Curvature coordinates in cosmology

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We develop cosmological theory from first principles starting with curvature coordinates (R,T) in terms of which the metric has the form $ds^2(R,T)=dR^2/A(R,T)+R^2d\Omega^2-B(R,T)dT^2$. The Einstein field equations, including cosmological constant, are given for arbitrary T^μ_ν, and the timelike geodesic equations are solved for radial motion. We then show how to replace T with a new time coordinate τ that is equal to the time measured by radially moving geodesic clocks. Cosmology is brought into the picture by setting T^μ_ν equal to the stress-energy tensor for a perfect fluid composed of geodesic particles, and letting τ be the time measured by clocks coincident with the fluid particles. We solve the field equations in terms of (R,τ) coordinates to get the metric coefficients in terms of the pressure and density of the fluid. The metric on the subspace $\tau=\text{const}$ is equal to $dR^2+R^2d\Omega^2$, and so is flat, with R having the physical significance that it is a measure of proper distance in this subspace. As specific examples, we consider the de Sitter and Einstein—de Sitter universes. On an (R,τ) spacetime diagram, all trajectories in an Einstein—de Sitter universe are emitted from $R=0$ at the "big bang" at $\tau=0$. Further, a light signal coming toward $R=0$ at some time $\tau>0$ will, in its past history, have started from $R=0$ at $\tau=0$, and have turned around on the line $2R=3\tau$. A consequence of this is a "tilting" of the null cones along the trajectory of a cosmological particle. The turnaround line $2R=3\tau$ marks the transition where an $R=\text{const}$ line changes from spacelike to timelike in character. We show how to apply the techniques developed here to the inhomogeneous problem of a Schwarzschild mass imbedded in a given universe in the paper immediately following this one.

I. INTRODUCTION

A usual starting place for discussions of cosmological solutions to the Einstein field equations of general relativity is the Robertson-Walker isotropic metric form

$$ds^2(r,t)=[1+K(r/2b)^2]^{-2}e^{2h(t)}(dr^2+r^2d\Omega^2)-dt^2\,,$$
$$d\Omega^2=d\theta^2+\sin^2\theta\,d\phi^2\,. \tag{1.1}$$

The coordinates (r,t) used in (1.1) are "comoving" in the sense that r stays constant along a particle in the cosmological fluid which is the source of the gravitational field described by (1.1), and t is measured by clocks at rest in the fluid. A particular universe is described by specifying the function $h(t)$, whereupon $K=-1$, 0, or $+1$ defines the intrinsic curvature of the three-dimensional subspace $t=\text{constant}$, and the constant b is a measure of the radius of curvature of this subspace. The metric (1.1) is "isotropic" in form because the spatial part is proportional to $dr^2+r^2d\Omega^2$. In this paper we will treat only zero-curvature cosmologies ($K=0$).

When discussing spherically symmetric situations, it is useful to write the metric in curvature form as

$$ds^2(R,T)=\frac{dR^2}{A(R,T)}+R^2d\Omega^2-B(R,T)dT^2\,. \tag{1.2}$$

The metric is "curvature" in form because the angular part is equal to $R^2d\Omega^2$. With the exception of the de Sitter universe, described in Sec. VI, to my knowledge little or no attention has been given in the literature to using the curvature form (1.2) in cosmology.

In this paper we examine cosmological theory from the point of view of the curvature coordinates (R,T) used in (1.2). In Sec. II, we show how to transform zero-curvature cosmologies from the isotropic form (1.1) to the curvature form (1.2). We next show how to develop cosmological theory starting afresh from the curvature metric form (1.2). In Sec. III we give the field equations and solution to the radial timelike geodesic equations for a completely general spherically symmetric field described in curvature coordinates. We then show in Sec. IV how to use the geodesic solutions to change the time coordinate from T to a geodesic time coordinate τ measured by radially moving geodesic clocks. In Sec. V we bring in cosmological principles by taking the source of the field, which as yet has not been specified, to be a perfect fluid whose particles move along geodesics. We then apply the cosmological principle of homogeneity of the Universe to develop the relationships between the metric components and the pressure and density of the cosmological fluid. Examples of how our approach applies to the de Sitter and the Einstein—de Sitter universes are given in Secs. VI and VII.

When finally done, the equations relating pressure, density, and metric components in the fluid will be the same as obtained with the standard approach that starts from the metric form (1.1). However, it will be found that new insight and previously unrecognized features will be obtained by looking at cosmology from the point of view of curvature coordinates. Moreover, the methods developed here will allow generalizations to cosmological problems where homogeneity is not applicable, so that (1.1) can not be used. As an example, the paper immediately following this one shows how the methods of this paper can be used for attacking the problem of imbedding a Schwarzschild mass into a given universe.

II. CHANGING FROM ISOTROPIC TO CURVATURE COORDINATES

In this section we show how to bring the isotropic metric form (1.1) into the curvature form (1.2). We will here consider only zero-curvature cosmologies, so that, with $K=0$, the isotropic metric form (1.1) is

$$ds^2(r,t)=e^{2h(t)}(dr^2+r^2d\Omega^2)-dt^2 \ . \tag{2.1}$$

We will convert this to the curvature form in two steps. We first define a curvature radial coordinate R by

$$R=re^{h(t)} \tag{2.2}$$

in terms of which (2.1) assumes the nondiagonal form

$$ds^2(R,t)=[dR-RH(t)dt]^2+R^2d\Omega^2-dt^2 \ , \tag{2.3}$$

where $H(t)=dh/dt$.

The final step is to introduce a curvature time coordinate $T(R,t)$, or equivalently $t(R,T)$, that diagonalizes (2.3). As can be checked by direct substitution, the required $t(R,T)$ must satisfy

$$t_{,R}=\frac{RH}{(RH)^2-1} \ . \tag{2.4}$$

The metric (2.3) then assumes the diagonal curvature form

$$ds^2=\frac{dR^2}{1-(RH)^2}+R^2d\Omega^2 -[1-(RH)^2](t_{,T})^2dT^2 \ , \tag{2.5}$$

where H is now to be regarded as a function of (R,T) obtained from the transformation $t(R,T)$ defined by (2.4).

At this point, the expression $t(R,T)$ satisfying (2.4) will involve an arbitrary function $F(T)$. We will specify $F(T)$ by requiring that the (R,T) coordinates go over to Minkowski coordinates at $R=0$, which means

$$t_{,T}\to 1 \quad \text{as } R\to 0 \ . \tag{2.6}$$

III. FORMALISM OF CURVATURE COORDINATES

The curvature metric form (2.5) has been obtained starting from the cosmological isotropic metric form (2.1). It is certainly permissible, though, to reverse the ordering, and to start cosmological discussions from the curvature metric form (2.5). We shall here develop cosmological theory from first principles starting with curvature coordinates, and without assuming (2.1). In this manner, we shall obtain a new view of cosmology, and will find that it is unnecessary to bring the isotropic coordinates (r,t) into the picture. Besides being a worthwhile endeavor in its own right, the methods developed here will allow cosmological ideas to be extended to areas where the isotropic formalism does not apply.

We begin by developing the relevant equations for a metric written in the completely general curvature form

$$ds^2(R,T)=\frac{dR^2}{A(R,T)}+R^2d\Omega^2 -A(R,T)f^2(R,T)dT^2 \ . \tag{3.1}$$

The Einstein field equations $G^\mu_\nu=-8\pi T^\mu_\nu+\Lambda\delta^\mu_\nu$ involving only first-order derivatives of the metric components of (3.1) give

$$[R(1-A)]_{,R}=-8\pi R^2T^4_4+R^2\Lambda \ , \tag{3.2a}$$

$$[R(1-A)]_{,R}-2AR(f_{,R}/f)=-8\pi R^2T^1_1+R^2\Lambda \ , \tag{3.2b}$$

$$[R(1-A)]_{,T}=8\pi R^2T^1_4 \ . \tag{3.2c}$$

The remaining equations for $T^2_2=T^3_3$, which involve second-order derivatives, can be obtained from the Bianchi identities $G^\mu_{\nu;\mu}=0$, and therefore do not have to be written down. Thus, it is sufficient to satisfy only the three equations (3.2).

