Equation of motion for bubble boundaries

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The equation of motion of a (non-null) bubble boundary, in the thin-wall approximation, is obtained directly from the Lanczos equations for all spherically symmetric, cylindrically symmetric, and pseudo-spherically symmetric embedding four-geometries. No explicit form for the intrinsic surface energy three-tensor is used up to the evolution of the surface mass.

Recently, Wu Zhong Chao[1] has discussed the effects of gravitation in the collision of bubbles in the very early universe by using the thin-wall approximation.[2] The first integral of the motion was obtained for a pseudo–Schwarzschild–de Sitter embedding. The purpose of the present calculation is to point out that the equation of motion follows directly from the Lanczos equations without integration or any dependence on the explicit form of the intrinsic surface energy three-tensor. Moreover, it is shown that the spherically symmetric, cylindrically symmetric, and pseudo-spherically symmetric cases can be handled generally and simultaneously.[3] In view of the possibility[4] of a simultaneous transition of the vacuum, the case of a spacelike boundary is included here.

The surface layer (Σ) is characterized by a four-tangent u^α and four-normal n^α, where

$$n^\alpha n_\alpha = -u^\alpha u_\alpha \equiv \epsilon = \begin{cases} +1 & \text{timelike } \Sigma \\ -1 & \text{spacelike } \Sigma \end{cases} . \quad (1)$$

The general Lanczos equations[2] read

$$8\pi S_{ij} = \epsilon(\gamma g_{ij} - \gamma_{ij}) , \quad (2)$$

where g_{ij} is the metric intrinsic to Σ, $\gamma \equiv g^{ij}\gamma_{ij}$, and $\gamma_{ij} \equiv [K_{ij}]$. [We have used Israel's convention for the sign of K_{ij}, that is $K_{ij} \equiv -n_\alpha \delta(\partial x^\alpha/\partial \xi^j)/\delta\xi^i$, ξ^i the intrinsic coordinates of Σ.] In terms of the intrinsic tangent u^i, the surface density σ is defined by the eigenvalue equation

$$S_{ij}u^i = -\sigma u_j . \quad (3)$$

From Eqs. (1), (2), and (3) we have

$$8\pi\sigma = -(\epsilon\gamma + \gamma_{ij}u^i j^j) . \quad (4)$$

For the intrinsic metric write

$$ds_\Sigma{}^2 = R^2(\tau)d\Omega^2 - \epsilon\, d\tau^2 , \quad (5)$$

where

$$d\Omega^2 \equiv d\theta^2 + \frac{1}{k}\sin^2\sqrt{k}\,\theta\, d\phi^2 . \quad (6)$$

The spherically symmetric, cylindrically symmetric, and hyperbolically symmetric (pseudo-spherical) cases have $k = +1$, 0, and -1, respectively. The intrinsic three-tangent is given by

$$u^i = \delta^i_\tau . \quad (7)$$

For the metric in the enveloping four-geometries (V^+ and V^-) take

$$ds_\pm{}^2 = [e^{\alpha(r,t)}dr^2 + R^2(r,t)d\Omega^2 - e^{\beta(r,t)}dt^2]_\pm , \quad (8)$$

where θ and ϕ are taken continuous through Σ. Without loss of generality, only two of the functions α, β, and R are arbitrary. The four-normal to Σ is given by

$$n_r = \pm e^{(\alpha+\beta)/2}\dot{t}, \quad n_t = \mp e^{(\alpha+\beta)/2}\dot{r}, \quad n_\theta = n_\phi = 0 , \quad (9)$$

where an overdot denotes $d/d\tau$.

From Eqs. (4) through (9) we have

$$4\pi R^2\sigma = -\epsilon\gamma_{\theta\theta} \equiv M(\tau) \quad (10)$$

and

$$K_{\theta\theta}{}^2 = R^2\left\{\dot{R}^2 + \epsilon\left(1 - \frac{2m}{R}\right)\right\} , \quad (11)$$

where

$$m \equiv \frac{R}{2}\left\{1 + e^{-\beta}\left(\frac{\partial R}{\partial t}\right)^2 - e^{-\alpha}\left(\frac{\partial R}{\partial r}\right)^2\right\} . \quad (12)$$

From Eq. (10) we have the identity

$$K_{\theta\theta}{}^{+2} = \frac{1}{4M^2}(K_{\theta\theta}{}^{+2} - K_{\theta\theta}{}^{-2} + M^2)^2 . \quad (13)$$

Finally, from Eqs. (11) and (13) we obtain the equation of motion

$$\dot{R}^2 + \epsilon = \left(\frac{m_+ - m_-}{M}\right)^2 + \epsilon\left(\frac{m_+ + m_-}{R}\right) + \left(\frac{M}{2R}\right)^2 . \quad (14)$$

[Wu Zhong Chao's equation (4.23) follows directly from Eq. (14) with the substitutions $r \to x$, $R = t \to$ "s", $e^\alpha = e^{-\beta} \to f^{-1}$, and $\epsilon = +1$.] The history of Σ is completed by relating an explicit form of the intrinsic three-tensor S_{ij} to the jump across Σ of the energy-momentum tensors of the enveloping four-manifolds.[2] From the Gauss-Codazzi equations, the Einstein field equations, the Lanczos equations, and Eq. (3) we have

$$[T_{\alpha\beta}u^\alpha n^\beta] = \epsilon(\dot{\sigma} + \sigma\Theta + S^j_i \nabla_j u^i) , \quad (15)$$

where Θ gives the three-expansion of Σ. For a dust shell in vacuum, with the symmetries given by Eqs. (5) and (6), it follows from Eq. (15) that

$$\dot{M} = 0 . \quad (16)$$

Equation (14), with suitable initial conditions, then completely determines the history of Σ in this case.

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[3]The present calculation is a generalization (within the context of diagonal embeddings) of the procedure given by K. Lake, Phys. Rev. D 19, 2847 (1979).

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