PHYSICAL REVIEW D VOLUME 29, NUMBER 8 15 APRIL 1984

Arbitrary initial conditions of nonlocal hidden variables

Edward R. Floyd
Naval Ocean System Center, San Diego, California 92152
(Received 2 December 1983)

A deterministic nonlocal hidden-variable theory with arbitrary initial conditions is consistent with quantum theory. The sufficiency conditions for the hidden variables are developed.

While the recent observations of Aspect and co-workers[1,2] are very strongly consistent with quantum theory and offer compelling evidence to dismiss local hidden-variable theory, these experiments do not bear upon nonlocal hidden-variable theory. Nonlocal hidden variables such as Bohm's 1951 theory[3] are also consistent with quantum mechanics for equilibrium conditions. Keller had noted the important role that probability and equilibrium have in Bohm's 1951 theory.[4] Bohm subsequently showed that random collisions would drive even an initial arbitrary nonequilibrium probability density to an equilibrium probability density consistent with quantum theory.[5] Some workers have concluded that nonlocal hidden-variable theory would deviate from quantum theory for tests that are sufficiently fast to measure motion still in a nonequilibrium condition before the hidden variables could relax to their equilibrium distributions from their arbitrary initial nonequilibrium distributions.[3,5-7]

Herein we show that, although quantum mechanics is incomplete (the Schrödinger wave function is not exhaustive as it has microstates[8]), faster or even instantaneous tests on a set of nonlocal hidden variables with arbitrary initial conditions would still yield observations consistent with the predictions of the Schrödinger equation and that probability need not be evoked for a nonlocal hidden-variable theory. We also develop the set of nonlocal hidden variables whose initial conditions are sufficient to determine continuous quantum motion. For this exposition, one dimension suffices.

The Hamilton-Jacobi equation for continuous quantum motion is given in one dimension, x, for stationary states by[8]

$$\frac{1}{2\mu}\left(\frac{\partial S}{\partial x}\right)^2 + U(x,\partial S/\partial t) + \frac{\partial S}{\partial t} = 0 \ , \tag{1}$$

where $S(x,E,t)$ is Hamilton's principal function for continuous quantum motion, t is time, E is the Hamilton-Jacobi transformed momentum and a constant of the motion, μ is the mass, and U is a modified potential which is the quantum-mechanical continuation[9] of the classical potential that exactly accounts for the nonlocal wave nature of the finite test charge needed to establish the field for a force rather than the local infinitesimal point charge assumed for the classical field. We note that the relationship between the modified potential U and Bohm's quantum-mechanical potential $\mathcal{U}_b$ is given for and only for $U < E$ by $U = V + \mathcal{U}_b$, where $V(x)$ is the classical potential.[8] For stationary states, Halpern had noted[10] that the constants of motion for Hamilton's principal function for continuous quantum motion are not nontrivial integration constants. This is manifested by U being an explicit function of $\partial S/\partial t$. Nevertheless, for stationarity we may still choose $S = W(x,E) + Et$, where W is Hamilton's characteristic function. The constant of the motion E may now be identified as energy. Then Eq. (1) may be expressed as

$$\frac{1}{2\mu}\left(\frac{\partial W}{\partial x}\right)^2 + U(x,E) - E = 0 \ , \tag{1'}$$

where U is explicitly energy dependent. In order to determine a unique trajectory for a particle, $x(t)$, Eq. (1) or (1') must be supplemented by the initial conditions and U must be described. Keller[4] had given the initial conditions for Eq. (1) explicitly and Eq. (1') by extrapolation as

$$x(0) = x_0 \ , \tag{2}$$

$$S(x,E,0) = S_0(x,E) = W_0(x,E) \ , \tag{3}$$

where S and W contain an arbitrary integration constant, and

$$P(x,E,0) = P_0(x,E) \ , \tag{4}$$

where P, which is the amplitude factor for Bohm's ansatz for the wave function [i.e., $P\exp(iS)$], has been identified by Keller[4] to be the "quantum-mechanical field" and by Bohm[3] to be the probability density.