A scheme for solving the field equations (3.2) is to regard T^4_4 and T^1_1 as given functions of (R,T).[1] We then obtain $A(R,T)$ from (3.2a) as

$$A(R,T)=1+(8\pi/R)\int_0^R T^4_4R^2dR-(\Lambda/3)R^2 \ , \tag{3.3}$$

where the integral is evaluated over a $T=$const surface. To obtain $f(R,T)$, we combine (3.2a) and (3.2b) to get

$$\ln f^2(R,T)=8\pi\int_0^R (R/A)(T^1_1-T^4_4)dR \ . \tag{3.4}$$

From (3.2c) the component T^1_4 is given by

$$T^1_4=-(1/R^2)\int_0^R T^4_{4,T}R^2dR \ . \tag{3.5}$$

Although it is not necessary to write (3.5), it is instructive to see its form, as it represents transfer of energy across an $R=$const surface. The remaining components $T^3_3=T^4_4$ are obtained from the Bianchi identities $G^\mu_{\nu;\mu}=0$, with $\nu=1$, by differentiation as

$$\begin{aligned}T^2_2=T^3_3=&(R/2)T^1_{1,1}-(R/2A^2f^2)T^1_{4,4}+T^1_1\\&+(R/2A^2f^2)(2A_{,4}/A+f_{,4}/f)T^1_4\\&+(R/4)(A_{,1}/A+2f_{,1}/f)(T^1_1-T^4_4) \ .\end{aligned} \tag{3.6}$$

Thus, specification of T^4_4 and T^1_1 determines the metric coefficients $A(R,T)$ and $f(R,T)$ from (3.3) and (3.4), with the Einstein field equations being consistently satisfied.

We shall now solve the timelike geodesic equations for radial motion. The relevant Christoffel symbols from the metric form (3.1) are

$$\begin{aligned}&\Gamma^1_{11}=-A_{,R}/2A, \quad \Gamma^1_{14}=-A_{,T}/2A \ ,\\&\Gamma^1_{44}=f^2AA_{,R}/2+A^2ff_{,R} \ ,\\&\Gamma^4_{11}=-A_{,T}/2f^2A^3, \Gamma^4_{14}=A_{,R}/2A \ ,\\&\Gamma^4_{44}=A_{,T}/2A+f_{,T}/f \ .\end{aligned} \tag{3.7}$$

Direct substitution (see the Appendix) shows that a solution to the geodesic equations

$$\delta V^\mu/\delta\tau = dV^\mu/d\tau + \Gamma^\mu_{\rho\sigma}V^\rho V^\sigma = 0 \,, \tag{3.8}$$

$$g_{\rho\sigma}V^\rho V^\sigma = (V^1)^2/A - Af^2(V^4)^2 = -1 \,, \tag{3.9}$$

$$V^\mu = dx^\mu/d\tau$$

for radially moving particles is

$$V^1 = dR/d\tau = m(k^2 - A)^{1/2} \quad (m = \pm 1) \,, \tag{3.10a}$$

$$V^4 = dT/d\tau = k/fA \,, \tag{3.10b}$$

where $m = +1$ or -1 depending, respectively, on whether the particle is moving in the sense of increasing or decreasing R, and the parameter k is related to the proper time τ along the particle's trajectory and the metric coefficients by

$$\tau_{,T} = kF \,, \tag{3.11a}$$

$$\tau_{,R} = -mA^{-1}(k^2 - A)^{1/2} \,. \tag{3.11b}$$

The quantity $k(R,T)$ is associated with the energy per unit mass of the radially moving geodesic particle.

To get a feeling for k, let us turn momentarily to the cosmological situation described by the metric form (2.5), so that $A = 1 - (RH)^2$. There are two possibilities for k. The particle may have some minimum turning radius R_i where $dR/d\tau = 0$ in (3.10a), for which case

$$k^2 = 1 - (R_i H)^2 \,. \tag{3.12a}$$

Such a particle with $k < 1$ will come in radially from $R = \infty$, stop momentarily at R_i given by (3.12a), and then head back out to infinity again. If the cosmology is not static, i.e., if H depends on T, particles coming from infinity with the same value k_0 of the energy parameter will have turning radii R_i that vary with time. The other possibility is for the energy parameter to be $k > 1$, so that the particle reaches the origin at $R = 0$. At $R = 0$ the cosmological metric form (2.5) is Minkowski in form, so that at $R = 0$ the coordinate velocity dR/dT of a particle is equal to the velocity v_0 of the particle that is actually measured by an observer fixed at $R = 0$:

$$dR/dT\,|_{R=0} = v_0 \,, \tag{3.13}$$

We then find for particles with $k > 1$,

$$k^2 = (1 - v_0{}^2)^{-1/2} \,. \tag{3.12b}$$

A transition value particle with $k = 1$, so that $R_i = 0$ or $v_0 = 0$, corresponds to a particle that stays forever at $R = 0$. Thus, the origin $R = 0$ is a timelike geodesic.

IV. A GEODESIC TIME COORDINATE

With the (R,T) coordinates used in the curvature metric form (3.1), the time coordinate T has the physical significance that it is associated with clocks that are located at fixed values of R. The coordinate time lapse ΔT is related to the time lapse $\Delta\tau_R$ on a clock fixed at some $R =$ const point by, from (3.1),

$$\Delta\tau_R = fA^{1/2}\Delta T \,. \tag{4.1}$$

Instead of measuring time with clocks located at fixed values of R, it is sometimes advantageous to measure time with geodesically moving clocks. In this manner, we have previously shown, for example, that new insight was gained about the Schwarzschild field[2–6] and the de Sitter universe.[7] Further, we shall see below that measuring time with geodesic clocks will prove to be important in developing cosmological theory.

We can change from the curvature time coordinate T to a geodesic time coordinate τ in the following manner. We consider a congruence of radially moving geodesic clocks, whose trajectories in terms of (R,T) coordinates are given by (3.10) with (3.11). The energy parameter k varies smoothly from one clock to the next in the congruence. The time τ assigned to any event is then equal to the time recorded by the particular geodesic clock that is coincident with the event.

The variation of k is determined from the type of reference system we choose to build with the geodesic clocks. For example, we have developed in the Schwarzschild field a geodesic-clock analog to Eddington-Finkelstein coordinates by treating a reference system where each clock falls from a fixed radius R_i.[2–6] For this situation, k was the same constant for each clock of the reference system. A different type of reference system in the Schwarzschild field where a swarm of clocks moves outward from $R = 0$ to some maximum radius R_i, different for each clock, and then falls back to $R = 0$, gives rise to a geodesic-clock analog to the Kruskal-Szekeres coordinate system.[3–6] For this reference system, k has a different value for each clock. We have also treated similar geodesic-clock reference systems in a de Sitter universe.[7]

In the following, we shall leave the reference system formed from the radially moving geodesic clocks unspecified. Thus, k will be taken as an arbitrary function of the coordinates.

The transformation replacing T with τ is obtained from (3.11), but cannot be written down explicitly until the metric coefficients and the type of reference system are specified. However, the metric form in terms of (R,τ) coordinates is determinable from the transformation implied from (3.11), and has the nondiagonal form

$$ds^2(R,\tau) = k^{-2}[dR - m(k^2 - A)^{1/2}d\tau]^2 + R^2 d\Omega^2 - d\tau^2 \,. \tag{4.2}$$

In (4.2) we see the quantities $A(R,\tau)$ and the energy parameter $k(R,\tau)$, which are to be determined from the field equations once a source and reference system are specified. In terms of the coordinates (R,τ), the trajectories of the clocks measuring the time coordinate τ are given by, from (3.10) with (3.11),

$$V^1 = dR/d\tau = m(k^2 - A)^{1/2}, \quad V^4 = d\tau/d\tau = 1 \,. \tag{4.3}$$

We will use greek letters to indicate tensor components in terms of (R,τ) coordinates. The three relevant field equations from $\Gamma^\mu_\nu = -8\pi\tau^\mu_\nu + \Lambda\delta^\mu_\nu$, which can be obtained from (3.2) by applying the transformation defined by (3.11), are

$$\frac{\partial}{\partial R}[R(1-A)]=-8\pi R^2\tau_4^4+R^2\Lambda\ , \tag{4.4a}$$

$$\frac{\partial}{\partial R}[R(1-A)]+m(k^2-A)^{-1/2}\frac{\partial}{\partial\tau}[R(1-A)]+2m(RA/k)(k^2-A)^{-1/2}[\partial k/\partial\tau+m(k^2-A)^{1/2}\partial k/\partial R]$$
$$=-8\pi R^2\tau_1^1+R^2\Lambda\ , \tag{4.4b}$$

$$\frac{\partial}{\partial\tau}[R(1-A)]=8\pi R^2\tau_4^1\ . \tag{4.4c}$$

We will indicate partial derivatives using (R,τ) coordinates by $\partial/\partial x^\mu$ to distinguish them from the different partial derivatives when (R,T) coordinates are used, which will be indicated by commas.

