PHYSICAL REVIEW D VOLUME 29, NUMBER 8 15 APRIL 1984

Color confinement and the quantum-chromodynamic vacuum

Arthur R. Swift and Jose Luis Rodriguez Marrero
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003
(Received 22 July 1983)

The bound-state problem in QCD is studied in the modified Tamm-Dancoff formalism. The confining interaction is identified in momentum space as the vacuum expectation value of the instantaneous Coulomb interaction in the Coulomb gauge. If this vacuum expectation value is infrared singular, then color is automatically confined. All colored states have infinite energy. For color-singlet quark-antiquark and two-gluon states, the infrared singularity cancels out. The resulting finite-energy bound-state equations produce an energy spectrum comparable to that produced by a linear potential in the nonrelativistic Schrödinger equation.

I. INTRODUCTION

Now that quantum chromodynamics (QCD) has been accepted as the theory of strong interactions, there remains the problem of learning how to make meaningful calculations. Two aspects of QCD have received particular attention. On one hand there is the effort to calculate high-momentum processes where the theory becomes asymptotically free and perturbation theory is expected to apply.[1] At the other extreme there are the attempts to understand the hadronic particle spectrum. The fundamental fields of QCD are quarks and gluons, each of which carries a color charge. The dominant feature of the particle spectrum is the fact that all physical states are color singlet. Free quarks and gluons do not exist. The absence of colored states is summarized by the statement that quarks and gluons are confined. This paper is an attempt to formulate a theory for the confinement process within the framework of QCD.

The lattice formulation of QCD has come the closest to a first-principles calculation of the hadronic spectrum.[2] However, there is still a need for an analytic, as opposed to numerical, understanding of the problem. The physical picture based on two QCD vacuum phases is very attractive.[3] A perturbative vacuum exists in regions of high field strength. As a result, quarks and gluons appear as nearly free particles when probed at short distances. When the fields drop below a critical strength, there is a transition to the true vacuum phase which expels color flux. The analogy with the behavior of a superconductor is obvious. Color flux is confined into tubes of finite cross section. As the sources of the color flux are separated to infinity, an infinite amount of energy is stored in the flux tube. Although the details of this picture may not be correct, the basic idea is that confinement is a consequence of the structure of the QCD vacuum. There have been a number of attempts to model the vacuum.[4] What is often lacking is a prescription that connects the properties of the vacuum to the hadronic spectrum.

In this paper we start with the QCD Hamiltonian and develop a plausible, quantitative connection between the vacuum expectation value of a particular operator and the confining potential in a set of relativistic bound-state equations. The fact of confinement can be turned around and used as a probe of the structure of the QCD vacuum. The starting point is the Tamm-Dancoff[5] method as modified by Dyson[6] to incorporate relativistic effects.[7] The Tamm-Dancoff wave functions are the expectation values of normal-ordered products of free-particle operators between the true ground state and an excited state. The effects of interactions are introduced through equal-time commutators of the normal-ordered operator products with the interaction Hamiltonian. The Coulomb gauge is used to take advantage of the fact that only the physical degrees of freedom are quantized, and there are no residual gauge conditions to be imposed on states.[8] In addition, in the Coulomb gauge the interaction between color charge densities is modified by vacuum effects in a well-defined manner. The vacuum expectation value of the modified Coulomb interaction is identified as the source of confinement. Building on an earlier, more *ad hoc* model, we assume that this vacuum expectation value is infrared singular.[9] As a function of momentum transfer, it diverges as $q^2 \to 0$ faster than $(q^2)^{-3/2}$. The Tamm-Dancoff equations in momentum space are truncated to the two-particle sector. The modified Coulomb interaction contributes both to self-energy terms and to scattering terms in the bound-state equations. When the quark-antiquark pair is in a color-singlet state, the infrared singularity cancels between the two terms and leads to a set of bound-state equations with finite energy. For nonsinglet states of one or two quarks, the energy is infinite due to the infrared singularity. Gluons exhibit the same cancellation phenomenon in color-singlet states. An immediate prediction of this model is that colored states cannot exist.

In the next section, after a brief discussion of the Tamm-Dancoff method, it is applied to the quark-antiquark system. The Coulomb-gauge Hamiltonian is written explicitly in momentum space, and the Coulomb term is identified as the source of the confining interaction. There are four two-particle amplitudes that are coupled together in a set of integral equations. After demonstration of the infrared cancellation, two of the amplitudes are absorbed into higher-order corrections that do not affect the confinement process. The final analysis is carried out in the center-of-momentum frame, although the validity of the equations is more general. Each approximation

is identified and can, in principle, be tested for self-consistency once solutions of the final equations are available. The third section is devoted to the study of gluon bound states. Each step in the discussion of quark states has an analog in the analysis of two-gluon states. The final result is a finite-energy bound-state equation for massless gluons in a color-singlet state. The fourth section is concerned with verifying that the bound-state equations do indeed produce a discrete spectrum of bound state for reasonable values of the QCD coupling constant and the scale parameter that enters through the vacuum expectation value of the Coulomb interaction. The Appendix deals with the details of reducing the quark and gluon equations to a tractable form through the use of an angular momentum expansion of appropriate scalar, vector, and tensor amplitudes.

II. BOUND-STATE EQUATIONS FOR QUARKS

The proper description of quark bound states in QCD requires a set of equations which are derived from field theory and which account for relativistic effects. Moreover, since it is impossible to solve the theory exactly, the equations must lend themselves to a set of approximations which can be justified. The Bethe-Salpeter equation has long been favored because it is manifestly covariant.[9] However, the price of covariance is a relative time coordinate which complicates the analysis and is usually set equal to zero. The Bethe-Salpeter equation is tractable only in the ladder approximation, an approximation which is motivated more by the requirement of solubility than the physics of the problem.

Here we use the Tamm-Dancoff method[5] as modified by Dyson[6] to remove the vacuum energy and to include the effects of annihilation and antiparticles. If $|\Psi_0\rangle$ is the physical ground state and $|\Psi\rangle$ is an excited state of energy $\bar{E}$, then

$$(H_0+H_I)|\Psi_0\rangle=E_0|\Psi_0\rangle \tag{1}$$

and

$$(H_0+H_I)|\Psi\rangle=\bar{E}|\Psi\rangle\ . \tag{2}$$

The free part of the Hamiltonian describes noninteracting fields. The Tamm-Dancoff wave functions are defined by

$$\Psi_{NN'}=\langle\Psi_0|C(N')A(N)|\Psi\rangle\ , \tag{3}$$

where $C(N')$ is a product of creation operators specified by N' and $A(N)$ is a product of annihilation operators specified by N. The physical energy, $E=\bar{E}-E_0$, is calculated from

$$E\Psi_{NN'}=\langle\Psi_0|[C(N')A(N),H_0+H_I]|\Psi\rangle\ . \tag{4}$$

The commutator of $C(N')A(N)$ with H_0 is easily evaluated; it is just the sum of the free-particle energies of the annihilated particles minus the sum of the free-particle energies of the created particles. Thus, (4) becomes

$$\left[E-\sum^{N}w(k_i)+\sum^{N'}w(k_i')\right]\Psi_{NN'}=\langle\Psi_0|[C(N')A(N),H_I]|\Psi\rangle\ . \tag{5}$$

When the commutator on the right-hand side (RHS) is expanded in a sum of terms with creation operators on the left-hand side (LHS), the result is an infinite set of coupled integral equations for the amplitudes $\Psi_{NN'}$. In practice, the equations are truncated to produce a finite set of time-independent bound-state equations.[10]

To apply these equations to quarks, we start with the QCD Hamiltonian in the Coulomb gauge in momentum space.[8] The Coulomb gauge is used because only the physical degrees of freedom are quantized; there are no residual gauge conditions to be imposed on the states. Moreover, in the Coulomb gauge, there is an explicit mechanism by which the Coulomb interaction of ordinary QED is modified to produce a confining interaction. In addition the Coulomb interaction has the spinor structure that produced confinement in earlier work.[9]

The free QCD Hamiltonian is

$$H_0=\int d^3p\,w(\vec{p})\sum_r[b_r^\dagger(\vec{p})b_r(\vec{p})-d_r(\vec{p})d_r^\dagger(\vec{p})]+\tfrac{1}{2}\int d^3p\,|\vec{p}|\sum_\lambda[a_\lambda^\dagger(\vec{p})a_\lambda(\vec{p})+a_\lambda(\vec{p})a_\lambda^\dagger(\vec{p})]\ . \tag{6}$$

Color indices are suppressed. The quark energy is $w(\vec{p})=(p^2+m^2)^{1/2}$ and $p=|\vec{p}|$ is the gluon energy. The operator $b_{ri}^\dagger(\vec{p})$ $[d_{ri}^\dagger(\vec{p})]$ creates a quark [antiquark] of momentum $\vec{p}$, color index i, and spin projection r; $a_\lambda^{e\dagger}(\vec{p})$ creates a transverse gluon of momentum $\vec{p}$, polarization λ, and color index e. The interaction part of the Hamiltonian is

$$H_I=\frac{ig}{[8(2\pi)^3]^{1/2}}\int\frac{d^3p_1d^3p_2d^3p_3}{(p_1p_2p_3)^{1/2}}\delta^3(\vec{p}_1+\vec{p}_2+\vec{p}_3)f_{abc}\vec{A}_+^{\,a}(\vec{p}_1)\cdot\vec{A}_+^{\,c}(\vec{p}_3)\vec{p}_1\cdot\vec{A}_+^{\,b}(\vec{p}_2)$$
$$+\frac{g^2}{16(2\pi)^3}\int\frac{d^3p_1d^3p_2d^3p_3d^3p_4}{(p_1p_2p_3p_4)^{1/2}}\delta^3(\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4)f_{abc}f_{ade}\vec{A}_+^{\,b}(\vec{p}_1)\cdot\vec{A}_+^{\,d}(\vec{p}_3)\vec{A}_+^{\,c}(\vec{p}_2)\cdot\vec{A}_+^{\,e}(\vec{p}_4)$$
$$+\frac{g(2\pi)^3}{[2(2\pi)^3]^{1/2}}\int\frac{d^3p}{(p)^{1/2}}\vec{J}^{\,a}(p)\cdot\vec{A}_+^{\,a}(\vec{p})+\frac{g^2(2\pi)^9}{2}\int d^3p_1d^3p_2K^a(\vec{p}_1)F_{ab}(\vec{p}_1,\vec{p}_2)K^b(\vec{p}_2)\ , \tag{7}$$

where

$$\vec{A}_\pm^{\,a}(\vec{p})=\sum_\lambda[\vec{\epsilon}_\lambda(\vec{p})a_\lambda^a(p)\pm\vec{\epsilon}_\lambda^{\,*}(-\vec{p})a_\lambda^{a\dagger}(-\vec{p})]$$

and $\vec{\epsilon}_\lambda(\vec{p})$ is a polarization vector transverse to $\vec{p}$. If

$$\psi(\vec{p})=\sum_r [b_r(\vec{p})u_r(\vec{p})+d_r^\dagger(-\vec{p})v_r(-\vec{p})] ,$$

the current $\vec{J}^{\,a}(p)$ is the three-vector part of the four-vector

$$J_\mu^a(\vec{p})=\frac{1}{(2\pi)^3}\int \frac{d^3q\, m}{[w(\vec{q})w(\vec{q}-\vec{p})]^{1/2}}\bar{\psi}(\vec{q})\gamma_\mu\lambda_a\psi(\vec{q}-\vec{p}) . \tag{8}$$

