Symmetries of the Dirac equation

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(Received 12 September 1983)

The symmetry of the free Dirac equation recently discussed by Radford is regarded as a subgroup of a larger SL(2,C) group of transformations. The Radford Lagrangian is shown to be unsatisfactory for a field theory quantized according to Fermi-Dirac statistics, and is replaced by a suitable alternative. The symmetry of the Dirac equation, of the equal-time field anticommutation relations, and of the Lagrangian, is studied under SL(2,C) and various subgroups. In particular, it is found that the Radford subgroup is not a symmetry of the quantized field theory.

I. INTERNAL SYMMETRY OPERATIONS

Radford[1] has recently presented an interesting symmetry of the free Dirac equation. This symmetry is not new.[2] In this Brief Report we present the detailed results only sketched in Ref. 2. We regard the SU(1,1) symmetry group considered by Radford as a subgroup of a larger SL(2,C) group which mixes the two independent left-handed fields of the Dirac bispinor in the most general way possible. This enlarged group is a symmetry of the free Dirac equation for a zero-mass particle (i.e., a Dirac neutrino). However, we show that the canonical anticommutation rules do not share the full SL(2,C) symmetry of the Dirac equation, but are invariant only under the maximal unitary subgroup SU(2) of SL(2,C), which is just the Pauli group.[3] In particular, the SU(1,1) symmetry discussed by Radford is incompatible with the canonical anticommutation rules. We also show that the Lagrangian proposed by Radford is incompatible with canonical quantization according to Fermi-Dirac statistics, whereas a modified Lagrangian, which is compatible with canonical quantization, is invariant only under the Pauli SU(2) subgroup. Hence although the full SL(2,C) group is a symmetry of the Dirac equation for a neutrino, and the subgroup SU(1,1) is a symmetry of the Dirac equation for a neutral massive particle, neither group is a symmetry of the quantized field theory. We use the notation of Itzykson and Zuber.[4]

The Dirac field ψ can be written

$$\psi=\psi_L+\psi_R \ ,$$
$$\psi_L=\tfrac{1}{2}(1-\gamma_5)\psi \ , \tag{1}$$
$$\psi_R=\tfrac{1}{2}(1+\gamma_5)\psi \ .$$

The charge conjugate field ψ^c is defined by $\psi^c=C\bar{\psi}^T$. As is well known, $(\psi^c)_L=(\psi_R)^c$, $(\psi^c)_R=(\psi_L)^c$. We define two independent "left-handed" fields ψ_1 and ψ_2 by

$$\psi_1\equiv\psi_L=\tfrac{1}{2}(1-\gamma_5)\psi \ ,$$
$$\psi_2\equiv(\psi^c)_L=\tfrac{1}{2}(1-\gamma_5)\psi^c \ . \tag{2}$$

Then

$$\psi=\psi_1+\psi_2^c=\psi_1+C\bar{\psi}_2^T \ . \tag{3}$$

The Dirac equation for a particle of charge e and mass m in an external electromagnetic field A^μ is

$$(i\not\partial-e\not A-m)\psi=0,\quad (i\not\partial+e\not A-m)\psi^c=0 \ , \tag{4}$$

from which

$$(i\not\partial-e\not A)\psi_1=m\psi_2^c,\quad (i\not\partial+e\not A)\psi_2=m\psi_1^c \ . \tag{5}$$

These are equivalent to Eqs. (1) and (2) of Radford's paper. We now wish to consider transformations which arbitrarily mix ψ_1 and ψ_2. We are not interested in a mere scaling of Eqs. (5) and therefore we impose the condition that the transformation be unimodular. The group of all such transformations is SL(2,C). The group operations are conveniently expressed as

$$\Psi\rightarrow e^{-i\vec{\alpha}\cdot\vec{\tau}/2}\Psi,\quad \dot{\Psi}\rightarrow e^{-i\vec{\alpha}^*\cdot\vec{\tau}/2}\dot{\Psi} \ , \tag{6}$$

where

$$\Psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\quad \dot{\Psi}=\begin{pmatrix}-\psi_2^c\\ \psi_1^c\end{pmatrix} \ , \tag{7}$$