As a scheme for solving the field equations (4.4), we first integrate (4.4a) to obtain

$$A(R,\tau)=1+(8\pi/R)\int_0^R\tau_4^4R^2dR-(\Lambda/3)R^2\ , \tag{4.5}$$

where the integral is evaluated over a $\tau=$const surface [which is different from a $T=$const surface used in (3.3)]. Equation (4.5) determines $A(R,\tau)$ once τ_4^4 is specified. From (4.5) and (4.4c) we obtain

$$\tau_4^1=-(1/R^2)\int_0^R\partial\tau_4^4/\partial\tau\, R^2dR \tag{4.6}$$

so that τ_4^1 is determined once τ_4^4 is specified. It is seen from (4.6) that there will be a flow of energy across an $R=$const surface if $A(R,\tau)$ has a time dependence. Substituting (4.4a) and (4.4c) into (4.4b), we find

$$-m\,2A[\partial k/\partial\tau+m(k^2-A)^{1/2}\partial k/\partial R]=8\pi Rk(k^2-A)^{1/2}(\tau_1^1-\tau_4^4)+m\,8\pi Rk\tau_4^1 \tag{4.7a}$$

or equivalently, after using (4.6),

$$-m\,2A[\partial k/\partial\tau+m(k^2-A)^{1/2}\partial k/\partial R]=8\pi Rk(k^2-A)^{1/2}(\tau_1^1-\tau_4^4)-m(8\pi k/R)\int_0^R\partial\tau_4^4/\partial\tau\, R^2dR\ . \tag{4.7b}$$

The relationship (4.7) gives a condition on $k(R,\tau)$ once τ_4^4 and τ_1^1 are specified. The energy parameter k cannot be determined from the field equations, because we have the freedom of specifying the type of geodesic reference system we are going to use to measure the time coordinate τ.

For completeness, we note that the τ_ν^μ in (R,τ) coordinates are related to the T_ν^μ in (R,T) coordinates by

$$T_4^4=\tau_4^4+mA^{-1}(k^2-A)^{1/2}\tau_4^1\ , \tag{4.8a}$$

$$T_1^1=\tau_1^1+mA^{-1}(k^2-A)^{1/2}\tau_4^1\ , \tag{4.8b}$$

$$T_4^1=kf\tau_4^1\ , \tag{4.8c}$$

$$T_2^2=T_3^3=\tau_2^2=\tau_3^3\ . \tag{4.8d}$$

V. COSMOLOGICAL PRINCIPLES

Thus far, we have only made calculations with the metric forms (3.1) and (4.2), and have made no mention of cosmology. We here bring cosmology into the picture.

One assumption made in cosmological theory is that the source of the gravitational field of the Universe corresponds to a "smearing out" of the galaxies in the Universe into a perfect fluid, whose pressure p and density ρ generate the stress-energy tensor

$$T_\nu^\mu=(\rho+p)V^\mu V_\nu+p\delta_\nu^\mu\ , \tag{5.1}$$

where the particles of the fluid move along geodesics. In terms of the coordinates (R,T) of (3.1), the components of (5.1) for radially moving geodesic particles given by (3.10) are

$$T_4^4=-(k^2/A)(\rho+p)+p\ , \tag{5.2a}$$

$$T_1^1=-(k^2/A-1)(\rho+p)+p\ , \tag{5.2b}$$

$$T_4^1=-mkf(k^2-A)^{1/2}(\rho+p)\ , \tag{5.2c}$$

$$T_2^2=T_3^3=p\ , \tag{5.2d}$$

which are to be substituted into the field equations (3.2). It should be noted that the T_ν^μ in (5.2a)–(5.2c) contain mixtures of ρ and p, so if they are solved for ρ and p explicitly, ρ and p will involve mixtures of the T_ν^μ.

Instead of (R,T) coordinates, let us see how the picture appears in the (R,τ) coordinates of (4.2). In so doing, we will let the geodesic clocks measuring τ correspond to particles in the geodesic fluid. Since we are now specifying the geodesic clocks to coincide with the geodesic particles of the fluid, the energy parameter $k(R,\tau)$ will be fixed by the field equations. The V^μ that go into the perfect-fluid expression (5.1) are given by (4.3), and the corresponding stress-energy components in (R,τ) coordinates are

$$\tau_4^4=-\rho\ , \tag{5.3a}$$

$$\tau_1^1=p\ , \tag{5.3b}$$

$$\tau_4^1=-m(k^2-A)^{1/2}(\rho+p)\ , \tag{5.3c}$$

$$\tau_2^2=\tau_3^3=p\ . \tag{5.3d}$$

Thus, in contrast with (R,T) coordinates, when (R,τ) coordinates are employed $-\tau_4^4$ and τ_1^1 are, respectively, numerically equal to the density ρ and pressure p of the radially moving geodesic fluid. When Eqs. (5.3) are substituted into (4.7a), we obtain

$$\partial k/\partial\tau+m(k^2-A)^{1/2}\partial k/\partial R=0\ , \tag{5.4a}$$

which, using the V^μ from (4.3), can be rewritten as

$$(\partial k/\partial\tau)V^4+(\partial k/\partial R)V^1=(\partial k/\partial\tau)(d\tau/d\tau)+(\partial k/\partial R)(dR/d\tau)=dk/d\tau=0 \ . \tag{5.4b}$$

Thus, the field equations require that the energy parameter $k(R,\tau)$ must be a constant along each streamline of the cosmological fluid.

We have thus far talked only about a source that is a radially moving geodesic fluid, and have not yet invoked any cosmological principles. This we shall now do.

A basic principle of cosmology is that the Universe is isotropic and homogeneous. The metric forms (3.1) and (4.2) incorporate the isotropic requirement, for the angular part is written as $d\Omega^2$ and the metric components are independent of the angles, so the Universe appears the same in every direction.

For the Universe also to be homogeneous requires that the Universe should appear the same from all points, i.e., there should be nothing special about any origin $R=0$. This requirement puts a restriction on the energy parameter k of the streamlines of the cosmological fluid. If a grouping of neighboring streamlines had k values such that $k<1$, the fluid in these streamlines would not have sufficient energy to reach $R=0$. For these streamlines, there would then be a hole in the fluid around $R=0$, thereby giving the point $R=0$ a distinguishing feature, which correspondingly contradicts the principle of homogeneity. On the other hand, if $k>1$ for a grouping of neighboring streamlines, the fluid in these streamlines would have a velocity at $R=0$. This would mean that the point $R=0$ is a source or sink of fluid particles, again contradicting the principle of homogeneity.

Thus, in order to accommodate the cosmological principle that the Universe be homogeneous, we must have $k=1$ for all the streamlines of the fluid. In turn, the value $k=1$ means, from (3.12b), that the fluid will always be at rest relative to observers at any arbitrarily chosen geodesic origin $R=0$.

With $k=1$ and with the source terms (5.3), Eqs. (4.5), (4.6), and (4.7b) become

$$A(R,\tau)=1-(8\pi/R)\int_0^R \rho R^2 dR-(\Lambda/3)R^2 \ , \tag{5.5}$$

$$\tau_4^1=(1/R^2)\int_0^R \partial\rho/\partial\tau\, R^2 dR \ , \tag{5.6}$$

$$p+\rho+(m/R^2)(1-A)^{-1/2}\int_0^R \partial\rho/\partial\tau\, R^2 dR=0 \ . \tag{5.7}$$

The homogeneity requirement also puts a restriction on the time coordinate we should use for describing how quantities in the Universe vary with time. The time coordinate T used in the curvature metric form (3.1) is associated with clocks that are fixed at $R=$const points relative to the particular geodesic origin at $R=0$. This means that these clocks, which do not follow geodesic trajectories, would be moving relative to any other geodesically moving origin $R'=0$ which we might want to use for describing the Universe. Thus, T would not be an appropriate time coordinate for formulating a cosmological principle of time variation.

On the other hand, using the geodesic time coordinate τ for formulating cosmological time principles will be in accord with the principle of homogeneity. The time coordinate τ is measured directly with clocks that are fixed in the galaxies forming the cosmological fluid, so that these clocks move with the cosmological fluid. As such, the times τ measured by the clocks are not associated with any particular choice of galaxy for the origin for R, but rather are a manifestation of the overall Universe itself. Thus, in formulating cosmological principles regarding how quantities are to vary with time, the time we will use will be the geodesic time τ measured by clocks that are fixed relative to the particles of the cosmological fluid.

We now invoke the cosmological principle of homogeneity, and assume that at any given value of the cosmological time τ the fluid density ρ will be the same everywhere throughout the Universe. This means that ρ will be independent of R and depend only on τ, so that ρ and $\partial\rho/\partial\tau=d\rho/d\tau$ can be brought out of the spatial integrals in the previous equations, which then can be trivially integrated. From (5.5) we obtain

$$A(R,\tau)=1-(8\pi/3)\rho R^2-(\Lambda/3)R^2 \ , \tag{5.8}$$

so that once ρ is specified as a function of τ, the metric coefficient $A(R,\tau)$ is determined. The metric in (R,τ) coordinates then has the form, from (4.2),

$$ds^2(R,\tau)=[dR-mR(8\pi\rho/3+\Lambda/3)^{1/2}d\tau]^2+R^2d\Omega^2-d\tau^2 \ . \tag{5.9}$$

From (5.7), the pressure p in the fluid is related to the fluid density ρ by

$$p+\rho+m(24\pi\rho+\Lambda)^{-1/2}d\rho/d\tau=0 \tag{5.10}$$

so that, as expected, p will also depend only on τ. From (5.6), the energy flow across an $R=$const surface is determined by

$$\tau_4^1=(R/3)d\rho/d\tau \ . \tag{5.11}$$

This completes the formulation of cosmological theory in terms of (R,τ) coordinates.