The gluon charge density is

$$L^a(\vec{p})=\frac{-i}{2(2\pi)^3}\int d^3q\left[\frac{|\vec{q}|}{|\vec{p}+\vec{q}|}\right]^{1/2} f_{abc}\vec{A}^{\,b}_-(\vec{q})\cdot\vec{A}^{\,c}_+(-\vec{p}-\vec{q}) , \tag{9}$$

and the total charge density is

$$K^a(\vec{p})=J_0^a(\vec{p})+L^a(\vec{p}) . \tag{10}$$

The modified Coulomb interaction is described by $F_{ab}(\vec{p}_1,\vec{p}_2)$,

$$F_{ab}(\vec{p}_1,\vec{p}_2)=\int d^3k\, D_{ac}(\vec{p}_1,\vec{k})k^2D_{cb}(\vec{k},-\vec{p}_2) , \tag{11}$$

and

$$D_{ac}(\vec{p},\vec{k})=\frac{\delta_{ac}\delta^3(\vec{p}+\vec{k})}{p^2(2\pi)^3}+ig\frac{f_{ade}}{p^2}\int\frac{d^3q\,\vec{p}\cdot\vec{A}^{\,d}_+(\vec{p}-\vec{q})D_{ec}(\vec{q},\vec{k})}{[2(2\pi)^3|\vec{p}-\vec{q}|]^{1/2}} . \tag{12}$$

Since the quark creation and annihilation operators satisfy the usual anticommutation relations, $\{b_r(\vec{p}),b_s^\dagger(\vec{k})\}=\delta_{rs}\delta^3(\vec{p}-\vec{k})$, calculation of the commutator in (5) is a straightforward, though lengthy, process.

There are four two-quark wave functions which have the quantum numbers of the quark-antiquark system. We choose them to be

$$\Phi_A(\vec{k}_1,\vec{k}_2)=\sum_{r,s}u_s(\vec{k}_2)\langle\Psi_0|d_r(\vec{k}_1)b_s(\vec{k}_2)|\Psi\rangle\bar{v}_r(\vec{k}_1) , \tag{13a}$$

$$\Phi_B(\vec{k}_1,\vec{k}_2)=\sum_{r,s}v_s(\vec{k}_2)\langle\Psi_0|b_r^\dagger(\vec{k}_1)d_s^\dagger(\vec{k}_2)|\Psi\rangle\bar{u}_r(\vec{k}_1) , \tag{13b}$$

$$\Phi_C(\vec{k}_1,\vec{k}_2)=\sum_{r,s}u_s(\vec{k}_2)\langle\Psi_0|b_r^\dagger(\vec{k}_1)b_s(\vec{k}_2)|\Psi\rangle\bar{u}_r(\vec{k}_1) , \tag{13c}$$

$$\Phi_D(\vec{k}_1,\vec{k}_2)=\sum_{r,s}v_s(\vec{k}_1)\langle\Psi_0|d_s^\dagger(\vec{k}_1)d_r(\vec{k}_2)|\Psi\rangle\bar{v}_r(\vec{k}_2) . \tag{13d}$$

The external spinors have color indices as well as spin indices. Each $\Phi_x(\vec{k}_1,\vec{k}_2)$ is a 4×4 matrix in spin space and a 3×3 matrix in SU(3) color space. The wave equation for Φ_A requires the commutator of $d_r(\vec{k}_1)b_s(\vec{k}_2)$ with H_I. When the commutator is normal ordered, there will be terms with the two-quark and four-quark operators as well as terms with gluon operators. The two-quark sector is coupled to the four-quark sector which is in turn coupled to the six-quark sector, and so forth. Our first approximation is to keep only two-quark amplitudes so that the system of equations close. The physical assumption is that while multiquark states are important, confinement is a consequence of interactions between quark and the QCD vacuum and is insensitive to the multiquark component of the states. This approximation could be checked by calculating the overlap of two-quark bound-state amplitudes with four-quark amplitudes which are composed either of free quarks or a pair of two-quark bound states. Our second approximation is to drop contributions from states with physical gluons. Again the argument is that while the actual energy levels may be sensitive to such terms, the fact of confinement is not. The final approximation is to replace the modified Coulomb interaction defined by (11) and (12) with its vacuum expectation value in the physical vacuum. This is a natural step, since confinement, if it occurs, is a consequence of the complicated structure of the QCD vacuum. In this model confinement is produced by two charge densities interacting via an instantaneous Coulomb interaction that has been modified by vacuum processes. Only the last term in H_I is relevant for confinement, $K^a(\vec{p})$ is replaced by $J_0^a(\vec{p})$, and the operator $F_{ab}(\vec{p}_1,\vec{p}_2)$ is replaced by the function $F_{ab}(\vec{p}_1,\vec{p}_2)$.

The modified Tamm-Dancoff equations for the functions $\Phi_x(\vec{k}_1,\vec{k}_2)$ are

$$[E-w(\vec{k}_1)-w(\vec{k}_2)]\Phi_A(\vec{k}_1,\vec{k}_2)=-\Lambda_+(\vec{k}_2)\left\{I_{ab}\left[\left[\frac{w(\vec{k}_1)w(\vec{k}_2)}{w(\vec{k}_1-\vec{p}_1)w(\vec{k}_2-\vec{p}_2)}\right]^{1/2}\lambda^a[\Phi_A(\vec{k}_1-\vec{p}_1,\vec{k}_2-\vec{p}_2)+\Phi_B(\vec{p}_1-\vec{k}_1,\vec{p}_2-\vec{k}_2)+\Phi_C(\vec{p}_1-\vec{k}_1,\vec{k}_2-\vec{p}_2)-\Phi_D(\vec{p}_2-\vec{k}_2,\vec{k}_1-\vec{p}_1)]\lambda^b\right]\right\}\Lambda_-(\vec{k}_1)$$

$$+\tfrac{1}{2}I_{ab}\left[\left[\left|\frac{w(\vec{k}_1)}{w(\vec{k}_1-\vec{p}_1-\vec{p}_2)}\right|\right]^{1/2}[\Phi_A(\vec{k}_1-\vec{p}_1-\vec{p}_2,\vec{k}_2)+\Phi_C(\vec{p}_1+\vec{p}_2-\vec{k}_1,\vec{k}_2)]\right.$$

$$\left.\times\lambda_b[\Lambda_-(\vec{k}_1-\vec{p}_1)-\Lambda_+(\vec{k}_1-\vec{p}_1)]\lambda_a\right]\Lambda_-(\vec{k}_1)$$

$$+\tfrac{1}{2}\Lambda_+(\vec{k}_2)I_{ab}\left[\left[\left|\frac{w(\vec{k}_2)}{w(\vec{k}_2-\vec{p}_1-\vec{p}_2)}\right|\right]^{1/2}\lambda_a[\Lambda_+(\vec{k}_2-\vec{p}_1)-\Lambda_-(\vec{k}_2-\vec{p}_1)]\lambda_b\right.$$

$$\left.\times[\Phi_A(\vec{k}_1,\vec{k}_2-\vec{p}_1-\vec{p}_2)-\Phi_D(\vec{p}_1+\vec{p}_2-\vec{k}_2,\vec{k}_1)]\right], \tag{14}$$

$$[E+w(\vec{k}_1)+w(\vec{k}_2)]\Phi_B(\vec{k}_1,\vec{k}_2)$$

$$=\Lambda_-(-k_2)\left\{I_{ab}\left[\left[\left|\frac{w(\vec{k}_1)w(\vec{k}_2)}{w(\vec{k}_1+\vec{p}_1)w(\vec{k}_2+\vec{p}_2)}\right|\right]^{1/2}\lambda^a[\Phi_A(-\vec{p}_1-\vec{k}_1,-\vec{p}_2-\vec{k}_2)+\Phi_B(\vec{p}_1+\vec{k}_1,\vec{p}_2+\vec{k}_2)\right.\right.$$

$$\left.\left.+\ \Phi_C(\vec{p}_1+\vec{k}_1,-\vec{p}_2-\vec{k}_2)-\Phi_D(\vec{p}_2+\vec{k}_2,-\vec{p}_1-\vec{k}_1)]\lambda^b\right]\right\}\Lambda_+(-\vec{k}_1)$$

$$-\tfrac{1}{2}\Lambda_-(-\vec{k}_2)I_{ab}\left[\left[\left|\frac{w(\vec{k}_2)}{w(\vec{k}_2+\vec{p}_1+\vec{p}_2)}\right|\right]^{1/2}\lambda^a[\Lambda_-(-\vec{k}_2-\vec{p}_2)-\Lambda_+(-\vec{k}_2-\vec{p}_2)]\lambda^b\right.$$

$$\left.\times[\Phi_B(\vec{k}_1,\vec{k}_2+\vec{p}_1+\vec{p}_2)+\Phi_C(\vec{k}_1,-\vec{p}_1-\vec{p}_2-\vec{k}_2)]\right]$$

$$-\tfrac{1}{2}I_{ab}\left[\left[\left|\frac{w(\vec{k}_1)}{w(\vec{k}_1+\vec{p}_1+\vec{p}_2)}\right|\right]^{1/2}[\Phi_B(\vec{k}_1+\vec{p}_1+\vec{p}_2,\vec{k}_2)-\Phi_D(\vec{k}_2,-\vec{p}_1-\vec{p}_2-\vec{k}_1)]\right.$$

$$\left.\times\lambda^a[\Lambda_+(-\vec{k}_1-\vec{p}_2)-\Lambda_-(-\vec{k}_1-\vec{p}_2)]\lambda^b\right]\Lambda_+(-\vec{k}_1), \tag{15}$$

$$[E+w(\vec{k}_1)-w(\vec{k}_2)]\Phi_C(\vec{k}_1,\vec{k}_2)=\tfrac{1}{2}\Lambda_+(\vec{k}_2)I_{ab}\left[\left[\left|\frac{w(\vec{k}_2)}{w(\vec{k}_2-\vec{p}_1-\vec{p}_2)}\right|\right]^{1/2}\lambda^a[\Lambda_+(\vec{k}_2-\vec{p}_1)-\Lambda_-(\vec{k}_2-\vec{p}_1)]\lambda^b\right.$$

$$\left.\times[\Phi_C(\vec{k}_1,\vec{k}_2-\vec{p}_1-\vec{p}_2)+\Phi_B(\vec{k}_1,\vec{p}_1+\vec{p}_2-\vec{k}_2)]\right]$$