$$\vec{\tau}=\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\} \tag{8}$$

is a set of Pauli matrices acting on the space of $\{\psi_1,\psi_2\}$, and $\vec{\alpha}$ is an arbitrary complex three-component "vector." The notation $\Psi,\dot{\Psi}$, reminiscent of Van der Waerden's undotted and dotted spinors, is adapted from Pursey.[5] Equations (5) can be written as

$$(i\not\partial-e\tau_3\not A)\Psi+m\tau_3\dot{\Psi}=0 \ . \tag{9}$$

The transformations (6) are locally isomorphic to Lorentz transformations[6] in a fictitious "space-time," which is related to physical space-time only by mathematical analogy. Following Refs. 5 and 6, and using notations adapted from Ref. 5, we define τ^μ, $\dot{\tau}^\mu$, $\mu=0,1,2,3$, by

$$\tau^\mu=(1,\vec{\tau}),\quad \dot{\tau}^\mu=(1,-\vec{\tau}) \ . \tag{10}$$

The generators of infinitesimal "Lorentz transformations" acting on $\Psi, \dot{\Psi}$ are $S^{\mu\nu}, \dot{S}^{\mu\nu}$, respectively, where

$$S^{\mu\nu} = \tfrac{1}{4} i (\dot{\tau}^\mu \tau^\nu - \dot{\tau}^\nu \tau^\mu), \quad \dot{S}^{\mu\nu} = \tfrac{1}{4} i (\tau^\mu \dot{\tau}^\nu - \tau^\nu \dot{\tau}^\mu) \quad . \tag{11}$$

Corresponding to a four-vector m^μ, we can define matrices $M, \dot{M}$ by

$$M = m_\mu \tau^\mu, \quad \dot{M} = m_\mu \dot{\tau}^\mu \quad . \tag{12}$$

A "Lorentz transformation" of m^μ is equivalent to[5,6]

$$M \rightarrow e^{-i \vec{\alpha}^* \cdot \vec{\tau}/2} M e^{-i \vec{\alpha} \cdot \vec{\tau}/2} \;, \qquad \dot{M} \rightarrow e^{-i \vec{\alpha} \cdot \vec{\tau}/2} \dot{M} e^{i \vec{\alpha}^* \cdot \vec{\tau}/2} \;. \tag{13}$$

Hence Eq. (9) can be written as

$$(i \not{\partial} - e_{\mu\nu} S^{\mu\nu} \not{A}) \Psi + m_\mu \dot{\tau}^\mu \dot{\Psi} = 0 \quad , \tag{14}$$

where

$$(e^{01}, e^{02}, e^{03}, e^{23}, e^{31}, e^{12}) = (0,0,0,0,0,e) \;, \qquad m^\mu = (0,0,0,m) \;. \tag{15}$$

The general form of Eq. (14) is uniquely determined by Eq. (9) and the requirement that Eq. (14) be formally covariant under SL$(2,C)$, i.e., that the effect of an SL$(2,C)$ transformation of $\Psi, \dot{\Psi}$ can be exactly compensated by a "Lorentz transformation" of the c-number "tensor" $e^{\mu\nu}$ and "four-vector" m^μ. In particular, it is not possible to interpret the charge as a "four-vector" or the mass as a "second-rank skew tensor." The particular values for $e^{\mu\nu}, m^\mu$ given by Eq. (15) are uniquely determined by the conditions that Eq. (14) is identically the same as Eq. (9) and that $e^{\mu\nu}$ is real. (The second of these conditions may be relaxed, leading to a one-parameter ambiguity in $e^{\mu\nu}$, without significantly changing the conclusions of this paper.) Of course, if Ψ and $\dot{\Psi}$ are replaced by functions transformed according to Eq. (6), then $e^{\mu\nu}$ and m^μ must be replaced by the corresponding "Lorentz-transformed" quantities in order that the transformed Eq. (14) be identical to Eq. (9). This freedom is analogous to the "form invariance" exploited in Refs. 3 and 7.

We now draw attention to various subgroups of SL$(2,C)$.