Once a metric for a particular Universe has been determined in (R,τ) coordinates by specifying ρ and p, the same universe can be described with (R,T) curvature coordinates by using the transformation defined by (3.11).

A usual way of proceeding from this point is to assume there is some equation of state of the fluid relating ρ and p, i.e., $p=p(\rho)$. One then substitutes this equation of state into (5.10), solves this for ρ, which then determines the metric coefficient $A(R,\tau)$ from (5.8).

To relate our approach to the standard cosmological formulation in terms of the isotropic coordinates (r,t) of (2.1), we identify τ with t, and set $H^2(t)$ in (2.3) equal to

$$H^2(t)=(8\pi/3)\rho+\Lambda/3 \ . \tag{5.12}$$

However, we will have no need to develop this here.

VI. THE EMPTY-SPACE DE SITTER UNIVERSE

A de Sitter universe corresponds to a situation where there is no cosmological fluid, i.e., the Universe is a vacuum empty of matter. Setting $\rho=0$ in (5.10) gives $p=0$, which is to be expected since if there is no matter there can be no pressure. With ρ and p zero, i.e., with $T^{\mu}_{\nu}=0$, we obtain $A(R,T)$ in (R,T) coordinates from (3.3), or $A(R,\tau)$ in (R,τ) coordinates from (5.8) as

$$A(R,T)=A(R,\tau)=1-(\Lambda/3)R^2 \; . \tag{6.1}$$

From (3.4) we obtain

$$f=1 \tag{6.2}$$

so that the curvature form (3.1) of the metric describing the de Sitter universe is

$$ds^2(R,T)=\frac{dR^2}{1-(\Lambda/3)R^2}+R^2d\Omega^2 -[1-(\Lambda/3)R^2]dT^2 \; . \tag{6.3}$$

We know from Birkhoff's theorem that any spherically symmetric field *in vacuo* is static, which means that it must be possible to find coordinates that explicitly exhibit this intrinsic static property. Curvature coordinates are these coordinates, for it is seen that the metric coefficients in the diagonal form (6.3) are independent of T.

The cosmological constant Λ in (6.3), which we assume to be positive, plays the role of a "source" term, and because of its presence geodesic particles move away from each other.

It is seen that the curvature metric form (6.3) has a coordinate singularity at $R=(3/\Lambda)^{1/2}=R_0$. Because of this, a particle or light signal cannot be tracked across R_0 in (R,T) coordinates, so $R=R_0$ constitutes an event horizon.

It is possible, though, to follow particle trajectories across R_0 by changing the time coordinate from T to the geodesic time coordinate τ described in Sec. IV. In terms of τ, the metric (4.2) for the de Sitter universe has the form, using (6.1),

$$ds^2(R,\tau)=k^{-2}[dR-m(\Lambda R^2/3+k^2-1)^{1/2}d\tau]^2 +R^2d\Omega^2-d\tau^2 \; . \tag{6.4}$$

We have discussed this in detail elsewhere,[7] where we have shown that it is possible to develop in the de Sitter universe geodesic-clock analogs to Eddington-Finkelstein and Kruskal-Szekeres coordinates that are used in the Schwarzschild field. In developing these geodesic-clock coordinate systems, we are not restricted by the homogeneity condition $k=1$, described in Sec. V, because there is no cosmological fluid, and an emtpy-space universe is automatically homogeneous. In fact, it turns out that in order to have physically sensible geodesic-clock reference systems, we must not set $k=1$.[7] This, in turn, raises questions about the mathematical maneuverings that are required to cast the de Sitter metric form (6.4) into the isotropic metric form (2.1), which requires setting $k=1$. We have discussed this in detail in Ref. 7, and will not develop this further here.

VII. THE ZERO-PRESSURE EINSTEIN–DE SITTER UNIVERSE

A model that has achieved widespread acceptance as giving a good description of our Universe is the Einstein–de Sitter (ES) universe. In addition, Dirac claims that this particular universe is in agreement with his large numbers hypothesis.[8]

An ES universe is characterized by $\Lambda=0$ and an equation of state where the cosmological fluid has zero pressure everywhere. Setting $p=0$ in (5.10), we obtain the corresponding density as

$$\rho=\frac{1}{6\pi(\tau-\tau_0)^2} \; , \tag{7.1}$$

where $\tau<\tau_0$ corresponds to a collapsing fluid ($m=-1$), and $\tau>\tau_0$ corresponds to an expanding fluid ($m=+1$). Substituting (7.1) into (5.8), and setting $\tau_0=0$, we obtain $A(R,\tau)$ as

$$A(R,\tau)=1-(2R/3\tau)^2 \tag{7.2}$$

so that the metric form (5.9) becomes

$$ds^2(R,\tau)=[dR-m(2R/3\tau)d\tau]^2+R^2d\Omega^2-d\tau^2 \; . \tag{7.3}$$

To transform the nondiagonal form (7.3) into the diagonal curvature form (3.1), we use the differential requirement (3.11b):

$$\tau_{,R}=m\frac{2R/3\tau}{(2R/3\tau)^2-1} \; . \tag{7.4}$$

As can be checked by differentiation, the transformation $\tau(R,T)$ satisfying (7.4) and the Minkowski condition $\tau_{,T}\to 1$ as $R\to 0$ is given implicitly by

$$T=\tau[1+\tfrac{1}{2}(2R/3\tau)^2]^{3/2} \; , \tag{7.5}$$

In turn, (7.5) is an explicit expression for the transformation $T(R,\tau)$. We now find $f(R,T)$ from (3.11a) and (7.5) to be

$$f=\tau_{,T}=[1-(2R/3\tau)^2]^{-1}[1+\tfrac{1}{2}(2R/3\tau)^2]^{-1/2} \; . \tag{7.6}$$

The curvature metric form (3.1) correspondingly becomes

$$ds^2(R,T)=\frac{dR^2}{1-(2R/3\tau)^2}+R^2d\Omega^2 -\frac{dT^2}{[1-(2R/3\tau)^2][1+\frac{1}{2}(2R/3\tau)^2]} \tag{7.7}$$

in which $\tau(R,T)$ is given implicitly by (7.5). From (7.7) it is seen that there is a coordinate singularity at values of R and T defined by

$$(2R/3\tau)^2=1 \; . \tag{7.8}$$

We will now look at various trajectories in the ES universe, dealing first with trajectories in terms of (R,τ) coordinates, and then next showing how these same trajectories appear in (R,T) coordinates. For simplicity, we will treat only an expanding universe ($m=+1$); the extension to a collapsing universe ($m=-1$) is straightforward.

Along the trajectory of a particle in the cosmological fluid (a galaxy) we must have $ds^2(R,\tau)=-d\tau^2$, and we find from (7.3)

$$dR/d\tau=2R/3\tau\ , \tag{7.9}$$

which yields upon integration

$$R=b\tau^{2/3}\ . \tag{7.10}$$

The constant b is a measure of the energy of the galaxy, measured relative to the origin $R=0$, at the "big bang" at $\tau=0$.

The equation for a radially moving light signal is obtained by setting $ds^2(R,\tau)=0$ and $d\Omega=0$ in (7.3) to obtain

$$dR/d\tau=2R/3\tau\pm 1\ . \tag{7.11}$$

One obtains for the "plus" sign

$$R=3\tau-3\tau_0{}^{1/3}\tau^{2/3}\ , \tag{7.12}$$

and for the "minus" sign

$$R=-3\tau+3\tau_0{}^{1/3}\tau^{2/3}\ . \tag{7.13}$$

World lines of galaxies and light signals are shown on the (R,τ) spacetime diagram of Fig. 1. It is seen that all trajectories occurring at the "big bang" at $\tau=0$ are emitted from $R=0$ in an explosive manner with infinite values of $dR/d\tau$. The constant b in (7.10) determines the "spread" of the trajectories of the galaxies. The larger the value of b, the greater is the energy, measured relative to $R=0$, that a galaxy has when it is emitted from $R=0$ at $\tau=0$.