$$-\tfrac{1}{2}I_{ab}\left[\left[\left|\frac{w(\vec{k}_1)}{w(\vec{k}_1+\vec{p}_1+\vec{p}_2)}\right|\right]^{1/2}[\Phi_C(\vec{k}_1+\vec{p}_1+\vec{p}_2,\vec{k}_2)+\Phi_A(-\vec{k}_1-\vec{p}_1-\vec{p}_2,\vec{k}_2)]\right.$$

$$\left.\times\lambda^a[\Lambda_+(-\vec{k}_1-\vec{p}_2)-\Lambda_-(-\vec{k}_1-\vec{p}_2)]\lambda^b\right]\Lambda_+(-\vec{k}_1), \tag{16}$$

$$[E+w(\vec{k}_1)-w(\vec{k}_2)]\Phi_D(\vec{k}_1,\vec{k}_2)=-\tfrac{1}{2}\Lambda_-(-\vec{k}_1)I_{ab}\left[\left[\frac{w(\vec{k}_1)}{w(\vec{k}_1+\vec{p}_1+\vec{p}_2)}\right]^{1/2}\lambda^a[\Lambda_-(-\vec{k}_1-\vec{p}_1)-\Lambda_+(-\vec{k}_1-\vec{p}_1)]\lambda^b\right.$$
$$\left.\times[\Phi_D(\vec{k}_1+\vec{p}_1+\vec{p}_2,\vec{k}_2)-\Phi_A(\vec{k}_2,-\vec{k}_1-\vec{p}_1-\vec{p}_2)]\right]$$
$$+\tfrac{1}{2}I_{ab}\left[\left[\frac{w(\vec{k}_2)}{w(\vec{k}_2-\vec{p}_1-\vec{p}_2)}\right]^{1/2}[\Phi_D(\vec{k}_1,\vec{k}_2-\vec{p}_1-\vec{p}_2)-\Phi_B(\vec{p}_1+\vec{p}_2-\vec{k}_2,\vec{k}_1)]\right.$$
$$\left.\times\lambda^a[\Lambda_-(\vec{k}_2-\vec{p}_1)-\Lambda_+(\vec{k}_2-\vec{p}_1)]\lambda^b\right]\Lambda_-(\vec{k}_2)\,, \tag{17}$$

where

$$I_{ab}[G_{ab}(\vec{p}_1,\vec{p}_2)]=g^2(2\pi)^3\int d^3p_1d^3p_2F_{ab}(\vec{p}_1,\vec{p}_2)G_{ab}(\vec{p}_1,\vec{p}_2)\,.$$

In these equations λ^a is an SU(3) matrix normalized so that $\sum_a\lambda^a_{ij}\lambda^a_{lm}=-2\delta_{ij}\delta_{lm}/3+2\delta_{im}\delta_{jl}$. The matrices $\Lambda_+(\vec{k})$ and $\Lambda_-(\vec{k})$ are positive- and negative-energy projection operators:

$$\Lambda_+(\vec{k})=\frac{w(\vec{k})+\gamma_0(\vec{\gamma}\cdot\vec{k}+m)}{2w(\vec{k})}$$
$$=\frac{m}{w(\vec{k})}u(\vec{k})\bar{u}(\vec{k})\gamma_0$$
$$=\frac{m}{w(\vec{k})}u(\vec{k})\bar{u}(-\vec{k})\,. \tag{18a}$$

$$\Lambda_-(\vec{k})=1-\Lambda_+(\vec{k})=\frac{w(\vec{k})-\gamma_0(\vec{\gamma}\cdot\vec{k}+m)}{2w(\vec{k})}$$
$$=\frac{m}{w(\vec{k})}\gamma_0v(\vec{k})\bar{v}(\vec{k})$$
$$=\frac{m}{w(\vec{k})}v(-\vec{k})\bar{v}(\vec{k})\,. \tag{18b}$$

The energy projection operators arise naturally in the evaluation of the commutators. The γ_0 is supplied by the vector nature of the effective Coulomb interaction. Only in the Coulomb gauge do the equations have this comparatively simple form. The projection operators match the properties of the $\Phi_x(\vec{k}_1,\vec{k}_2)$. For example, from (18), (13a), and (13b) we see that

$$\Lambda_+(\vec{k}_2)\Phi_A(\vec{k}_1,\vec{k}_2)\Lambda_-(\vec{k}_1)=\Phi_A(\vec{k}_1,\vec{k}_2)\,,$$
$$\Lambda_-(\vec{k}_2)\Phi_A(\vec{k}_1,\vec{k}_2)=\Phi_A(\vec{k}_1,\vec{k}_2)\Lambda_+(\vec{k}_1)=0\,,$$
$$\Lambda_-(-\vec{k}_2)\Phi_B(\vec{k}_1,\vec{k}_2)\Lambda_+(-\vec{k}_1)=\Phi_B(\vec{k}_1,\vec{k}_2)\,,$$
$$\Lambda_+(-\vec{k}_2)\Phi_B(\vec{k}_1,\vec{k}_2)=\Phi_B(\vec{k}_1,\vec{k}_2)\Lambda_-(-\vec{k}_1)=0\,.$$

The projection properties of Φ_C and Φ_D can be deduced from those for Φ_A and Φ_B.

The integral terms in (14)—(17) are of two types. In (13) and (14) the factor $\Phi_x(\vec{k}_1\pm\vec{p}_1,\vec{k}_2\pm\vec{p}_2)$ indicates that the two quarks scatter off each other. The scattering terms cancel out of the equations for Φ_C and Φ_D. The other integrals are self-energy interactions that affect only one of the quarks.

To proceed further, we need information on the function $F_{ab}(\vec{p}_1,\vec{p}_2)$. Drawing on earlier work,[9] we assume that in $F_{ab}(\vec{p}_1,\vec{p}_2)$ there is a piece which is color conserving, momentum conserving, and infrared singular. The color and momentum properties are prescribed by a color-singlet and translationally invariant vacuum,

$$F_{ab}(\vec{p}_1,\vec{p}_2)=\frac{\delta_{ab}}{(2\pi)^6}\delta^3(\vec{p}_1+\vec{p}_2)F(\vec{p}_1)\,. \tag{19}$$

To lowest order in g, $F_{ab}(\vec{p}_1,\vec{p}_2)$ is a pure Coulomb interaction and $F(\vec{p})=1/p^2$. The condition that $F(\vec{p})$ is infrared singular implies that as $p\to0$, $F(\vec{p})\to p^{-2n}$ where $n>\frac{3}{2}$.[9] For large p, $F(\vec{p})$ is expected to approach $1/p^2$. Large p corresponds to small distances where vacuum screening should be unimportant. Renormalization via mass and wave-function counterterms in H_I will control ultraviolet divergences to order g^2. Since we are interested in the confinement mechanism, we assume that the large-p^2 behavior of $F(\vec{p})$ is such as to render all integrals ultraviolet finite. If $\frac{5}{2}>n>\frac{3}{2}$, the singularity at $p^2=0$ can be controlled by a single subtraction,[9]

$$F(\vec{p})=F(\vec{p})-\rho\delta^3(\vec{p})+\rho\delta^3(\vec{p})$$
$$=\bar{F}(\vec{p})+\rho\delta^3(\vec{p})\,. \tag{20}$$

The function $\bar{F}(\vec{p})$ is infrared finite and ρ is an infinite constant that must cancel out if the equations are to produce finite energies. When (19) and (20) are combined, Eq. (14) becomes

$$[E-w(\vec{k}_1)-w(\vec{k}_2)]\Phi_A(\vec{k}_1,\vec{k}_2)=\kappa\rho[-\lambda^a\Phi_A(\vec{k}_1,\vec{k}_2)\lambda^a+\tfrac{1}{2}\Phi_A(\vec{k}_1,\vec{k}_2)\lambda^a\lambda^a+\tfrac{1}{2}\lambda^a\lambda^a\Phi_A(\vec{k}_1,\vec{k}_2)]$$
$$+\text{ finite terms}\,, \tag{21}$$

with κ a constant. The projection operators remove the Φ_B, Φ_C, and Φ_D amplitudes. The color properties of Φ_A now come into play. For SU(3) Φ_A describes a quark and an antiquark which can either be in a singlet state or in an element of the eight-dimensional regular representation. If Φ_A is a singlet, (21) is automatically satisfied. If Φ_A belongs to the regular representation, the only finite-energy solution to (21) is $\Phi_A=0$. Thus, there can exist bound states only for color-singlet configurations of a quark and an antiquark. This result depends on the exact cancellation of the infrared singularities between the self-energy and the scattering parts of the equations. Additional calculations show that there do not exist finite-energy quark-quark states. The first prediction of this model is that if finite-energy bound states exist, they must be color singlets.

The infrared-singular terms in Eq. (15) for Φ_B drop out if Φ_B is a color singlet. The equations for Φ_C and Φ_D are automatically satisfied. In fact, when (19) is used for $F_{ab}(\vec{p}_1,\vec{p}_2)$, the Φ_C and Φ_D equations become algebraic. For color-singlet amplitudes

$$[E+w(\vec{k}_1)-w(\vec{k}_2)]\Phi_C(\vec{k}_1,\vec{k}_2)$$
$$=\frac{g^2}{2(2\pi)^3}\int d^3p\,\bar{F}(\vec{p})\lambda^a\lambda^a\{\Lambda_+(\vec{k}_2)[\Lambda_+(\vec{k}_2-\vec{p})-\Lambda_-(\vec{k}_2-\vec{p})][\Phi_C(\vec{k}_1,\vec{k}_2)+\Phi_B(\vec{k}_1,-\vec{k}_2)]$$
$$-[\Phi_C(\vec{k}_1,\vec{k}_2)+\Phi_A(-\vec{k}_1,\vec{k}_2)][\Lambda_+(-\vec{k}_1+\vec{p})-\Lambda_-(-\vec{k}_1+\vec{p})]\Lambda_+(-\vec{k}_1)\}\,. \tag{22}$$

The Φ_D equation is similar. Since both Φ_C and Φ_D are algebraic functions of Φ_A and Φ_B, they can be explicitly eliminated from the equations for Φ_A and Φ_B. The new terms of order g^4 are comparable to the four-quark terms that have been neglected. Hence, we drop the g^4 contributions due to Φ_C and Φ_D. Given a solution of our final equations, it is possible to calculate the error involved in this approximation. There is one final step before we write down our final equations. Since the infrared singularities cancel for color-singlet amplitudes, $F_{ab}(\vec{p}_1,\vec{p}_2)$ is replaced everywhere by (19) with $\bar{F}(\vec{p})$ in place of $F(\vec{p})$. When (14) and (15) are properly renormalized, part of the self-energy effect is absorbed into the physical mass. The residual momentum dependence will not affect the existence of finite-energy bound states.

