-t} U_D(t) \psi = \Omega_{\pm} \psi, \quad \forall \psi \in \mathscr{H}_q \tag{9}$$

and

$$\operatorname*{s-lim}_{t\to\pm\infty} U_D(-t) U_t P_M = \Omega^*_{\pm} P_M \quad . \tag{10}$$

Asymptotic completeness means that $P_M = P_{ac}(H)$, the projection on the continuous spectral part of H, which is true in this case. Since $\vec{P}$ commutes with $U_D(t)$ we get

$$\begin{aligned}\operatorname*{s-lim}_{t\to\pm\infty} U(-t)\vec{P}U(t)P_M &= \operatorname*{s-lim}_{t\to\pm\infty} U(-t)U_D(t)\vec{P} \\ &\quad \times U_D(-t)U(t)P_M \\ &= \Omega_{\pm}\vec{P}\Omega^*_{\pm} \equiv \vec{P}^{\pm} \quad ;\end{aligned} \tag{11}$$

hence asymptotic momentum exists. Moreover, since E_1 commutes with e_i, E_1 commutes with $U(t)$, and hence $E_1^{\pm}$ ($= E_1$) exist. Consider now the $E_2(E_3)$ behavior under the time evolution. We first show that $E_2^D(t)$ does not have a limit as $|t| \to \infty$. $E_2^D(t)$ gives the time dependence of E_2 (in the Heisenberg picture) under the modified free dynamics $U_D(t)$:

$$\begin{aligned} E_2^D(t) &= \exp\left[te_1(-\vec{\nabla}^2) + \frac{\alpha}{\sqrt{H_0}} E_1 \ln|4H_0 t|\right] E_2 \\ &\quad \times \exp\left[-e_1 t(-\vec{\nabla}^2) - \frac{\alpha}{\sqrt{H_0}} E_1 \ln|4H_0 t|\right] \\ &= \exp\left[\frac{2\alpha}{\sqrt{H_0}} E_1 \ln|4H_0 t|\right] E_2 \quad . \end{aligned} \tag{12}$$

The last expression is easily shown to converge weakly to zero as $|t| \to \infty$ and therefore the strong limit does not exist; we see that the $E_2(E_3)$ phase of the wave function oscillates logarithmically fast as $|t|$ approaches infinity. However, since $U_D(t)$ does not commute with E_2 (or E_3) it does not follow, in general, that the limit of $E_2(t) = U^*(t)E_2U(t)$ does not exist. To prove it we need an extra argument.

Asymptotic completeness implies that

$$\| U(t)\psi - U_D(t)\phi \| \to 0 \text{ as } |t| \to \infty, \quad \forall \psi \in P_M \mathscr{H}_q \tag{13}$$

for some ϕ, which depends on ψ.

Let us assume that the limit

$$E_2^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} U(t)^* E_2 U(t) P_M$$

exists; we then obtain

$$U(t)^* E_2 U_D(t) = U(t)^* U_D(t) \exp\left[\frac{2\alpha}{\sqrt{H_0}} E_1 \ln|4H_0 t|\right] E_2 \equiv A_t \tag{14}$$

is a uniformly bounded family of operators converging strongly to some operator, call it A, since

$$\begin{aligned} &\| E_2^+ \psi - U(t)^* E_2 U_D(t) \phi \| \\ &\qquad \leq \| U^*(t) E_2 \| \, \| U(t)\psi - U_D(t)\phi \| \end{aligned} \tag{15}$$

converges to zero for all ϕ by asymptotic completeness (for suitably chosen ψ which depends on ϕ). Since $A\phi = E_2^+$, $(1-P_M)A\phi - (1-P_M)E_2^+\psi = 0$. Since asymptotic completeness holds, $U_D^*(t)U(t)P_M=\Omega_t^*P_M$ converges strongly. Hence

$$\operatorname*{s-lim}_{t\to\infty}\Omega_t^*P_MA_t=\lim_{t\to\infty}\Omega_t^*A_t=\lim_{t\to\infty}U_D^*(t)U(t)U(t)^*U_D(t)\exp\left(\frac{2\alpha}{\sqrt{H_0}}E_1\ln|4H_0t|\right)E_2=\lim_{t\to\infty}\exp\left(\frac{2\alpha}{\sqrt{H_0}}E_1\ln|4H_0t|\right)E_2 \quad (16)$$

also converges strongly ($\|\Omega_t^*P_M\|\leq 1$), which is impossible. We therefore conclude that $E_2^\pm$, $E_3^\pm$ do not exist.

Remark. It is possible to find directly from estimates based on classical equations of motion which observables are asymptotic: For example

$$\frac{d\vec{P}}{dt}\sim\frac{\partial U}{\partial x}(\vec{x}(t))\sim\frac{1}{|\vec{x}(t)|^2}\sim\frac{1}{t^2}\ , \quad (17)$$

but

$$\frac{dE_2}{dt}\sim V(\vec{x}(t))\sim\frac{1}{t}\ , \quad (18)$$

and hence a logarithmic divergence in time is expected.