The line $\tau=\frac{1}{3}R$ ($\tau_0=0$) on Fig. 1 marks the changeover in outgoing light signals, given by (7.12), from those signals that are emitted from $R=0$ after $\tau=0$ and those signals that are emitted at $\tau=0$. For those signals emitted after $\tau=0$, τ_0 is equal to the time of emission from $R=0$. These signals leave $R=0$ with $dR/d\tau=+1$, and gradually bend downwards with an asymptotic slope of $dR/d\tau=+3$. For those signals emitted at the "big bang" at $\tau=0$, τ_0 is a negative number and is a measure of the "spread" of the light signals. These signals leave $R=0$ at $\tau=0$ with infinite values of $dR/d\tau$, and gradually bend upwards with an asymptotic slope of $dR/d\tau=+3$.

Because galaxies as well as outward-moving light signals given by (7.12) are emitted from $R=0$ with infinite speeds at the "big bang," at some later $\tau>0$ some galaxies will have evolved to a larger distance than certain light signals, as shown in Fig. 1. However, all light signals given by (7.12) will eventually catch up to all galaxies.

The other class of null trajectories on Fig. 1 is given by (7.13). These represent light signals striking $R=0$ with $dR/d\tau=-1$ at $\tau=\tau_0>0$, and are the incoming light signals that an observer at $R=0$ would see through a telescope. When these incoming signals are followed backwards in their histories, a somewhat surprising feature is found. At some point the trajectory of each signal turns around and goes back to $R=0$ at $\tau=0$. From (7.11), with the minus sign, it is found that this turnaround point lies on the line $\tau=\frac{2}{3}R$, along which there is a coordinate singularity in the curvature metric form (7.7).

This turnaround feature gives rise to a "tilting" of the null cones along the trajectory of a cosmological particle, as shown in Fig. 2. Before a particle crosses the transition line $\tau=\frac{2}{3}R$, both radial null lines through an event on its trajectory are inclined away from $R=0$. As the particle crosses $\tau=\frac{2}{3}R$, one of the radial null lines is inclined vertically on the (R,τ) spacetime diagram at event d shown in Fig. 2. Above $\tau=\frac{2}{3}R$, both radial null lines are inclined "normally," with one inclined toward and the other inclined away from $R=0$.

Because both radial null lines through any event below $\tau=\frac{2}{3}R$ are inclined away from $R=0$, a line $R=$const in this region will lie outside null cones along it. This means that lines $R=$const are spacelike in the region below the line $\tau=\frac{2}{3}R$. Above the line $\tau=\frac{2}{3}R$, a line $R=$const will lie inside a null cone through an event on it, so that in this region lines $R=$const are timelike. Thus, the line $\tau=\frac{2}{3}R$ marks the transition where a line $R=$const changes from

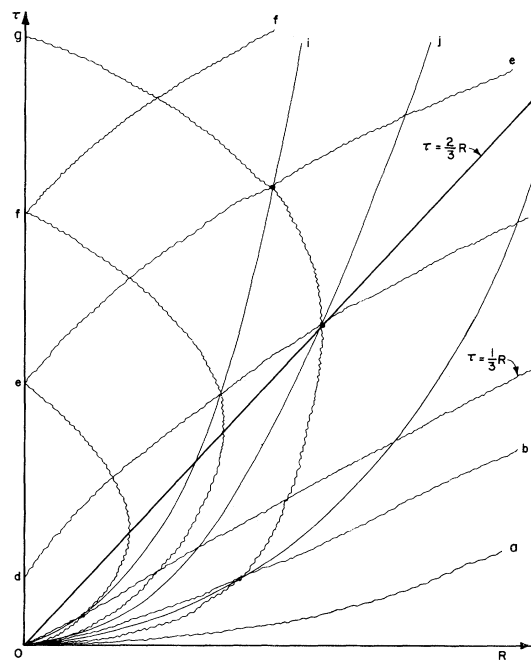

FIG. 1. Galaxy and light-signal trajectories on an (R,τ) spacetime diagram. Galaxies of the cosmological fluid follow trajectories such as $0i$, $0j$, $0k$. The null trajectories $0a$ and $0b$ represent light-signal trajectories emitted at the "big bang" at $\tau=0$ that always move towards increasing R. The null trajectory $0c$, given by $\tau=\frac{1}{3}R$, represents the limit of these light signals. Null trajectories dd, ee, ff represent outward-moving light signals emitted from $R=0$ after the "big bang." Null trajectories $0e$, $0f$, $0g$ represent light signals moving towards $R=0$ at various times $\tau>0$. These are the light signals that would be seen in a telescope at $R=0$. When these incoming signals are followed backwards in their past history, they turn around and go back to $R=0$ at $\tau=0$. The turnaround point lies on the timelike line $\tau=\frac{2}{3}R$, which marks the transition where $R=$const lines change from being spacelike to being timelike in character.

being spacelike to being timelike in character.

Difficulties arise when one tries to describe the above trajectories in terms of the coordinates (R,T), because the transformation (7.5) is double-valued. Figure 3 shows a plot of T versus τ for a fixed value of R, as given by (7.5). It is seen that the transition line $\tau=\frac{2}{3}R$ corresponds to the line $T=(\frac{3}{2})^{1/2}R$, and there are no values of (R,T) smaller than this line for the entire range $0\leq\tau\leq\infty$. This means that on an (R,T) spacetime diagram that repeats an (R,τ) spacetime diagram, one will never find a trajectory lying below the line $T=(\frac{3}{2})^{1/2}R$. For each value of $T>(\frac{3}{2})^{1/2}R$ there are two values of τ, one less than and one greater than $\tau=\frac{2}{3}R$. For example, Fig. 3 shows that the line $T=(3)^{1/2}R$ corresponds to both the null line $\tau=\frac{1}{3}R$ and the timelike line $\tau=\frac{1}{3}[1+(3)^{1/2}]^{3/2}R$.

Because of the double-valuedness, there will be two sets of trajectories through each point (R,T), corresponding to the two values of τ associated with the point. For example, a given point (R,T) will lie on both the trajectory of one cosmological particle before it has reached the transition line $\tau=\frac{2}{3}R$, and another different cosmological particle that has passed the transition line. Similarly, there will be two sets of radial null lines through each point (R,T), corresponding to two different (R,τ) events, one above and the other below the transition line on an (R,τ) spacetime diagram.

To get around this double-valuedness, it seems advisable to plot two separate (R,T) spacetime diagrams, one for events that lie above and the other for events that lie below the transition line $\tau=\frac{2}{3}R$. As an example, the events that lie below the transition line on the (R,τ) spacetime diagram of Fig. 2 are repeated in the (R,T) spacetime diagram of Fig. 4(a), while those events above the transition line on Fig. 2 are repeated in Fig. 4(b). The trajectories in Fig. 4 were obtained by applying the time transformation (7.5) to (7.10), (7.12), and (7.13) to get, respectively,

$$R=bT^{2/3}-\tfrac{2}{9}b^3\,, \tag{7.14}$$

which is the equation of a cosmological particle;

$$9R=-10\tau_0+9\tau_0{}^{1/3}T^{2/3}+(3T^{2/3}-2\tau_0{}^{2/3})^{3/2}\,, \tag{7.15}$$

which is the equation of an outgoing light signal; and

$$9R=10\tau_0-9\tau_0{}^{1/3}T^{2/3}\pm(3T^{2/3}-2\tau_0{}^{2/3})^{3/2}$$

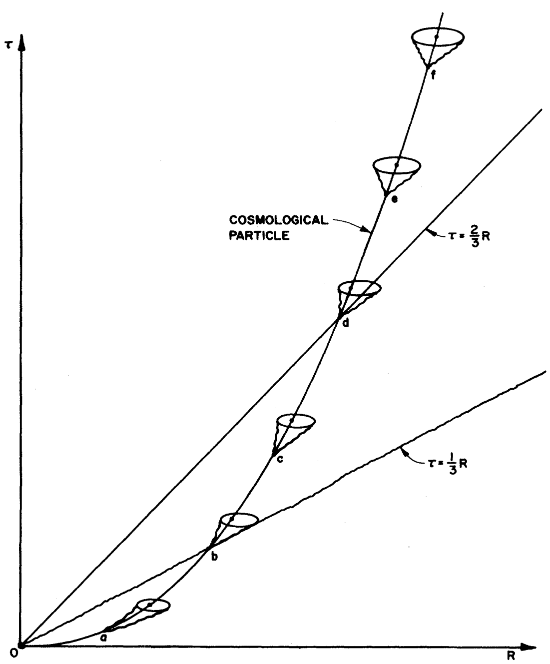

FIG. 2. Tilting of null cones. Null cones at various events along the trajectory of a cosmological particle "tilt" as the history of the particle evolves. Below the line $\tau=\frac{2}{3}R$ both radial portions of a null cone are inclined away from $R=0$. This means that $R=$const lines in this region lie outside the null cone, and so are spacelike. As the particle crosses $\tau=\frac{2}{3}R$ at d, one radial null line is inclined vertically. This marks the turnaround point of the global trajectory of this null line, as shown on Fig. 1. Above the line $\tau=\frac{2}{3}R$, one radial null line points toward and the other away from $R=0$. Since $R=$const lines lie inside null cones in this region, they are timelike here.