The final equations for the quark-antiquark simplify in the center-of-momentum frame where $\vec{k}_1=-\vec{k}_2=-\vec{k}$. (In principle, the equations could be studied for arbitrary total momentum.) If $\Psi_A(\vec{k})=\Phi_A(-\vec{k},\vec{k})$ and $\Psi_B(\vec{k})=\Phi_B(\vec{k},-\vec{k})$,

$$[E-2w(\vec{k})]\Psi_A(\vec{k})=-f\int d^3p\,\bar{F}(\vec{p}-\vec{k})\frac{w(\vec{k})}{w(\vec{p})}\Lambda_+(\vec{k})[\Psi_A(\vec{p})+\Psi_B(\vec{p})]\Lambda_-(-\vec{k})\,, \tag{23}$$

$$[E+2w(\vec{k})]\Psi_B(\vec{k})=f\int d^3p\,\bar{F}(\vec{p}-\vec{k})\frac{w(\vec{k})}{w(\vec{p})}\Lambda_-(\vec{k})[\Psi_B(\vec{p})+\Psi_A(\vec{p})]\Lambda_+(-\vec{k})\,, \tag{24}$$

where

$$f=\frac{4}{3}\frac{g^2}{(2\pi)^3}\,.$$

These equations describe energy levels of permanently confined quark-antiquark pairs in color-singlet states. In Sec. IV we discuss the problem of solving (23) and (24) to demonstrate confinement.

It is possible to go through the development of this section for a single-quark state. Although ultraviolet divergences are controlled by renormalization, the infrared divergence remains and gives single-quark states infinite energy.

III. GLUON BOUND STATES

A theory which binds quarks into color-singlet hadrons does not necessarily bind gluons and produce glueballs. It is possible to extend the analysis of the previous section and study gluon bound states. The only additional assumption that needs to be made is that the Coulomb term also dominates gluon-gluon interactions at low momentum transfer. We consider two-gluon amplitudes, use only the Coulomb part of H_I, and retain just the two-gluon terms in the commutator. Again it is possible, given a solution, to estimate the effect of neglected terms. The three Tamm-Dancoff wave functions are

$$\Phi^{ab}_{1\alpha\beta}(\vec{k}_1,\vec{k}_2)=\sum_{\sigma,\tau}\langle\Psi_0|\,a^a_\sigma(\vec{k}_1)a^b_\tau(\vec{k}_2)\,|\Psi\rangle\epsilon_\sigma(\vec{k}_1)_\alpha\epsilon_\tau(\vec{k}_2)_\beta\,, \tag{25a}$$

$$\Phi^{ab}_{2\alpha\beta}(\vec{k}_1,\vec{k}_2)=\sum_{\sigma,\tau}\langle\Psi_0|\,a^{a\dagger}_\sigma(\vec{k}_1)a^{b\dagger}_\tau(\vec{k}_2)\,|\Psi\rangle\epsilon^*_\sigma(\vec{k}_1)_\alpha\epsilon^*_\tau(\vec{k}_2)_\beta\,, \tag{25b}$$

$$\Phi^{ab}_{3\alpha\beta}(\vec{k}_1,\vec{k}_2)=\sum_{\sigma,\tau}\langle\Psi_0|\,a^{a\dagger}_\sigma(\vec{k}_1)a^b_\tau(\vec{k}_2)\,|\Psi\rangle\epsilon^*_\sigma(\vec{k}_1)_\alpha\epsilon_\tau(\vec{k}_2)_\beta\,. \tag{25c}$$

Since H_I contains several gluon interaction terms, the justification for the approximation of singling out a particular

term and keeping just two-gluon pieces from the commutator rests on the hypothesis that the mechanism for gluon binding is similar to the mechanism for quark binding.[11] Explicitly gluon charge densities (9) interact via the modified Coulomb interaction (19).

The Tamm-Dancoff equations for the gluon wave functions are

$$
\begin{aligned}
(E-k_1-k_2)\Phi^{ab}_{1\alpha\beta}(\vec{k}_1,\vec{k}_2)=&-P_{\alpha\gamma}(\vec{k}_1)P_{\beta\delta}(\vec{k}_2)I'_{cd}\left\{\frac{f_{ace}f_{bdg}}{(k_1k_2\,|\,\vec{k}_1-\vec{p}_1\,|\,|\,\vec{k}_2-\vec{p}_2\,|\,)^{1/2}}\right.\\
&\times[\,(k_1+|\,\vec{k}_1-\vec{p}_1\,|\,)(k_2+|\,\vec{k}_2-\vec{p}_2\,|\,)\Phi^{eg}_{1\gamma\delta}(\vec{k}_1-\vec{p}_1,\vec{k}_2-\vec{p}_2)\\
&+(k_1-|\,\vec{k}_1-\vec{p}_1\,|\,)(k_2-|\,\vec{k}_2-\vec{p}_2\,|\,)\Phi^{eg}_{2\gamma\delta}(\vec{p}_1-\vec{k}_1,\vec{p}_2-\vec{k}_2)\\
&+(k_1-|\,\vec{k}_1-\vec{p}_1\,|\,)(k_2+|\,\vec{k}_2-\vec{p}_2\,|\,)\Phi^{eg}_{3\gamma\delta}(\vec{p}_1-\vec{k}_1,\vec{k}_2-\vec{p}_2)\\
&\left.+(k_1+|\,\vec{k}_1-\vec{p}_1\,|\,)(k_2-|\,\vec{k}_2-\vec{p}_2\,|\,)\Phi^{ge}_{3\delta\gamma}(\vec{p}_2-\vec{k}_2,\vec{k}_1-\vec{p}_1)]\right\}\\
&-P_{\alpha\epsilon}(\vec{k}_1)I'_{cd}\left\{f_{ach}f_{hde}P_{\epsilon\gamma}(\vec{k}_1-\vec{p}_1)\left[\left[\mu_1+\frac{1}{\mu_1}\right]\Phi^{eb}_{1\gamma\beta}(\vec{k}_1-\vec{p}_1-\vec{p}_2,\vec{k}_2)\right.\right.\\
&\left.\left.+\left[\mu_1-\frac{1}{\mu_1}\right]\Phi^{eb}_{3\gamma\beta}(\vec{p}_1+\vec{p}_2-\vec{k}_1,\vec{k}_2)\right]\right\}\\
&-P_{\beta\epsilon}(\vec{k}_2)I'_{cd}\left\{f_{gce}f_{bdg}P_{\epsilon\delta}(\vec{k}_2-\vec{p}_2)\left[\left[\mu_2+\frac{1}{\mu_2}\right]\Phi^{ae}_{1\alpha\delta}(\vec{k}_1,\vec{k}_2-\vec{p}_1-\vec{p}_2)\right.\right.\\
&\left.\left.+\left[\mu_2-\frac{1}{\mu_2}\right]\Phi^{ea}_{3\delta\alpha}(\vec{p}_1+\vec{p}_2-\vec{k}_2,\vec{k}_1)\right]\right\},
\end{aligned}
\tag{26}
$$

where

$$\mu_1=|\,\vec{k}_1-\vec{p}_1\,|\,/(k_1\,|\,\vec{k}_1-\vec{p}_1-\vec{p}_2\,|\,)^{1/2}\text{ and }\mu_2=|\,\vec{k}_2-\vec{p}_2\,|\,/(k_2\,|\,\vec{k}_2-\vec{p}_1-\vec{p}_2\,|\,)^{1/2};$$

$$
\begin{aligned}
(E+k_1+k_2)\Phi^{ab}_{2\alpha\beta}(\vec{k}_1,\vec{k}_2)=&P_{\alpha\gamma}(\vec{k}_1)P_{\beta\delta}(\vec{k}_2)I'_{cd}\left\{\frac{f_{ace}f_{bdg}}{(k_1k_2\,|\,\vec{k}_1+\vec{p}_1\,|\,|\,\vec{k}_2+\vec{p}_2\,|\,)^{1/2}}\right.\\
&\times[\,(k_1+|\,\vec{k}_1+\vec{p}_1\,|\,)(k_2+|\,\vec{k}_2+\vec{p}_2\,|\,)\Phi^{eg}_{2\gamma\delta}(\vec{k}_1+\vec{p}_1,\vec{k}_2+\vec{p}_2)\\
&+(k_1-|\,\vec{k}_1+\vec{p}_1\,|\,)(k_2-|\,\vec{k}_2+\vec{p}_2\,|\,)\Phi^{eg}_{1\gamma\delta}(-\vec{k}_1-\vec{p}_1,-\vec{k}_2-\vec{p}_2)\\
&+(k_1-|\,\vec{k}_1+\vec{p}_1\,|\,)(k_2+|\,\vec{k}_2+\vec{p}_2\,|\,)\Phi^{ge}_{3\delta\gamma}(\vec{k}_2+\vec{p}_2,-\vec{k}_1-\vec{p}_1)\\
&\left.+(k_1+|\,\vec{k}_1+\vec{p}_1\,|\,)(k_2-|\,\vec{k}_2+\vec{p}_2\,|\,)\Phi^{eg}_{3\gamma\delta}(\vec{k}_1+\vec{p}_1,-\vec{k}_2-\vec{p}_2)]\right\}\\
&+P_{\alpha\epsilon}(\vec{k}_1)I'_{cd}\left\{f_{acg}f_{gde}P_{\epsilon\gamma}(\vec{k}_1+\vec{p}_1)\left[\left[\nu_1+\frac{1}{\nu_1}\right]\Phi^{eb}_{2\gamma\beta}(\vec{k}_1+\vec{p}_1+\vec{p}_2,\vec{k}_2)\right.\right.\\
&\left.\left.+\left[\nu_1-\frac{1}{\nu_1}\right]\Phi^{be}_{3\beta\gamma}(\vec{k}_2,-\vec{k}_1-\vec{k}_2)\right]\right\}\\
&+P_{\beta\epsilon}(\vec{k}_2)I'_{cd}\left\{f_{bdg}f_{gce}P_{\epsilon\gamma}(\vec{k}_2+\vec{p}_2)\left[\left[\nu_2+\frac{1}{\nu_2}\right]\Phi^{ae}_{2\alpha\gamma}(\vec{k}_1,\vec{k}_2+\vec{p}_1+\vec{p}_2)\right.\right.\\
&\left.\left.+\left[\nu_1-\frac{1}{\nu_1}\right]\Phi^{ae}_{3\alpha\gamma}(\vec{k}_1,-\vec{k}_2-\vec{p}_1-\vec{p}_2)\right]\right\},
\end{aligned}
\tag{27}
$$

where

$$\nu_1=|\vec{k}_1+\vec{p}_1|/(k_1|\vec{k}_1+\vec{p}_1+\vec{p}_2|)^{1/2} \text{ and } \nu_2=|\vec{k}_2+\vec{p}_2|/(k_2|\vec{k}_2+\vec{p}_1+\vec{p}_2|)^{1/2};$$

$$(E+k_1-k_2)\Phi^{ab}_{3\alpha\beta}(\vec{k}_1,\vec{k}_2)=P_{a\epsilon}(\vec{k}_1)I'_{cd}\left\{f_{ach}f_{hde}P_{\epsilon\gamma}(\vec{k}_1+\vec{p}_1)\left[\left[\nu_1-\frac{1}{\nu_1}\right]\Phi^{be}_{1\beta\gamma}(\vec{k}_2,-\vec{k}_1-\vec{p}_1-\vec{p}_2)\right.\right.$$

$$\left.\left.+\left[\nu_1+\frac{1}{\nu_1}\right]\Phi^{eb}_{3\gamma\beta}(\vec{k}_1+\vec{p}_1+\vec{p}_2,\vec{k}_2)\right]\right\}$$

$$-P_{\beta\epsilon}(\vec{k}_2)I'_{cd}\left\{f_{bcg}f_{gde}P_{\epsilon\gamma}(\vec{k}_2-\vec{p}_2)\left[\left[\mu_2+\frac{1}{\mu_2}\right]\Phi^{ae}_{3\alpha\gamma}(\vec{k}_1,\vec{k}_2-\vec{p}_1-\vec{p}_2)\right.\right.$$

$$\left.\left.+\left[\mu_2-\frac{1}{\mu_2}\right]\Phi^{ae}_{2\alpha\gamma}(\vec{k}_1,\vec{p}_1+\vec{p}_2-\vec{k}_2)\right]\right\} \tag{28}$$

and

$$I'_{cd}[G^{ab:cd}(\vec{p}_1,\vec{p}_2)]=\frac{g^2(2\pi)^3}{4}\int d^3p_1d^3p_2F_{cd}(\vec{p}_1,\vec{p}_2)G^{ab:cd}(\vec{p}_1,\vec{p}_2)\,.$$