Keller deduced first that P obeyed the equation of continuity of probability and consequently that if P_0, which represents the real quantum-mechanical field of some particular experiment, accidently represented the initial distribution of x for some ensemble of experiments, then only in this fortuitous case would Bohm's theory give probabilities consistent with quantum theory.[4] In response Bohm evoked many random interactions of large assemblies from the theory of measurements that would rapidly and irreversibly drive P toward an equilibrium distribution.[5] Since the initial conditions for the hidden variables of a particular experiment were in general arbitrary and different from those of an ensemble of experiments, nonlocal hidden-variable theory was believed to differ from quantum theory for the transition period necessary to establish equilibrium conditions of the hidden variables.[3,5-7] Furthermore, Bohm speculated that for small domains of the order 10^{-13} cm present theories may not be adequate.[3,5]

However, in Bohm's and Keller's deduction it was assumed that the conjugate momentum $\partial S/\partial x$ was equivalent to the mechanical momentum (i.e., $\mu\dot{x}$) which is not so[8] in general. As a result, P does not obey the equation of continuity of probability. This casts doubt on any conclusions that nonlocal hidden-variable theory differs from quantum theory for nonequilibrium conditions.

Keller also reported that while the classical trajectory is uniquely determined by knowing $\partial S/\partial x$ at $x = x_0$, for the quantum trajectory S_0 must be known for all x.[4] The partic-

ular trajectory specifies the particular microstate of the eigenfunction for the particle.[8]

Previously the set of hidden variables $[x_0,\dot{x}_0,\ddot{x}_0]$, whose initial values are necessary to specify a unique quantum trajectory, have already been determined.[8] We now proceed to show that this set is also sufficient and then to resolve whether nonlocal hidden-variable theory differs from quantum theory for nonequilibrium conditions.

The modified potential U is determined by[8]

$$U+\frac{\hbar^2}{8\mu}\frac{\partial^2 U/\partial x^2}{E-U}+\frac{5\hbar^2}{32\mu}\left(\frac{\partial U/\partial x}{E-U}\right)^2=V \ , \tag{5}$$

where $\hbar$ is Planck's constant. From Eqs. (1′) and (5) we may eliminate U to develop an alternative form of the Hamilton-Jacobi equation for continuous quantum motion expressed by

$$\frac{(\partial W/\partial x)^2}{2\mu}+V-E=-\frac{\hbar^2}{4\mu}\frac{\partial^3 W/\partial x^3}{\partial W/\partial x}+\frac{3\hbar^2}{8\mu}\left(\frac{\partial^2 W/\partial x^2}{\partial W/\partial x}\right)^2 \ . \tag{6}$$

In Eq. (6), $\partial W/\partial x$ remains real. Equation (6) has a nodal singularity where $\partial W/\partial x \rightarrow 0$ as $x \rightarrow \pm\infty$. The second and third derivative terms on the right-hand side of Eq. (6), which contain the nodal singularity, manifest the quantum effect while the terms on the left-hand side of Eq. (6) are the terms for classical motion. Thus the quantum effect raises the Hamilton-Jacobi equation from a first-order differential equation to a third-order differential equation. With this alternate form, some of the confusion about the initial conditions may now be resolved. As Eq. (6) is third order, the sufficient set of initial conditions for determining a unique solution becomes the following:

$$x_0=x(0) \ , \tag{2}$$

$$W_0(E)=W(x_0,E) \ , \tag{7}$$

where W contains an arbitrary integration constant,

$$\frac{\partial W_0(E)}{\partial x}=\left.\frac{\partial W(x,E)}{\partial x}\right|_{x=x_0} \ , \tag{8}$$

and

$$\frac{\partial^2 W_0(E)}{\partial x^2}=\left.\frac{\partial^2 W(x,E)}{\partial x^2}\right|_{x=x_0} \ , \tag{9}$$

where Eq. (2) still stands but the initial conditions specified by Eqs. (3) and (4) are replaced by those of Eqs. (7) through (9). This relaxes Keller's requirement that Eq. (3) be known everywhere. However, Eq. (9) represents the additional initial condition nescessary to describe quantum motion vis-à-vis classical motion. These initial conditions with a given energy specify the unique trajectories or, for bound states, orbits that are microstates of the wave function.[8] As such, these initial conditions of hidden variables form a sufficient set to specify continuous quantum motion.