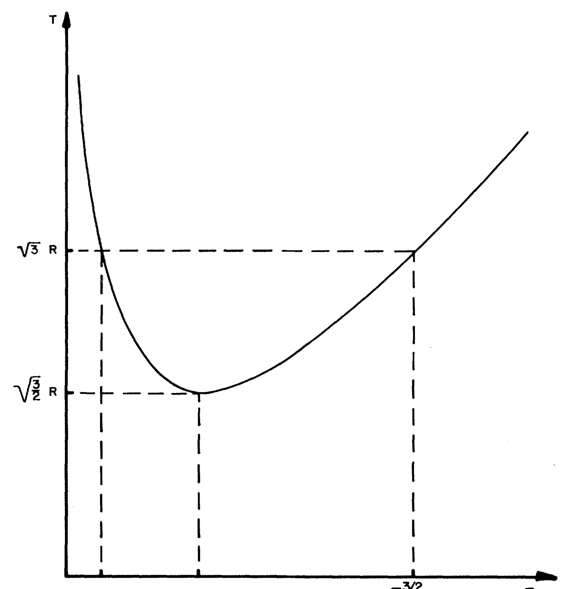

FIG. 3. The double-valuedness of the T-τ transformation for an EdS Universe. The figure shows a plot of T vs τ at constant R for the expression $T=\tau[1+\frac{1}{2}(2R/3\tau)^2]^{3/2}$ that relates the T and τ time coordinates in an ES universe. The transition line $\tau=\frac{2}{3}R$ corresponds to the line $T=(\frac{3}{2})^{1/2}R$. For all other values of T there are two values of τ, one corresponding to an event that lies below the line $\tau=\frac{2}{3}R$ and the other to an event that lies above this line. As an example, the time $T=(3)^{1/2}R$ corresponds to an event that lies on the null line $\tau=\frac{1}{3}R$ and a separate event that lies on the timelike line $\tau=\frac{1}{3}[1+(3)^{1/2}]^{3/2}R$. Within the range $0\leq\tau\leq\infty$ there are no events on an (R,T) spacetime diagram that lie below the transition line $T=(\frac{3}{2})^{1/2}R$. The values of T corresponding to the region $\tau<\frac{2}{3}R$ occur where the metric form (7.7) has an improper signature.

$$(+\text{ sign for } \tau<\tfrac{2}{3}R;\ -\text{ sign for } \tau>\tfrac{2}{3}R)\ , \qquad (7.16)$$

which is the equation of the light signal corresponding to (7.13).

It is seen in Fig. 4 that galaxies and outgoing light signals given by (7.15) that are moving in the same sense as the cosmological clocks measuring τ are tangent to the transition line $T=(\frac{3}{2})^{1/2}R$ at the (R,T) event corresponding to the crossing of the transition line when (R,τ) coordinates are used. On the other hand, light signals given by (7.16) moving with opposite sense to the cosmological clocks measuring τ undergo a discontinuity as they meet the transition line $T=(\frac{3}{2})^{1/2}R$, and move "backwards" in terms of T on the (R,T) spacetime diagram of Fig. 4(a).

Because of the upward trend of the T versus τ plot of Fig. 3 for values $\tau<\frac{2}{3}R$, straight lines through the origin below the transition line $\tau=\frac{2}{3}R$ on an (R,τ) spacetime diagram are mapped above the transition line $T=(\frac{3}{2})^{1/2}R$ on an (R,T) spacetime. As the inclination gets smaller on an (R,τ) spacetime diagram, the corresponding inclination gets larger on an (R,T) spacetime diagram, as shown in Fig. 4(a). Trajectories such as the world line of a cosmological particle that originate from the "big bang" at $(R,\tau)=(0,0)$ appear to emanate from different points along the T axis on an (R,T) spacetime diagram.

In obtaining the trajectories of Fig. 4(a), we have made use only of the time transformation (7.5), and have not considered the metrical properties of the spacetime region shown in Fig. 4(a). It is seen, however, that the metric form (7.7) results in g_{11} negative and g_{44} positive in the region $\tau<\frac{2}{3}R$, which corresponds to the spacetime of Fig. 4(a). Thus, describing trajectories on Fig. 4(a) is analogous to describing trajectories in the region $R<2M$ when curvature coordinates (R,T) are used in a Schwarzschild field. It is questionable in a Schwarzschild field whether it is physically valid to construct spacetime trajectories in the region $R<2M$ in terms of curvature coordinates (R,T), because the curvature metric form has an improper signature there. Similarly, the trajectories on Fig. 4(a) also are of questionable validity because the metric form (7.7) has an improper signature for this spacetime region. However, because the metric form (7.7) has a correct signature for $\tau>\frac{2}{3}R$, the trajectories on Fig. 4(b) give a correct description of the histories of particles and light signals.

The existence of the transition line $\tau=\frac{2}{3}R$ can be understood by a generalization from Schwarzschild field. We define the mass M inside a radius R from (4.5) by

$$M=-4\pi\int_0^R \tau_4^4 R^2 dR\ . \qquad (7.17)$$

For a Schwarzschild field, this definition yields the usual

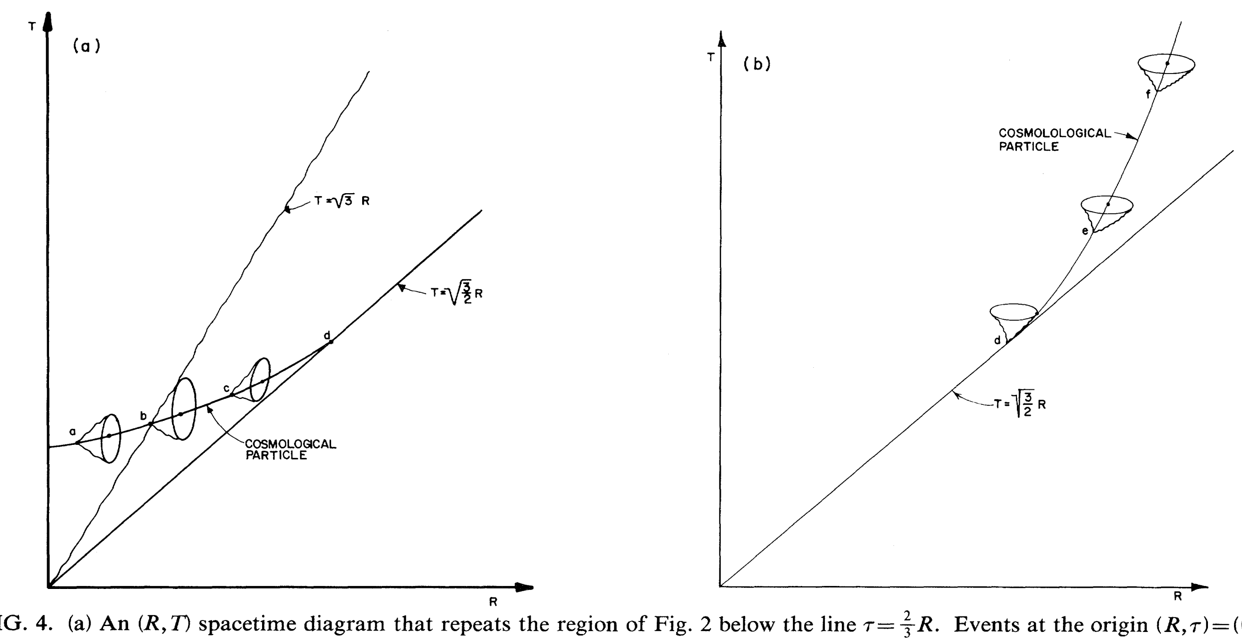

FIG. 4. (a) An (R,T) spacetime diagram that repeats the region of Fig. 2 below the line $\tau=\frac{2}{3}R$. Events at the origin $(R,\tau)=(0,0)$ on Fig. 2 are spread out along the T axis, so that the trajectory of the cosmological particle starts from a nonzero value of T at $R=0$. Straight lines through the origin below $\tau=\frac{2}{3}R$ on Fig. 2 are "inverted" on this diagram. The smaller the slope of a line on Fig. 2, the larger the slope of the corresponding line on this diagram. For example, the null line $\tau=\frac{1}{3}R$ on Fig. 2 is mapped into the line $T=(3)^{1/2}R$ on this diagram. It is seen that one of the radial null lines in a null cone evolves "backwards" in terms of T. The trajectories of the other radial null lines and the cosmological particle meet the transition line $T=(\frac{3}{2})^{1/2}R$ tangentially. Because of this, the null cones "collapse" as the particle approaches the tangent point d on the transition line. The validity of constructing (R,T) spacetime diagrams on this figure for the region $\tau<\frac{2}{3}R$ is questionable, because the curvature metric form (7.7) for the region of this figure has an improper signature. (b) An (R,T) spacetime diagram that repeats the region of Fig. 2 above the line $\tau=\frac{2}{3}R$. This figure is qualitatively similar to Fig. 2, except that outgoing light signals and cosmological particles emerge from the transition line $T=(\frac{3}{2})^{1/2}R$ at an inclination that is tangent to the line.