$P_{\alpha\beta}(\vec{k})=\delta_{\alpha\beta}-k_\alpha k_\beta/k^2$ is the spin-1 projection operator ($P_{\alpha\beta}\Phi_{\beta\gamma}P_{\gamma\delta}=\Phi_{\alpha\delta}$). There are scattering and self-interaction terms for Φ_1 and Φ_2 but only self-interaction terms for Φ_3.

Each step in the reduction of the quark wave equations has its counterpart in the gluon sector. The infrared singularity from the scattering terms cancels against the self-energy parts for color-singlet amplitudes. Thus, the finite-energy gluon bound states must be color singlets. When (19) is used for $F_{ab}(\vec{p}_1,\vec{p}_2)$, Eq. (28) becomes algebraic. Φ_3 is a function of Φ_1 and Φ_2 and induces a g^4 correction in the integral equations for Φ_1 and Φ_2. We drop this term as we did for quarks. Residual finite self-energy contributions are dropped. In the center-of-momentum frame with $\Psi_1(\vec{k})=\Phi_1(\vec{k},-\vec{k})$ and $\Psi_2(\vec{k})=\Phi_2(-\vec{k},\vec{k})$, the infrared-finite equations for color-singlet gluon wave functions are

$$(E-2k)\Psi_{1\alpha\beta}(\vec{k})=-f'\int d^3p\frac{\bar{F}(\vec{p}-\vec{k})}{pk}P_{\alpha\gamma}(\vec{k})P_{\beta\delta}(\vec{k})[(k+p)^2\Psi_{1\gamma\delta}(\vec{p})+(k-p)^2\Psi_{2\gamma\delta}(\vec{p})]\,, \tag{29}$$

$$(E+2k)\Psi_{2\alpha\beta}(\vec{k})=f'\int d^3p\frac{\bar{F}(\vec{p}-\vec{k})}{pk}P_{\alpha\gamma}(\vec{k})P_{\beta\delta}(\vec{k})[(k+p)^2\Psi_{2\gamma\delta}(\vec{p})+(k-p)^2\Psi_{1\gamma\delta}(\vec{p})]\,, \tag{30}$$

where $f'=3g^2/4(2\pi)^3$. In the next section these equations together with the corresponding quark equations are studied to determine if they produce a sequence of finite-energy bound states.

IV. THE ENERGY SPECTRUM

The finite-energy equations for color-singlet configurations of quarks or gluons have a deceptively simple appearance. For quarks, the energy projection operators reduce the sixteen components of Ψ_A and Ψ_B to four independent components for each. To demonstrate confinement, we need to solve coupled three-dimensional integral equations for eight amplitudes. When the functions are expanded in partial waves, parity conservation reduces (23) and (24) to sets of two, two, and four coupled equations in one variable. The angular momentum expansion reduces the gluon set from eight coupled integral equations to a more manageable two, two, and four. The details of the wave-function decomposition and angular momentum expansion are given in the Appendix.

The singular nature of the interaction creates additional problems in any numerical treatment of the equations.[9] $\bar{F}(\vec{p})$ is regular at $p^2=0$, but the subtraction procedure is cumbersome in practice. If we choose

$$F(\vec{p})=\lim_{\mu\to0}\frac{(\Lambda^2)^{n-1}}{(p^2+\mu^2)^n}\,, \tag{31}$$

with Λ a parameter with dimensions of mass, then the divergent constant ρ in (20) is

$$\rho=\int d^3p\,F(\vec{p})$$

$$=\lim_{\mu\to0}\pi^{3/2}\frac{\Gamma(n-\frac{3}{2})}{\Gamma(n)}(\Lambda^2)^{n-1}\mu^{3-2n}\,. \tag{32}$$

If $n>\frac{3}{2}$, ρ is infinite. The explicit subtraction procedure is avoided by calculating the Fourier transform of $\bar{F}(\vec{p})$ from the Fourier transform of (31):

$$\bar{F}(\vec{p})=\lim_{\mu\to 0}\left[\frac{(\Lambda^2)^{n-1}2\pi^{3/2}}{\Gamma(n)}\int\frac{d^3r}{(2\pi)^3}e^{i\vec{p}\cdot\vec{r}}\left[\frac{r}{2\mu}\right]^{n-3/2}K_{n-3/2}(\mu r)\right]-\rho$$

$$=-\frac{\pi^{3/2}(\Lambda^2)^{n-1}}{2^{2n-3}}\frac{(n-\frac{1}{2})\Gamma(\frac{5}{2}-n)}{\Gamma(n)}\int\frac{d^3r}{(2\pi)^3}e^{i\vec{p}\cdot\vec{r}}r^{2n-3}$$

$$=-\frac{\eta}{4\pi}\int d^3r\,e^{i\vec{p}\cdot\vec{r}}r^{2n-3}\,. \tag{33}$$

Thus, when $\frac{5}{2}>n>\frac{3}{2}$, the interaction is equivalent to an infinitely rising potential. In general $\bar{F}(\vec{p})$ will have other, shorter-range, terms, but we are looking for confinement, not a detailed fit to the hadronic spectrum. The Fourier transform in (33) is well defined only when used with rapidly converging functions.

When the quarks are very massive, the expansion $w(\vec{p})=m+p^2/2m$ together with $E=2m+\epsilon$ reduces (23) and (24) to the nonrelativistic Schrödinger equation with an infinitely rising potential. In that limit, quark confinement is obvious. However, the legitimacy of the expansion in powers of p^2/m^2 must be justified. Moreover, no such expansion is available for massless gluons or for very light quarks. We explore two complementary approaches to the problem of solving the bound-state equations. The lowest energy levels are calculated variationally.[12] An inner-product operation is defined such that the equations are generated by a variational principle. A set of linearly independent functions with arbitrary coefficients is used to produce a sequence of upper bounds on the energies of the system. The signal of confinement is rapid convergence to a set of discrete energy levels that are both nonzero and unbounded from above. A possible contrary result would be levels which all converge to zero energy. In that case there would be a continuous spectrum rather than the discrete spectrum that is expected from a confining interaction. The semiclassical approximation is used to prove analytically that the spectrum is discrete and unbounded from above for both quarks and gluons. There is quantitative agreement between the variational and semiclassical calculations.

When the constituents are light, they should be well separated in space so that the long-range part of the potential dominates the interaction. Thus, a final check on the consistency of our calculations is an order-of-magnitude comparison of the scale parameter in our linear potential with the values used in more quantitative calculations.

A complete discussion of all possible combinations of quarks or gluons would be overwhelming. Instead, the details of the angular momentum decomposition are relegated to the Appendix. Our variational procedure is sketched for the simplest set of quark equations. When the quark and antiquark are in a spin-0, orbital-angular-momentum-0 state, we have from (A10)

$$ER_+(k)-2w(k)R_-(k)=-4\pi f\int_0^\infty p^2dp\,\bar{F}_0(k,p)R_-(p)\,, \tag{34a}$$

$$ER_-(k)-2w(k)R_+(k)=-4\pi f\int_0^\infty\frac{p^2dp}{w(p)w(k)}[m^2\bar{F}_0(k,p)+kp\bar{F}_1(k,p)]F_+(p)\,. \tag{34b}$$

There is no easy way, based on its definition, to create an inner product for the Tamm-Dancoff wave functions. However, if

$$\Psi=\begin{bmatrix}R_-\\R_+\end{bmatrix}$$

and

$$I=\int_0^\infty k^2dk\,\Psi^\dagger(k)\begin{bmatrix}2w(k)&-E\\-E&2w(k)\end{bmatrix}\Psi(k)-4\pi f\int_0^\infty k^2dk\int_0^\infty p^2dp\,\Psi^\dagger(k)\begin{bmatrix}\bar{F}_0(k,p)&0\\0&\dfrac{m^2\bar{F}_0(k,p)+kp\bar{F}_1(k,p)}{w(k)w(p)}\end{bmatrix}\Psi(p)\,, \tag{35}$$

then variation of I with respect to $\Psi^\dagger(k)$ produces the integral equations in (34). It is possible to use a set of trial wave functions in I and generate a set of upper bounds on the energy E. If the infrared-finite interaction is given by (33), the partial-wave projection of $\bar{F}(\vec{p}-\vec{k})$ is equal to

$$\bar{F}_J(p,k)=-\eta\int_0^\infty r^2dr\,r^{2n-3}j_J(pr)j_J(kr)\,. \tag{36}$$

For massive quarks we choose the variational wave functions,

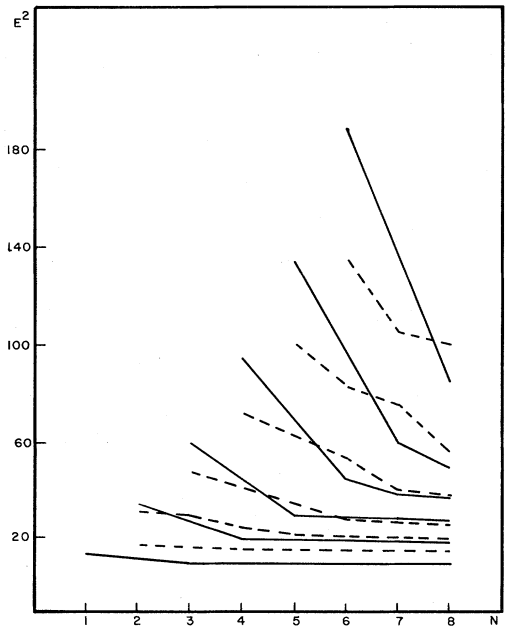

FIG. 1. The energy levels for massive S-wave and P-wave quarks are calculated for increasing N, the dimensionality of the variational wave function. Solid lines are for S waves and the dashed lines are for P waves. The energy is in units of the quark mass and the effective coupling constant is set equal to unity.

$$R_{\pm}(k)=\sum_{n=0}^{N}\frac{C_n^{\pm}}{(1+k^2/m^2)^{n+3/2}}\ , \tag{37}$$

so that the integrals can be evaluated analytically. The formally divergent transform in (36) becomes finite if we insist that all momentum-space integrations are done before the r integration. The expansion constants are fixed by the requirement that the energy is a minimum. In Fig. 1 we plot (energy)^2 versus N, the order of the approximation for $n=2$, a linear potential. Convergence is rapid, and the discrete-energy-level structure is exactly what would be expected in a confining theory.