Since the asymptotic algebras are $\{\vec{P}^\pm,1,E_1^\pm\}=\{\vec{P}^\pm,1,E_1\}$ the S-matrix operator intertwining them is an operator between two operator algebras over the complex numbers ($=\{1,E_1\}$), and can therefore be represented as a complex linear operator acting in a complex Hilbert space, which can be identified with the complex submanifold of $\mathcal{H}_q$ over $\{1,E_1\}$; this submanifold is left invariant under the action of S (and $\Omega_\pm$) and has the property that $\{\vec{P}^\pm,1,E_1\}$ are maximal Abelian algebras on it. The S matrix therefore describes the scattering of a particle which is initially finally structureless, although for finite times it is not.

Recently we developed a perturbation theory for the S matrix in the presence of long-range (Coulomb-type) interactions[8] which can be used in this case. The approach is based on finding a modified (momentum-dependent) potential V_m, in terms of which the Lippmann-Schwinger equations are cast in the usual form and then expanded in powers of V_m. One can then show that the only effect of the change $V\to V_m$ is an addition of a phase factor in each vertex in the usual Feynman graphs and that the resulting expansion is finite order by order.

In the case we are dealing with here V_m will depend explicitly on E_1: Following the procedure of Ref. 8 one finds (up to choice of phase in the definition of U_D)

$$V_m=(H_0-k^2-e_1\epsilon)^{-E_1\alpha/\sqrt{H_0}}V(x)(H_0-k^2+e_1\epsilon)^{E_1\alpha/\sqrt{H_0}}$$
$$\left(V(x)=\frac{\alpha}{|\vec{x}|}\right)\ . \quad (19)$$

To first order in V_m we get an oscillatory term of the form $e^{-E_1f(H_0)}$, for some real function f, which behaves like the Abelian Coulomb phase, but which causes transitions in (E_2, E_3) space in an unpredictable way since the phase factor is not uniquely determined by the dynamics.

In the case we are considering S (and $\Omega_\pm$) can be viewed as an operator in $\mathcal{H}_q$ and there is a question of whether this can used to extract extra information other than that contained in $\{\vec{P}^\pm,1,E_1\}$. One can, for example, *define* $E_2^\pm=\Omega_\pm E_2\Omega_\pm^*$; it gives the difference of the actions of $U_D(t)$ and $U(t)$ on E_2, and this difference is convergent as $|t|\to\infty$ since $U_D(t)$ and $U(t)$ rotate E_2 in (E_2,E_3) space with the *same* divergent angle (for t large enough) relative to the free dynamics. But since $U_D(t)$ is determined only up to multiplication by a phase factor of the form $e^{E_1f(H_0)}$, the phase difference between $U(t)$ and $U_D(t)$ is undetermined too, although finite.

In the Abelian Coulomb case the choice of $U_D(t)$ is also not unique, but can be found be considering Coulomb + short-range part, experimentally. The analog in the non-Abelian case is not so immediate and is associated with non-Abelian interference.[9] Another point to be noted is that when long-range fields are present with (asymptotically) fixed direction in color space they dictate a choice of E_i in the definition of the momentum, which is otherwise arbitrary. This might clarify the question of what is the analog of i (the complex unit) in quaternionic quantum-mechanical models.

For the analogous problem in non-Abelian quantum field theory, it is clear that we cannot take quantum effects of the gauge field itself explicitly into account, but let us assume that colored particles do emerge as asymptotic states. They interact through the gauge field which in the non-Abelian case will induce a long-range potential $U_{ij}=(g^2/4\pi)e_i^ae_j^a/r_{ij}$ where $\{e_i^a\}_{a=1,\ldots,N}$ are the gauge group [SU(N)] matrices for the jth particle and r_{ij} the distance between the i and j particles. For this case our approach applies and the construction of the S matrix will depend on the form of the asymptotic dynamics $U_D(t)$ of each particle. Because of the long-range character of the interaction $U_D(t)$ depends on the directions in color space of all other particles, which themselves rotate indefinitely in directions orthogonal to the potential they see from the others. Consequently, if asymptotic states do exist they are bleached from their color. Moreover, cluster decomposition cannot hold in this case since the direction of the field each particle sees depends on its position and on the other particles. Of course, it is expected that the continual rotation in color space of each particle causes a significant change in the gauge field itself[10] (namely, enhancing gluon emission) that can change the momentum of the particle or even stop it completely from going to infinity, thus leading to confinement.[11] Colorless configurations of particles, however, induce only short-range-type fields and are expected to show up in the final states. It should be remarked, however, that this analysis rests on the assumption that the asymptotic states can be well approximated by classical gauge field configurations, which need not have to be the case. What does seem to follow is that if confinement does not hold it is a purely quantum effect of the gauge field.

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