The initial conditions for the set of alternate hidden variables $[x_0=x(0)$, $\dot{x}_0=\dot{x}(0)$, and $\ddot{x}_0=\ddot{x}(0)]$ may be derived from the equations for continuous quantum motion[8]

$$\dot{x}=\frac{\partial W/\partial x}{\mu(1-\partial U/\partial E)} \tag{10}$$

and

$$\ddot{x}=\frac{-\partial U/\partial x}{\mu(1-\partial U/\partial E)}+\frac{2(E-U)}{\mu(1-\partial U/\partial E)^3}\frac{\partial^2 U}{\partial x\partial E} \ .$$

Hence for given energy the set $[x_0,\dot{x}_0,\ddot{x}_0]$ is not only nescessary[8] but sufficient to specify unique quantum motion. We note again that Eq. (10) manifests that the conjugate momentum is not the mechanical momentum (i.e., $\partial S/\partial x \neq \mu\dot{x}$).[8] With this set of initial conditions, x_0 and $\partial S/\partial x$ at $x=x_0$ must be consistent in accordance with Eq. (10). This had been a point of concern for previous workers because the velocity at an initial point must be established by Bohm's quantum-mechanical potential.[3,10] Now we see that a modified potential may always be determined from the set of initial values of the hidden variables which includes the initial velocity as a member. Thus "the tables are turned": the set of initial values of the hidden variables determines the particular form of the modified potential.

For given energy the set $[x_0,\dot{x}_0,\ddot{x}_0]$, which is sufficient to determine a particular microstate among the set of microstates for the Schrödinger wave function represents for statistical purposes a δ-function distribution of the initial values as shown elsewhere.[8] Thus a deterministic theory may be obtained from quantum theory by assuming a δ-function distribution for the hidden variables which overcomes Keller's objections[4] on this aspect of Bohm's hidden-variable theory.

A conclusion that is the converse to the above is now given. For given energy the arbitrary set of initial conditions $[x_0,\dot{x}_0,\ddot{x}_0]$ is sufficient to determine a unique microstate of the Schrödinger wave function and its associated particular modified potential. The Schrödinger wave function may be recovered immediately from any one of the particular modified potentials by itself as shown elsewhere.[8] Accordingly there is no need to postulate an assembly of many microstates of arbitrary initial conditions which must interact randomly in order to achieve irreversibly in an appropriate relaxation period a statistical equilibrium that is consistent with the probability density of quantum mechanics.[3,5-7] The individual deterministic microstates of an energy eigenfunction are always consistent with the probability densities of quantum mechanics for that energy eigenvalue regardless of the particular values of the initial conditions and there is no need to bestow upon probability a role inherent to nonlocal hidden-variable theory. Furthermore, no test can be devised to show any difference at any time between a deterministic nonlocal hidden-variable theory for an arbitrary set of initial conditions and quantum mechanics. And finally, although quantum theory is not exhaustive, its statistical predictions remain consistent with arbitrary initial conditions and Bohm's speculation[3,5] for modifying the Schrödinger equation to cover small domains (i.e., 10^{-13} cm) is unnecessary.

Added note. Equation (6) also shows that the constant of the motion E for a particle of mass μ may be deduced in one dimension if somehow $V(x)$, $\partial W/\partial x$, $\partial^2 W/\partial x^2$, and $\partial^3 W/\partial x^3$ could all be evaluated at the same arbitrary point $x=x_0$. This is in contrast with classical mechanics where for time-independent conservative systems E is a first integral of the motion and may be deduced by only evaluating the potential V and the conjugate momentum $\partial W/\partial x$ both at the same arbitrary point.

[1]A. Aspect, P. Grangier, and G. Roger, Phys. Rev. Lett. 49, 91 (1982).
[2]A. Aspect, J. Delibard, and G. Roger, Phys. Rev. Lett. 49, 1804 (1982).
[3]D. Bohm, Phys. Rev. 85, 166 (1952).
[4]J. B. Keller, Phys. Rev. 89, 1040 (1953).
[5]D. Bohm, Phys. Rev. 89, 458 (1953).
[6]D. J. Bohm and B. J. Hiley, Found. Phys. 12, 1001 (1982).
[7]F. J. Belinfante, *A Survey of Hidden Variable Theories* (Pergamon, New York, 1973), pp. 12–16.
[8]E. R. Floyd, Phys. Rev. D 26, 1339 (1982).
[9]E. R. Floyd, Phys. Rev. D 25, 1547 (1982).
[10]O. Halpern, Phys. Rev. 87, 389 (1952).