Confinement means that the quark and antiquark should have a similar set of bound states for any spin and angular momentum configuration. The $L=1$, spin-0 levels are also shown in Fig. 1. Even more interesting is the spectrum for massless quarks. There could be a continuum of states starting at $E=0$ in the absence of a quark mass scale. In fact, for massless particles, the energy scale is set by the mass scale from the confining interaction [see (33)]. The trial wave functions for massless quarks are

$$R_{\pm}(k)=\sum_{n=0}^{N}C_n^{\pm}\left[\frac{\alpha k}{\Lambda}\right]^n e^{-\alpha k/\Lambda}\ , \tag{38}$$

and α is an additional variational parameter. The results for massless quarks appear in Fig. 2 for $n=2$, a linear potential. The convergence is rapid, and there are discrete energy levels comfortably separated from $E=0$.

Except for kinematic differences, the gluon equations are similar to the massless-quark equations. If

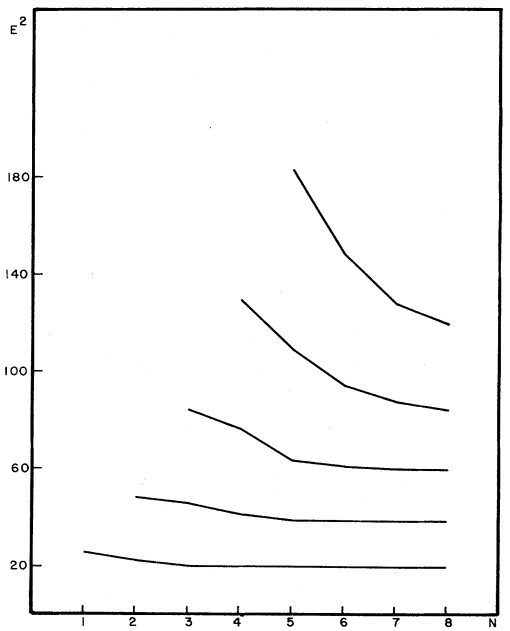

FIG. 2. The energy levels for massless S-wave quarks are calculated for increasing N, the dimensionality of the variational wave functions. The energy is scaled by the coupling constant which has dimensions of (mass)^2.

$$\Psi=\begin{bmatrix}X_+\\X_-\end{bmatrix}\quad\text{with } X_{\pm}=X_1\pm X_2\ ,$$

the variation of

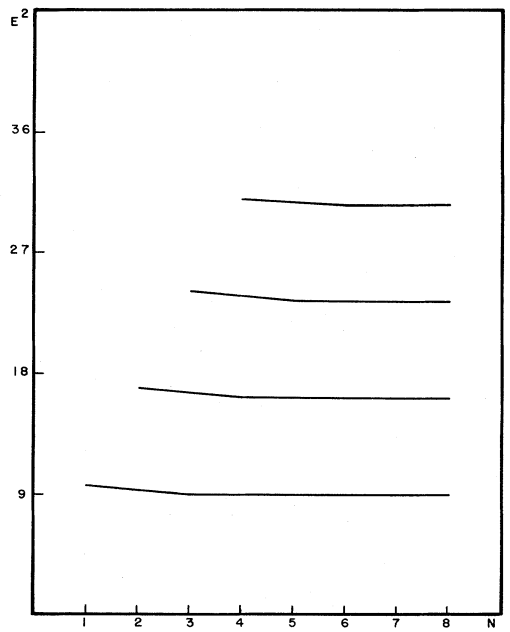

FIG. 3. The energy levels for massless S-wave gluons are calculated for increasing N, the dimensionality of the variational wave function. The energy scale is arbitrary; it is fixed by the dimensional coupling constant. In configuration space the confining interaction is linear.

$$I=\int_0^\infty k^2dk\,\Psi^\dagger(k)\begin{bmatrix}2k & -E\\ -E & 2k\end{bmatrix}\Psi(k)$$

$$+8\pi f'\int_0^\infty k^2dk\int_0^\infty p^2dp\begin{bmatrix}\dfrac{2\bar{F}_0(p,k)+\bar{F}_2(p,k)}{3pk}\end{bmatrix}\Psi^\dagger(k)\begin{bmatrix}\dfrac{p}{k}+\dfrac{k}{p} & 0\\ 0 & 2\end{bmatrix}\Psi(k) \tag{39}$$

with respect to $\Psi^\dagger$ produces the $J=0$ limit of (A19). The variational wave functions are given by (38), and the results appear in Fig. 3. The spectrum is basically the same as for massless quarks.

The calculation of energy levels in the semiclassical limit follows closely the methods developed to study the Schrödinger equation with relativistic kinematics.[13] The apparent complexity of the quark and gluon equations is due to spin. However, the analysis is still quite straightforward. Reference 13 should be consulted for details and for justification of the steps which are outlined here. For quark-antiquark bound states the starting point is (A10) which can be rewritten in the form

$$\left[w(k)^2-\frac{E^2}{4}\right]R_-^J(k)=\pi Ef\int_0^\infty\frac{p^2dp}{w(p)w(k)}[m^2\bar{F}_J(k,p)+kpK_J(k,p)]R_+^J(p)$$

$$+2\pi f\int_0^\infty p^2dp\,w(k)\bar{F}_J(k,p)R_-^J(p)\,, \tag{40a}$$

$$\left[w(k)^2-\frac{E^2}{4}\right]R_+^J(k)=\pi Ef\int_0^\infty p^2dp\,\bar{F}_J(k,p)R_-^J(p)$$

$$+2\pi f\int_0^\infty\frac{p^2dp}{w(p)}[m^2\bar{F}_J(k,p)+kpK_J(k,p)]R_+^J(p)\,, \tag{40b}$$

where $\bar{F}_J(k,p)$ is given by (36) and $K_J(k,p)$ is defined by (A13a). When configuration-space amplitudes are introduced via

$$R_\pm^J(r)=\int_0^\infty p^2dp\,j_J(pr)R_\pm^J(p)\,, \tag{41}$$

the complete configuration-space integral-differential equations are

$$\left[m^2-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}+\frac{J(J+1)}{r^2}-\frac{E^2}{4}\right]R_-^J(r)=-2\eta f\int_0^\infty k^2dk\int_0^\infty R^2dR\,R^{2n-3}\int_0^\infty p^2dp\int_0^\infty r'^2dr'j_J(kr)j_J(pr')$$

$$\times\left[\frac{E}{w(p)w(k)}\left[m^2j_J(kR)j_J(pR)+kp\frac{J+1}{2J+1}j_{J+1}(kR)j_{J+1}(pR)\right.\right.$$

$$\left.+kp\frac{J}{2J+1}j_{J-1}(kR)j_{J-1}(pR)\right]R_+^J(r')$$

$$\left.+2w(k)[j_J(kR)j_J(pR)]R_-^J(r')\right]\,, \tag{42a}$$

$$\left[m^2-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}+\frac{J(J+1)}{r^2}-\frac{E^2}{4}\right]R_+^J(r)=-2\eta f\int_0^\infty k^2dk\int_0^\infty R^2dR\,R^{2n-3}\int_0^\infty p^2dp\int_0^\infty r'^2dr'j_J(kr)j_J(pr')$$

$$\times\left[E[j_J(kR)j_J(pR)]R_-^J(r')\right.$$

$$+\frac{2}{w(p)}\left[m^2j_J(kr)j_J(pR)+kp\frac{J+1}{2J+1}j_{J+1}(kR)j_{J+1}(pR)\right.$$

$$\left.\left.+kp\frac{J}{2J+1}j_{J-1}(kR)j_{J-1}(pR)\right]R_+^J(r')\right]\,. \tag{42b}$$

The semiclassical approximation enters when we write $R^J_{\pm}(r)=\exp[i\sigma_{\pm}(r)]/r$ and replace the spherical Bessel functions by their asymptotic form

$$j_J(kr)\approx\frac{e^{ikr}e^{-i\pi J/2}-e^{-ikr}e^{i\pi J/2}}{2ikr}\ , \tag{43}$$

an approximation that is valid when $kr\gg[J(J+1)]^{1/2}$. The integrals are evaluated in the stationary-phase approximation. In each term of the integrals in (42) there is a product of four spherical Bessel functions. When each of the four is approximated by (43), there is a sum of sixteen phase factors, the variable parts of which have the form

$$\phi=\epsilon_1 kr+\epsilon_2 kR+\epsilon_3 pR+\epsilon_4 pr'+\sigma(r')\ , \tag{44}$$

and each $\epsilon_i=\pm1$. The requirement that $\sigma'=d\sigma/dr$ be positive and ϕ be stationary as a function of the four variables k,p,R,r' fixes $\epsilon_1=-\epsilon_2=\epsilon_3=-\epsilon_4=1$, $r'=R=r$, and $p=k=\sigma'(r)$. The integrals are evaluated around the stationary point to generate

$$\left[m^2+\sigma'^2_{-}-i\sigma''_{-}+\frac{J(J+1)}{r^2}-\frac{E^2}{4}\right]\frac{e^{i\sigma_-}}{r}$$

$$=-\frac{E}{2}U(r)\frac{e^{i\sigma_+}}{r}-(m^2+\sigma'^2_{-})^{1/2}U(r)\frac{e^{i\sigma_-}}{r}\ , \tag{45a}$$

$$\left[m^2+\sigma'^2_{+}-i\sigma''_{+}+\frac{J(J+1)}{r^2}-\frac{E^2}{4}\right]\frac{e^{i\sigma_+}}{r}$$

$$=-\frac{E}{2}U(r)\frac{e^{i\sigma_-}}{r}-(m^2+\sigma'^2_{+})^{1/2}U(r)\frac{e^{i\sigma_+}}{r}\ , \tag{45b}$$

where $U(r)=\pi^2\eta f r^{2n-3}$. In the semiclassical approximation $\sigma'^2\gg\sigma''$. In addition to be consistent with (43), we drop $J(J+1)/r^2$ on the LHS of Eqs. (45), although it was shown in Ref. 13 that a better approximation is obtained if instead $J(J+1)/r^2$ is added to σ'^2 on the RHS of Eqs. (45). The solution of Eqs. (45) has $\sigma_+=\sigma_-=\sigma$, and

$$(m^2+\sigma'^2)+U(r)(m^2+\sigma'^2)^{1/2}=\frac{E^2}{4}-\frac{E}{2}U(r)\ . \tag{46}$$

From this expression we obtain

$$\sigma'=\left[\frac{[E-2U(r)]^2}{4}-m^2\right]^{1/2}\ . \tag{47}$$

The quantization condition for energy levels is

$$\int_0^{r_+}\sigma'(r)dr=\pi(M+\tfrac{1}{2})\ , \tag{48}$$

where r_+ is the turning point defined by $\sigma'(r_+)=0$, $U(r_+)=E/2-m$. Since we are interested in both E and M large, the mass factor can be dropped. For a linear interaction with mass $m=0$, the turning point is at $r_+=E/2\pi^2\eta f$, and (48) becomes

$$\tfrac{1}{2}\int_0^{r_+}(E-2\pi^2\eta f r)dr=\frac{E^2}{8\pi^2\eta f}=\pi M$$

or

$$E^2_{q\bar{q}}=8\pi^3\eta f M\ . \tag{49}$$

The curves in Fig. 1 are plotted for $2\pi^2\eta f=1$ and should, for large M, match $E^2=4\pi M$. Remembering that M starts with $M=0$, we find that the agreement between the variational and semiclassical calculations is better than 1% for s waves at $M=3,4$. The variational calculation in Fig. 1 is not accurate for larger M due to the limit on the number of parameters in the variational wave function. The curves for massless quarks in Fig. 2 are for $4\pi\eta f=1$ and the agreement with (49) for the spacing is good.