(i) The SU(1,1) subgroup. The subgroup which leaves m^μ unaltered is SU(1,1), the universal covering group of R(2,1). If $e=0$, this is an exact symmetry of the Dirac equation, Eq. (14) or Eq. (4), for a neutral particle. This is the symmetry discussed by Radford.[1]

(ii) The Pauli subgroup. The maximal compact subgroup of SL$(2,C)$ is SU(2), defined by $\vec{\alpha}$ real in Eq. (6). This is a symmetry of Eq. (14) if $e=m=0$, i.e., for a (massless) Dirac neutrino. This group was first discovered by Pauli.[3]

(iii) The QED gauge group. The only subgroup of SL$(2,C)$ which leaves Eq. (14) invariant when $e \neq 0$, $m \neq 0$, is the U(1) subgroup generated by $S^{12} = \frac{1}{2}\tau_3$. This is the group of (global) gauge transformations of the Dirac field. It is also the maximal compact subgroup of the SU(1,1) group.[1]

(iv) A new symmetry. If $e \neq 0$ but $m=0$, then Eq. (14) is invariant under the two-parameter group locally isomorphic with rotations about and boosts along the third spacelike axis of the fictitious "space-time." This corresponds to $\vec{\alpha} = (0,0,\alpha)$ in Eq. (6) with α an arbitrary complex number. The restriction to real α corresponds to the QED gauge group, *(iii)* above.

II. CANONICAL ANTICOMMUTATION RELATIONS

The equal-time anticommutation relations for the quantized Dirac field are

$$\{\psi_\alpha(\vec{r},t), \psi^\dagger_\beta(\vec{r}\,',t)\} = \delta_{\alpha\beta}\delta^3(\vec{r} - \vec{r}\,') \quad . \tag{16}$$

These relations must be preserved by a unitary transformation of the quantized field theory. In terms of $\psi_1 \equiv \psi_L$ and $\psi_2 \equiv (\psi^c)_L$, Eq. (16) becomes

$$\{\psi_{j\alpha}(\vec{r},t), \psi^\dagger_{k\beta}(\vec{r}\,',t)\} = \delta_{jk}\tfrac{1}{2}(1-\gamma_5)_{\alpha\beta}\delta^3(\vec{r} - \vec{r}\,'), \qquad j,k = 1,2 \;. \tag{17}$$

From this it is apparent that the equal-time anticommutator of the fields is invariant under the transformations (6) only if $\vec{\alpha}$ is real, i.e., only for elements of the maximal compact subgroup SU(2), namely, the Pauli group. Hence, *only elements of the Pauli group can be produced by a unitary transformation of the theory.* In particular, *only the compact U(1) subgroup* (the QED gauge group) *of the SU(1,1) symmetry group considerd by Radford can correspond to a unitary transformation of the theory.* This is in contrast to real Lorentz transformations in physical space-time, which do correspond to unitary transformations of the quantized field theory even for the noncompact "boost" operations. This difference of behavior can be traced to the fact that physical Lorentz transformations act on the coordinates x^μ of the field point as well as the field components, whereas the transformations (6) leave space-time coordinates unaltered.

III. LAGRANGIAN FORMULATION

The Lagrangian density proposed by Radford[1] is equivalent to

$$L = i(\bar{\psi}_1 \not{\partial} \psi_1 - \bar{\psi}_2 \not{\partial} \psi_2) - m(\psi_1^T C^{-1} \psi_2 + \bar{\psi}_1 C \bar{\psi}_2^T) \quad . \tag{18}$$

This Lagrangian leads to the Eqs. (5) (with $e=0$) *only if the fields commute rather than anticommute.* Likewise, *the mass term of Eq. (18) is invariant under the full SU(1,1) group only if the fields commute rather than anticommute.* [These same assertions may be verified directly using Radford's Eq. (2). There it must be remembered that $u^A u_A = 0$ only if the field components commute.] We conclude that Radford's Lagrangian is unsuitable for use in quantized field theory.

In order to find a suitable Lagrangian, we start from the conventional Lagrangian density

$$L = \bar{\psi}(i\not{\partial} - m)\psi = i(\bar{\psi}_L \not{\partial} \psi_L + \psi_R \not{\partial} \psi_R) - m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \;, \tag{19}$$

or, better,[8]

$$\begin{aligned} L &= \tfrac{1}{2}\bar{\psi}(i\not{\partial} - m)\psi + \tfrac{1}{2}\psi^T(i\not{\partial}^T + m)\bar{\psi}^T \\ &= \tfrac{1}{2} i (\bar{\psi}_L \not{\partial} \psi_L + \psi_L^T \not{\partial} \bar{\psi}_L^T + \bar{\psi}_R \not{\partial} \psi_R + \psi_R^T \not{\partial}^T \bar{\psi}_R^T) \\ &\quad - \tfrac{1}{2} m (\bar{\psi}_L \psi_R - \psi_R^T \bar{\psi}_L^T + \bar{\psi}_R \psi_L - \psi_L^T \bar{\psi}_R^T) \;. \end{aligned} \tag{20}$$