Schwarzschild mass. Substituting $A(R,\tau)$ from (7.2) into (4.5) and using (7.17), we obtain

$$2M/R=(2R/3\tau)^2 \ . \tag{7.18}$$

In the spacetime region where $\tau<\frac{2}{3}R$, we have $R<2M$, so that this region is analogous to the region inside the Schwarzschild radius in a Schwarzschild field.

The time dependence of M is understandable. In the intrinsically nonstatic ES universe, the cosmological fluid is flowing across a sphere $R=$const. Since the amount of matter inside a sphere is changing with time τ, so too will the size of the associated Schwarzschild-type radius, and corresponding coordinate singularity, change with time. Thus, at a given epoch τ, if we go to sufficiently large values of R we will eventually encounter sufficient mass so that the spacetime region is similar to the region inside $2M$ in a Schwarzschild field. Or, if we go far enough in the past at a particular value of R we will also encounter such a region.

The change of signature in the metric form (7.7) at $\tau=\frac{2}{3}R$ can be understood by extending to the ES universe reasoning that we have previously applied to the Schwarzschild field[4–6] and the de Sitter universe.[7] The curvature metric form (7.7) is predicated on the assumption that one can have clocks fixed at $R=$const points, where these clocks read $(-g_{44})^{1/2}\Delta T$. However, in the region $\tau<\frac{2}{3}R$, $R=$const points are spacelike in character, resulting in negative values of $-g_{44}$. One cannot have a physical object such as a clock located at $R=$const points in this region, as it would be moving faster than light. Thus, the formalism leading to the metric form (7.7) breaks down at the transition line $\tau=\frac{2}{3}R$ where $R=$const points change from being timelike to being spacelike in character. What is then needed is a new way of measuring time in the region $\tau<\frac{2}{3}R$, which is accomplished by changing from the curvature time coordinate T to the geodesic time coordinate τ.

VIII. DISCUSSION

We have shown how cosmology can be described with curvature coordinates (R,T), where the metric for a universe assumes the diagonal curvature form (3.1). While temporal variations of physical quantities such as the density and pressure of the cosmological fluid can certainly be described with T, the coordinate T is not a suitable time coordinate for formulating cosmological principles. The homogeneity premise of cosmology requires that the description of cosmology should not be tied to any one particular galaxy in the Universe. This is equivalent to saying that any galaxy should be suitable for serving as the origin $R=0$ of a spatial coordinate. Once we have chosen a spatial origin at some galaxy, which moves along a geodesic in the Universe, the curvature time coordinate T used in the curvature metric form (3.1) is uniquely associated with that particular galaxy, because T is related to times recorded by clocks located at fixed values of R, which are not geodesics. These clocks will then necessarily be moving relative to the clocks determining T' in another reference system based on a new geodesic origin at some other galaxy.

In formulating a cosmological principle such as homogeneity of density of the cosmological fluid, we must measure this homogeneity relative to some time which should not be associated with any particular point in the Universe. Thus the time coordinate for describing homogeneity cannot be the curvature time coordinate T, because T is associated with some particular choice of geodesically moving galaxy as an origin. In formulating cosmological principles, therefore, we must necessarily change from T to some different time coordinate. The most natural way to proceed seems to be to measure time with clocks fixed in the galaxies forming the cosmological fluid. These clocks are not associated with any particular origin, and are intrinsically related to the source of the Universe.

We thus changed from (R,T) to (R,τ) coordinates, where τ is measured directly by clocks fixed in the cosmological fluid composed of geodesically moving galaxies. The cosmological principle of homogeneity was then formulated by assuming that at any given $\tau=$const time, the cosmological fluid density ρ was the same everywhere in the Universe, i.e., that ρ depends only on τ and is independent of R. This resulted in the set of equations (5.8) through (5.11) relating ρ, p, and the metric coefficient $A(R,\tau)$.

With (R,T) coordinates the reference system was composed of clocks located at fixed values of R, while with (R,τ) coordinates the reference system is formed from clocks moving relative to fixed values of R along geodesics. A consequence of using clocks moving relative to $R=$const points is that the corresponding metric form (5.9) is nondiagonal. However, both the R and τ coordinates of (5.9) have a very natural physical interpretation. We have already noted that τ is measured by clocks fixed in the cosmological geodesic fluid. Consider the subspace $\tau=$const. The metric on this subspace is, from (5.9),

$$ds^2(R,\tau=\text{ const})=dR^2+R^2d\Omega^2 \ , \tag{8.1}$$

which is seen to be flat. Thus the curvature coordinate R is the spatial coordinate that explicitly exhibits the flatness of the subspace $\tau=$const. Moreover, it follows from (8.1) that the curvature coordinate R has the physical significance that it is equal to the proper distance measured between τ-simultaneous events in the Universe. We have previously pointed out this significance of R for a Schwarzschild field.[2–6]

The coordinates (R,τ) give a new way of viewing the evolution of the Universe after the "big bang." The usual evolutionary picture is based upon expressing the metric for the Universe in isotropic coordinates. For example, the isotropic form of the ES universe can be written as

$$ds^2(r,\tau)=(\tau/\tau_n)^{4/3}(dr^2+r^2d\Omega^2)-d\tau^2 \ . \tag{8.2}$$

In terms of the comoving isotropic coordinates (r,τ), the galaxies are located at $r=$const points, so that their trajectories are vertical lines on an (r,τ) spacetime diagram, as shown in Fig. 5(a). The history of our galaxy is the line $r=0$, and the time τ_n is our present epoch measured from the "big bang" at $\tau=0$. Since the comoving coordinate r

remains constant along each galaxy, the galaxies are seen to be spread out infinitely far along the r axis at the "big bang." The light signal def from the "big bang" that is reaching us now at τ_n comes from the furthest galaxy we can see, described by the world line dh on Fig. 5(a). A galaxy represented by a world line such as ij will not be seen by us until some time in the future. Thus, at the present epoch τ_n we can see only a finite number of galaxies in the Universe. The boundary of the sphere defined by the coordinate radius fh is referred to as the spatial horizon at the present epoch.

The trajectories on the (r,τ) spacetime diagram of Fig. 5(a) are repeated in Fig. 5(b) in terms of (R,τ) coordinates. It is seen that when R is used as the spatial coordinate, all trajectories "explode" with infinite speeds from $R=0$ at the "big bang" at $\tau=0$. This is to be expected since, unlike r which measues only coordinate distance, R is a measure of proper distance relative to the cosmological time τ, and at the "big bang" there is zero separation between all galaxies. The relationship between the proper distance R and the coordinate r can be obtained by setting $d\tau^2=0$ in (8.2) to get

$$\int_0^r ds\,(R,\tau=\text{const})=R=(\tau/\tau_n)^{2/3}r\ . \tag{8.3}$$

The $\tau^{2/3}$ term in (8.3) has the effect of taking all the points along the r axis of Fig. 5(a) and moving them to the origin in Fig. 5(b). It thus follows that an incoming light signal such as df on the r coordinate plot of Fig. 5(a) will turn around on the R proper distance plot of Fig. 5(b).

When one sets $r=r_i=$const, which defines the trajectory of a galaxy, (8.3) gives

$$R=(r_i/\tau_n{}^{2/3})\tau^{2/3}=b\tau^{2/3}\ , \tag{8.4}$$

which is our previous expression (7.10). Since R is the proper distance measured at our present epoch τ_n, $dR/d\tau$ evaluated at τ_n will be equal to the actual velocity of an object measured by us now. From (8.4), the velocity of a galaxy is found to be

$$v=(dR/d\tau)\,|_{\tau_n}=(2/3\tau_n)R=HR,\quad H=2/3\tau_n\ , \tag{8.5}$$

which is the Hubble relationship showing that the velocity of a galaxy is proportional to its distance away from us.