The semiclassical method can be extended to the quark-antiquark states described by (A11) and (A12). The asymptotic energy spectrum in both cases is given by (49). The energies of the highly excited states are independent of spin as is expected in a confining theory.

The structure of the gluon equations is nearly identical to that of the quark-antiquark equations. In the semiclassical limit the four coupled equations in (A19) separate into two identical sets of coupled pairs of equations. The kinematic factors which distinguish gluons from quarks drop out in the stationary-phase approximation since $p=k=\sigma'$. The energy eigenvalues are given by (48) with σ' fixed by (47) with $m=0$. $U(r)$ is scaled by a different coupling constant. When $M\to\infty$ and $E\to\infty$

$$E_{gg}{}^2=32\pi^3\eta f' M=\tfrac{9}{4}8\pi^3\eta f M \tag{50}$$

and $E_{gg}=3E_{q\bar{q}}/2$ in this limit. The curves in Fig. 3 correspond to $4\pi\eta f=0.1$.

No attempt has been made to fit actual hadronic or glueball states. The emphasis of this paper is on the confinement process, and it is clear that the equations produce confinement. There are other, less singular, terms in $F(\vec{p})$ which must be included before one seeks to compare the details of the spectrum with the real world. However, there are comparisons that can be made. If the lowest quark-antiquark bound-state energy is adjusted to be on the order of 1 GeV, then the strength of the linear potential is fixed and can be compared with the linear potential used in other calculations. If that strength is denoted by β, we obtain $\beta=g^2\Lambda^2/4\pi=5.2\times10^{-2}$ GeV.^{2} In the nonrelativistic limit of (23) and (24) the coefficient of the linear potential is fixed such that $V(r)=\pi\beta r/2$. Eichten, Gottfried, Kinoshita, Lane, and Yan[14] in their charmonium fits use $\beta=1.16\times10^{-1}$ GeV^2, a factor of 2.2 larger. If we use the r^{-1} potential from Ref. 14 we can extract a value of $g^2/4\pi=0.39$. Combining this number with the value of β, we calculate the scale mass Λ to be on the order of 0.65 GeV. The final comparison we can make is to note that the ratio of the mass of the lightest pseudoscalar meson to the lightest glueball is 0.6. All of these numbers confirm the fact that our picture of confinement is phenomenologically reasonable.

V. CONCLUSIONS

The modified Tamm-Dancoff method has been used to study bound states in quantum chromodynamics. Starting with the QCD Hamiltonian in the Coulomb gauge, we derived homogeneous integral equations that describe both quark-antiquark and gluon-gluon color-singlet bound states. In the course of the derivation several approximations were made: (i) The momentum-space equations were

truncated to the two-particle sector, (ii) the modified Coulomb operator was replaced by its vacuum expectation value, (iii) higher-order interactions were dropped, and (iv) the vacuum expectation value was identified as the source of the confining potential, and other terms in the Hamiltonian were dropped. Except for the replacement of the operator by its vacuum expectation value, these approximations are, in principle, subject to quantitative justification. Confinement is produced by an assumed infrared singularity in the interaction, a singularity which produces infinite energy unless the constituent particles are in a color-singlet state. In that case the singularity cancels between the scattering and the self-energy terms in the bound-state equations. Both a variational calculation and a semiclassical calculation demonstrated the existence of finite-energy bound states. Hence, the hadronic spectrum was connected directly to the properties of the QCD vacuum through the vacuum expectation value of the modified Coulomb interaction.

This calculation is not a true deviation of confinement. The infrared singularity was postulated. Rather, this work should be viewed as establishing another constraint on the QCD vacuum state. If the vacuum expectation value of the modified Coulomb interaction is responsible for confinement in the Coulomb gauge, then the fact of confinement implies that the vacuum state must be such as to create that infrared singularity. The remaining step in proving confinement is the demonstration that the vacuum has that property. Any sufficiently explicit model of the vacuum can be tested against this criteria.[4] If the singularity is there, confinement is automatic.

One extension of this work is to use it as probe of the vacuum. For example, if the vacuum expectation value of the product of two-gluon fields is

$$\langle A^a_{+i}(\vec{p})A^b_{+j}(\vec{q})\rangle=\delta_{ab}\left[\delta_{ij}-\frac{p_ip_j}{p^2}\right]\delta^3(\vec{p}+\vec{q})A(\vec{p}) , \tag{51}$$

then it is possible to derive a nonlinear integral equation for the function $D_{ab}(\vec{p},\vec{k})$ in (12). The infrared behavior of D is controlled by the infrared properties of $A(\vec{p})$.

A second extension of this work is to use the bound-state equations in a unified calculation of the hadronic spectrum for both light and heavy quarks as well as for gluons. In this program the infrared-singular part of the interaction would need to be augmented with nonsingular terms. In addition, one must have a quantitative evaluation of the errors in the approximations used to derive the bound-state equations.

ACKNOWLEDGMENTS

We have benefited from discussions with E. Haqq, J. Donoghue, and G. Knightly. One of us (A.R.S.) would like to thank the Theory Group at SLAC for its hospitality during the early stages of this work. This research was supported in part by a grant from the National Science Foundation.

APPENDIX

1. Quarks

The most general 4×4 spinor amplitude which satisfies $\Lambda_+(\vec{k})\Psi_A(\vec{k})\Lambda_-(-\vec{k})=\Psi_A(\vec{k})$ has the form[9]

$$\Psi_A=\frac{1}{m}\begin{bmatrix}\vec{k}\cdot\vec{G}_A+\vec{\sigma}\cdot\vec{k}R_A-i\vec{\sigma}\cdot(\vec{k}\times\vec{F}_A) & (w+m)[R_A+\vec{\sigma}\cdot(\vec{G}_A+\vec{F}_A)]\\ (w-m)[R_A+\vec{\sigma}\cdot(\vec{G}_A-\vec{F}_A)] & \vec{k}\cdot\vec{G}_A+\sigma\cdot\vec{k}R_A+i\vec{\sigma}\cdot(\vec{k}\times\vec{F}_A)\end{bmatrix} . \tag{A1}$$

The vector function $\vec{G}_A$ is parallel to $\vec{k}$ ($G_A=\vec{k}\vec{k}\cdot\vec{G}_A/k^2$), and $\vec{F}_A$ is perpendicular to $\vec{k}$ ($\vec{k}\cdot\vec{F}=0$). The corresponding decomposition of $\Psi_B(\vec{k})$, $\Lambda(\vec{k})\Psi_B(\vec{k})\Lambda_+(-\vec{k})=\Psi_B(\vec{k})$, is obtained from (A1) by the change of w to $-w$:

$$\Psi_B=\frac{1}{m}\begin{bmatrix}\vec{k}\cdot\vec{G}+\vec{\sigma}\cdot\vec{k}R_B-i\vec{\sigma}\cdot(\vec{k}\times\vec{F}_B) & -(w-m)[R_B+\vec{\sigma}\cdot(\vec{G}_B+\vec{F}_B)]\\ -(w+m)[R_B+\vec{\sigma}\cdot(\vec{G}_B-\vec{F}_B)] & \vec{k}\cdot\vec{G}_B+\vec{\sigma}\cdot\vec{k}R_B+i\vec{\sigma}\cdot(\vec{k}\times\vec{F}_B)\end{bmatrix} . \tag{A2}$$

Equations (A1) and (A2) are used in (23) and (24) to produce a set of scalar and vector integral equations. With the definitions

$$I\{R(\vec{p})\}=f\int\frac{d^3p\,\bar{F}(\vec{p}-\vec{k})}{w(\vec{k})w(\vec{p})}R(\vec{p}) \tag{A3}$$

and

$$R_\pm(\vec{p})=R_A(\vec{p})\pm R_B(\vec{p}) , \tag{A4}$$

the 4×4 spinor equations reduce to the set

$$ER_+-2wR_-=-I\{w(\vec{k})w(\vec{p})R_-(\vec{p})\} , \tag{A5a}$$

$$ER_--2wR_+=-I\{(m^2+\vec{k}\cdot\vec{p})R_+(\vec{p})\} , \tag{A5b}$$

$$E\vec{F}_- - 2w\vec{F}_+ = -I\left[m^2\left[\vec{F}_+(\vec{p}) - \frac{\vec{k}\vec{k}}{k^2}\cdot\vec{F}_+(\vec{p})\right] + [\vec{k}\cdot\vec{p}\vec{F}_+(\vec{p}) - \vec{p}\vec{k}\cdot\vec{F}_+(\vec{p})] + mw(\vec{p})\left[\vec{G}_-(\vec{p}) - \frac{\vec{k}\vec{k}}{k^2}\cdot\vec{G}_-(\vec{p})\right]\right], \tag{A6a}$$

$$E\vec{F}_+ - 2w\vec{F}_- = -I\left[w(\vec{k})w(\vec{p})\left[\vec{F}_-(\vec{p}) - \frac{\vec{k}\vec{k}}{k^2}\cdot\vec{F}_-(p)\right] + mw(\vec{k})\left[\vec{G}_+(\vec{p}) - \frac{\vec{k}\vec{k}}{k^2}\cdot\vec{G}_+(\vec{p})\right]\right], \tag{A6b}$$

$$E\vec{G}_- - 2w\vec{G}_+ = -\vec{k}I\left[\vec{p}\cdot\vec{G}_+(\vec{p}) + \frac{m^2}{k^2}\vec{k}\cdot\vec{G}_+(\vec{p}) + \frac{mw(\vec{p})}{k^2}\vec{k}\cdot\vec{F}_-(\vec{p})\right], \tag{A6c}$$

$$E\vec{G}_+ - 2w\vec{G}_- = -\vec{k}I\left[\frac{w(\vec{k})}{k^2}[w(p)\vec{k}\cdot\vec{G}_-(\vec{p}) + m\vec{k}\cdot\vec{F}_+(\vec{p})]\right]. \tag{A6d}$$