Since $\psi_2 = (\psi^c)_L = (\psi_R)^c = C\bar{\psi}_R^T$, we have

$$\bar{\psi}_R^T = C^{-1}\psi_2, \quad \bar{\psi}_R = \psi_2^T C^{T-1} = -\psi_2^T C^{-1} \;, \qquad \psi_R = -\beta C \psi_2^* = C\bar{\psi}_2^T = \psi_2^c \;. \tag{21}$$

Hence, from Eq. (20),

$$L = \tfrac{1}{2} i (\bar{\psi}_1 \partial\!\!\!/ \psi_1 + \psi_1^T \partial\!\!\!/^T \bar{\psi}_1^T + \psi_2^T \partial\!\!\!/^T \bar{\psi}_2^T + \bar{\psi}_2 \partial\!\!\!/ \psi_2)$$
$$- \tfrac{1}{2} m (\bar{\psi}_1 C \bar{\psi}_2^T + \bar{\psi}_2 C \bar{\psi}_1^T - \psi_2^T C^{-1} \psi_1 - \psi_1^T C^{-1} \psi_2) \ . \qquad (22)$$

In Eq. (22), derivatives always act to the right, and $\partial\!\!\!/^T \equiv \gamma_\mu^T \partial^\mu$. If the fields anticommute, the Lagrangian density given by Eq. (22) differs from that of Eq. (19) only by a divergence. If the fields commute, then the Lagrangians given by Eqs. (19) and (18) differ only by a divergence. It is easily verified that when one considers infinitesimal variations $\delta\psi_j$ which anticommute with the fields, then the Lagrangian density of Eq. (22) does indeed yield the field equations (5).

We now investigate the invariance of L given by Eq. (22) under the transformations Eq. (6). It is immediately clear that the kinetic part of L is not invariant under the full SL(2,C) group or even under the SU(1,1) subgroup. Instead, the kinetic part of L is invariant only under the Pauli SU(2) subgroup. It is also readily seen that the mass term of L is invariant only under the QED (global) gauge transformations. A more careful investigation shows that the kinetic energy term of the Lagrangian transforms like the "time" component of a four-vector in the fictitious "space-time" when the fields Ψ and $\dot{\Psi}$ undergo an SL(2,C) transformation as in Eq. (13). This is true also for the full Lagrangian, including the mass terms and interaction terms with an external electromagnetic field, provided the c-number parameters m^μ and $e^{\mu\nu}$ are also "Lorentz transformed." Thus the correct Lagrangian is not even form invariant (in the sense of Refs. 3 and 7) under SL(2,C) or any noncompact subgroup of SL(2,C).

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[1] Chris Radford, Phys. Rev. D 27, 1970 (1983).

[2] The SU(1,1) symmetry discussed by Radford was already known to one of us (J.F.P.) in the 1950s, but was not published. The treatment of SU(1,1) as a subgroup of SL(2,C) was reported in J. F. Plebański and D. L. Pursey, Bull. Am. Phys. Soc. 5, 81 (1960). See also Feza Gürsey, Nuovo Cimento 8, 411 (1958).

[3] W. Pauli, Nuovo Cimento 6, 204 (1957).

[4] Claude Itzykson and Jean-Bernard Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[5] D. L. Pursey, in *Encyclopedic Dictionary of Physics* (Pergamon, Oxford, 1962), Vol. 6, p. 777.

[6] A. J. Macfarlane, J. Math. Phys. 3, 1116 (1962).

[7] D. L. Pursey, Nuovo Cimento 6, 266 (1957); S. Kahana and D. L. Pursey, *ibid.* 6, 1469 (1957); C. P. Enz, *ibid.* 6, 250 (1957); G. Lüders, *ibid.* 7, 171 (1958); N. Kemmer, J. C. Polkinghorne, and D. L. Pursey, Rep. Prog. Phys. 22, 368 (1959).

[8] This is a variant of the Lagrangian density given in Ref. 4.