From (8.2) it is seen that it is possible to diagonalize the metric form (5.9) by replacing R with a new comoving spatial coordinate that remains constant along the trajectory of each geodesic clock measuring the time coordinate τ. The form of the resulting diagonal metric will depend on how we define the comoving spatial coordinate, and will not necessarily be of the isotropic form (2.1). In fact, we have shown for a de Sitter universe that questionable mathematical maneuvers are required to cast the metric

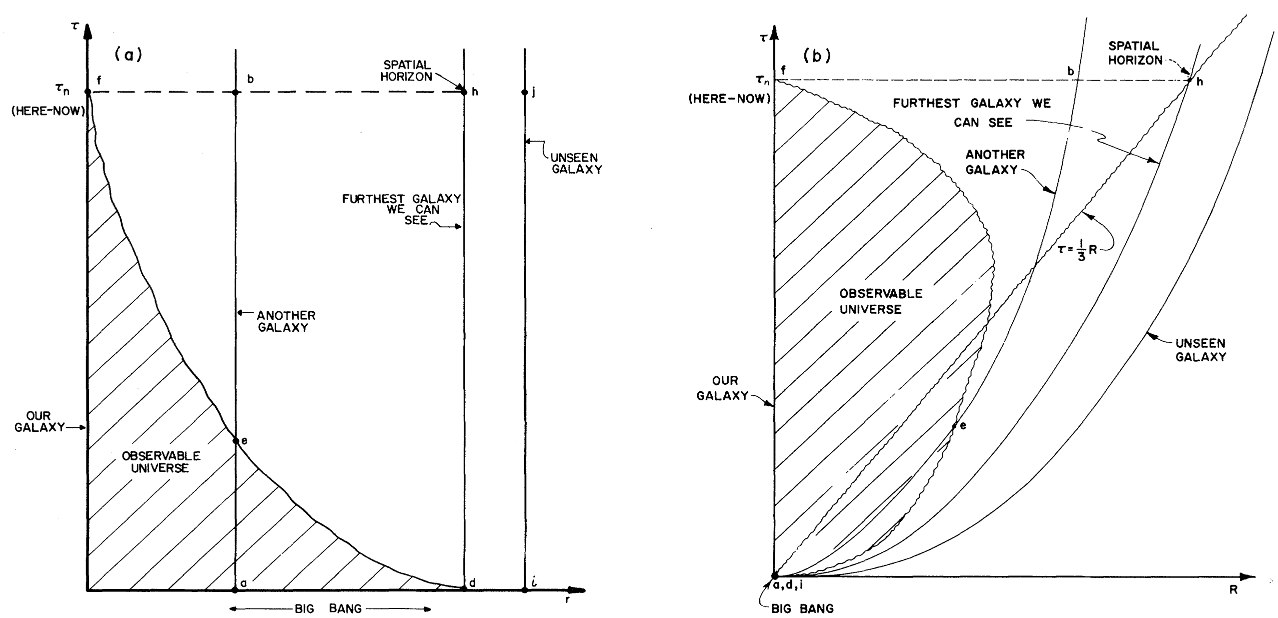

FIG. 5. (a) The evolution of the ES universe in terms of the comoving radial coordinate r. Since the comoving radial coordinate r used in the isotropic metric form (8.2) stays constant at the location of a galaxy, the trajectories of galaxies on this (r,τ) spacetime diagram are straight vertical lines. Consequently, the world lines of the galaxies intersect the r axis at different points at the "big bang" at $\tau=0$. The null trajectory def is the path of photons from galaxies that we, at $r=0$, see at the present epoch τ_n. The galaxy dh, whose light reaches us from the "big bang," is the furthest galaxy we can now see. The radius corresponding to fh is the boundary of our spatial horizon at the present epoch, and a galaxy such as ij beyond the horizon cannot be seen by us now. (b) The evolution of the ES universe in terms of the curvature radial coordinate R. This figure repeats the world lines of Fig. (a). At the "big bang" at $\tau=0$, all galaxies and light signals "explode" with infinite speeds from $R=0$. This happens because R measures proper or true distance, and at the "big bang" there is zero separation between all galaxies. Since the replacement of r with R via the transformation (8.3) moves all the points along the r axis on Fig. (a) to the origin of the (R,τ) coordinates, the incoming light signal def on Fig. (a) will have a turnaround point on this figure.

into the isotropic form (2.1),[7] so that the validity of using (2.1) for describing the de Sitter universe can be questioned.

From the method of approach of this paper, it seems that the only reason to replace R with a comoving spatial coordinate such as r is to produce a diagonal metric form. But the fact that we are used to seeing diagonal metric forms hardly seems a compelling reason for replacing R as the spatial coordinate. As we have shown above, R has the physical significance that it measures proper distance in the subspace τ=const, where τ is measured by clocks fixed in the geodesic fluid. Since there seems to be no good reason for doing otherwise, we will keep R as the spatial coordinate, and consider our development of cosmology in (R,T) and (R,τ) coordinates as complete.

Besides giving a new way of viewing cosmology, the methods developed here are extendable to areas where the standard approach based upon the isotropic metric form (2.1) is not applicable. For instance, the metric form (2.1) can not be used to deal with cosmological problems where inhomogeneities are present, whereas our approach can be applied to such problems. As an example, in the paper following this one, we show how to use the methods developed here to attack the inhomogeneous problem of describing the field of a Schwarzschild mass imbedded in a given Universe.

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APPENDIX

We show here that (3.10) with (3.11) is a solution of the geodesic equation (3.9) for radial motion. For radial motion we have

$$\delta V^\mu/\delta\tau=(\partial V^\mu/\partial R)V^1+(\partial V^\mu/\partial T)V^4+\Gamma^\mu_{\rho\sigma}V^\rho V^\sigma \ . \tag{A1}$$

Setting $\mu=1$ and substituting $\Gamma^1_{\rho\sigma}$ from (3.7), we obtain

$$\begin{aligned}\delta V^1/\delta\tau=&(\partial V^1/\partial R)V^1+(\partial V^1/\partial T)V^4\\&-(A_{,R}/2A)(V^1)^2-(A_{,T}/A)V^1V^4\\&+(\tfrac{1}{2}AA_{,R}f^2+A^2ff_{,R})(V^4)^2 \ .\end{aligned} \tag{A2}$$

Rearranging, and making use of (3.9), this becomes

$$\begin{aligned}\delta V^1/\delta\tau=&(\partial V^1/\partial R)V^1+(\partial V^1/\partial T)V^4+\tfrac{1}{2}A_{,R}\\&-(A_{,T}/A)V^1V^4+A^2ff_{,R}(V^4)^2 \ .\end{aligned} \tag{A3}$$

As an ansatz, assume (3.10) with (3.11) is a solution of (A3). From (3.10) we find

$$\partial V^1/\partial R=\frac{m}{2}(k^2-A)^{-1/2}(k^2-A)_{,R} \ , \tag{A4a}$$

$$\partial V^1/\partial T=\frac{m}{2}(k^2-A)^{-1/2}(k^2-A)_{,T} \ . \tag{A4b}$$

From (3.11a) we obtain

$$f_{,R}=\tau_{,TR}/k-(f/k)k_{,R} \tag{A5a}$$

and from (3.11b) we obtain

$$\begin{aligned}\tau_{,TR}=&-(m/2A)(k^2-A)^{-1/2}(k^2-A)_{,T}\\&+(m/A^2)(k^2-A)^{1/2}A_{,T} \ .\end{aligned} \tag{A5b}$$

Substituting (A5b) into (A5a), we find

$$\begin{aligned}f_{,R}=&-(m/2kA)(k^2-A)^{-1/2}(k^2-A)_{,T}\\&+(m/kA^2)(k^2-A)^{1/2}A_{,T}-(f/k)k_{,R} \ .\end{aligned} \tag{A6}$$

When (A4) and (A6) are substituted into (A3), we find

$$\delta V^1/\delta\tau\equiv 0 \ , \tag{A7}$$

thereby verifying that V^1 and V^4 given by (3.10) is a solution to the radial geodesic equations, provided that the condition (3.11) holds. It is sufficient to show (A7) for V^1 only, since V^4 can be obtained from (3.9). However, the above procedure can readily be repeated for $\mu=4$ to show directly $\delta V^4/\delta\tau=0$.

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[1]The development here follows very closely the treatment in J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1964), Ch. VII.

[2]R. Gautreau and B. Hoffmann, Phys. Rev. D 17, 2552 (1978) See also C. Ftaclas and J. M. Cohen, *ibid.* 21, 2103 (1980).

[3]R. Gautreau, Nuovo Cimento 56B, 49 (1980).

[4]R. Gautreau, in *After Einstein: Proceedings of the Einstein Centennial Celebration at Memphis State University, 1979,* edited by Peter Barker and Cecil G. Shugart (Memphis State University Press, Memphis, Tennessee, 1981).

[5]R. Gautreau, in *Proceedings of the Einstein Centennial Symposium on Fundamental Physics,* edited by S. M. Moore *et al.*, (Universidad de los Andes Press, Bogota, Columbia, 1981).

[6]R. Gautreau, report (unpublished).

[7]R. Gautreau, Phys. Rev. D 27, 764 (1983).

[8]P. A. M. Dirac, Proc. R. Soc. London A365, 19 (1979).