The angular momentum decomposition of these equations begins with

$$\bar{F}(\vec{p}-\vec{k}) = \sum_J (2J+1)\bar{F}_J(k,p)P_J(\cos\theta_{pk}), \tag{A7}$$

and

$$R_\pm(\vec{k}) = \sum_J R^J_\pm(k)Y^{JM}(\Omega_k), \tag{A8a}$$

$$\vec{G}_\pm(\vec{k}) = \sum_J G^J_\pm(k)\vec{Y}^{JM}_{||}(\Omega_k), \tag{A8b}$$

$$\vec{F}_\pm(\vec{k}) = \sum_J [F^{0J}_\pm(k)\vec{Y}^{JJM}(\Omega_k) + F^{1J}_\pm(k)\vec{Y}^{JM}_\perp(\Omega_k)]. \tag{A8c}$$

$\vec{Y}^{JM}_{||}$ and $\vec{Y}^{JM}_\perp$ are the linear combinations of the vector spherical harmonics that are parallel and perpendicular to $\vec{k}$:[15]

$$\vec{Y}^{JM}_{||} = -\left[\frac{J+1}{2J+1}\right]^{1/2}\vec{Y}^{JJ+1M} + \left[\frac{J}{2J+1}\right]^{1/2}\vec{Y}^{JJ-1M}, \tag{A9a}$$

$$\vec{Y}^{JM}_\perp = \left[\frac{J}{2J+1}\right]^{1/2}\vec{Y}^{JJ+1M} + \left[\frac{J+1}{2J+1}\right]^{1/2}\vec{Y}^{JJ-1M}. \tag{A9b}$$

The coupled partial-wave equations are

$$ER^J_+(k) - 2w(k)R^J_-(k) = -4\pi f\int_0^\infty p^2dp\,\bar{F}_J(k,p)R^J_-(p), \tag{A10a}$$

$$ER^J_-(k) - 2w(k)R^J_+(k) = -4\pi f\int_0^\infty \frac{p^2dp}{w(p)w(k)}[m^2\bar{F}_J(k,p) + kpK_J(k,p)]R^J_+(p), \tag{A10b}$$

$$EF^{0J}_+(k) - 2w(k)F^{0J}_-(k) = -4\pi f\int_0^\infty p^2dp\,\bar{F}_J(k,p)F^{0J}_-(p), \tag{A11a}$$

$$EF^{0J}_-(k) - 2w(k)F^{0J}_+(k) = -4\pi f\int_0^\infty \frac{p^2dp}{w(k)w(p)}[m^2\bar{F}_J(k,p) + kpL_J(k,p)]F^{0J}_+(p), \tag{A11b}$$

$$EF^{1J}_+(k) - 2w(k)F^{1J}_-(k) = -4\pi f\int_0^\infty p^2dp\left[L_J(k,p)F^{1J}_-(p) - \frac{m}{w(p)}M_J(k,p)G^J_+(p)\right], \tag{A12a}$$

$$EF^{1J}_-(k) - 2w(k)F^{1J}_+(k) = -4\pi f\int_0^\infty p^2dp\left[\frac{m^2L_J(k,p) + kp\bar{F}_J(k,p)}{w(k)w(p)}F^{1J}_+(p) - \frac{m}{w(k)}M_J(k,p)G^J_-(p)\right], \tag{A12b}$$

$$EG^J_+(k) - 2w(k)G^J_-(k) = -4\pi f\int_0^\infty p^2dp\left[K_J(k,p)G^J_-(p) - \frac{m}{w(p)}M_J(k,p)F^{1J}_+(p)\right], \tag{A12c}$$

$$EG^J_-(k) - 2w(k)G^J_+(k) = -4\pi f\int_0^\infty p^2dp\left[\frac{m^2K_J(p,k) + kp\bar{F}_J(k,p)}{w(k)w(p)}G^J_+(p) - \frac{m}{w(k)}M_J(k,p)F^{1J}_-(p)\right]. \tag{A12d}$$

The kernels K_J, L_J, and M_J are defined by

$$K_J=\frac{(J+1)\bar{F}_{J+1}+J\bar{F}_{J-1}}{2J+1}\ , \tag{A13a}$$

$$L_J=\frac{J\bar{F}_{J+1}+(J+1)\bar{F}_{J-1}}{2J+1}\ , \tag{A13b}$$

$$M_J=\frac{[J(J+1)]^{1/2}}{2J+1}(\bar{F}_{J+1}-\bar{F}_{J-1})\ . \tag{A13c}$$

Equations (A10), (A12c), and (A12d) are valid for $J\geq 0$, and the others require $J\geq 1$. When the quarks are massless, the F^{1J} and G^J amplitudes decouple, and there are just two different types of equations. Solutions to (A10) have space parity $P=(-1)^{J+1}$ and charge parity $C=(-1)^J$; the quarks are in a singlet-spin state. Quarks in the coupled triplet state, (A11), have $P=(-1)^{J+1}$ and $C=(-1)^{J+1}$. In the coupled triplet state, (A12), $P=(-1)^J$ and $C=(-1)^J$.

2. Gluons

The gluon wave functions in (29) and (30) have two vector indices. A complete angular momentum expansion of $\Psi_{\alpha\beta}$ has the form

$$\Psi_{\alpha\beta}=\sum_J[\,\delta_{\alpha\beta}A^JY^{JM}+\epsilon_{\alpha\beta\gamma}(B^JY_\gamma^{JJ+1M}+C^JY_\gamma^{JJM}+D^JY_\gamma^{JJ-1M})+E^JY_{\alpha\beta}^{JJ+2M}+F^JY_{\alpha\beta}^{JJ+1M}+G^JY_{\alpha\beta}^{JJM}$$

$$+H^JY_{\alpha\beta}^{JJ-1M}+I^JY_{\alpha\beta}^{JJ-2M}]\ . \tag{A14}$$

Each amplitude $A^J,B^J,\ldots$ is a function of k, and each spherical harmonic is a function of Ω_k. The functions $Y_{\alpha\beta}^{JLM}$ are tensor spherical harmonics constructed in a manner analogous to the vector spherical harmonics. They are symmetric and traceless in the $\alpha\beta$ indices. The transversality conditions $k_\alpha\Psi_{\alpha\beta}=\Psi_{\beta\alpha}k_\alpha=0$ constrain the nine amplitudes. We have

$$\begin{aligned}
&C^J=0\ ,\\
&-\left[\frac{J+1}{2J+1}\right]^{1/2}A^J-\left[\frac{J+2}{2J+3}\right]^{1/2}E^J\\
&\qquad+\left[\frac{J(2J-1)}{6(2J+1)(2J+3)}\right]^{1/2}G^J=0\ ,\\
&\left[\frac{J}{2J+1}\right]^{1/2}A^J+\left[\frac{J-1}{2J-1}\right]^{1/2}I^J\\
&\qquad-\left[\frac{(J+1)(2J+3)}{6(2J-1)(2J+1)}\right]^{1/2}G^J=0\ ,\\
&\left[\frac{J}{2J+1}\right]^{1/2}B^J+\left[\frac{J+1}{2J+1}\right]^{1/2}D^J=0\ ,\\
&\left[\frac{J+2}{2J+1}\right]^{1/2}F^J-\left[\frac{J-1}{2J+1}\right]^{1/2}H^J=0\ .
\end{aligned} \tag{A15}$$

The defining equations for $\Psi_{1\alpha\beta}$ and $\Psi_{2\alpha\beta}$ imply the symmetry property

$$\Psi_{\alpha\beta}(\vec{k})=\Psi_{\beta\alpha}(-\vec{k})\ . \tag{A16}$$

When J is odd, only F^J and H^J are nonzero. When J is even, there are six nonzero amplitudes and three constraint equations. The antisymmetric amplitudes B and D from (A14) combine to yield

$$\Psi'_{\alpha\beta}(\vec{k})=\epsilon_{\alpha\beta\gamma}\hat{k}_\gamma\sum_J B'^J(k)Y^{JM}(\Omega_k)\ . \tag{A17}$$

Another set of even-J amplitudes has the form

$$\Psi''_{\alpha\beta}(\vec{k})=(\delta_{\alpha\beta}-\hat{k}_\alpha\hat{k}_\beta)\sum_J A'^J(k)Y^{JM}(\Omega_k)\ . \tag{A18}$$

Except for $J=0$, A'^J is coupled to another even-J amplitude.

When the partial-wave expansion for $\Psi_{\alpha\beta}$ is introduced into (29) and (30), a set of coupled one-dimensional integral equations results. For even J, with $\rho_\pm=(p\pm k)^2/pk$ and $I'[g(p)]=4\pi f'\int_0^\infty p^2dp$,

$$\begin{aligned}
&(E-2k)X_1^J(k)=-I'\{U_J(k,p)[\rho_+X_1^J(p)+\rho_-X_2^J(p)]+S_J(k,p)[\rho_+Y_1^J(p)+\rho_-Y_2^J(p)]\}\ ,\\
&(E+2k)X_2^J(k)=I'\{U_J(k,P)[\rho_+X_2^J(p)+\rho_-X_1^J(p)]+S_J(k,p)[\rho_+Y_2^J(p)+\rho_-Y_1^J(p)]\}\ ,\\
&(E-2k)Y_1^J(k)=-I'\{V_J(k,p)[\rho_+Y_1^J(p)+\rho_-Y_2^J(p)]+T_J(k,p)[\rho_+X_1^J(p)+\rho_-X_2^J(p)]\}\ ,\\
&(E+2k)Y_2^J(k)=I'\{V_J(k,p)[\rho_+Y_2^J(p)+\rho_-Y_1^J(p)]+T_J(k,p)[\rho_+X_2^J(p)+\rho_-X_1^J(p)]\}\ ,
\end{aligned} \tag{A19}$$

with

$$U_J=\frac{(J+2)(3J+1)\bar{F}_J+(J^2+J+1)\bar{F}_{J+2}}{(2J+1)(2J+3)}\,,$$
$$V_J=\frac{(J-1)(3J+2)\bar{F}_J+(J^2+J+1)\bar{F}_{J-2}}{(2J+1)(2J-1)}\,,$$
$$S_J=\omega(\bar{F}_{J+2}-\bar{F}_J)\,, \tag{A20}$$
$$T_J=\omega(\bar{F}_{J-2}-\bar{F}_J)\,,$$

and

$$\omega=\left[\frac{(J-1)J(J+1)(J+2)}{(2J-1)(2J+1)^2(2J+3)}\right]^{1/2}.$$

A^J, G^J, E^J, and I^J are linear functions of X^J and Y^J that satisfy the constraint equations

$$A^J=-\frac{1}{3}\left[\frac{(J+1)(J+2)}{(2J+1)(2J+3)}\right]^{1/2}X^J-\frac{1}{3}\left[\frac{J(J-1)}{(2J-1)(2J+1)}\right]^{1/2}Y^J,$$
$$E^J=\frac{J^2+J+1}{(2J+1)(2J+3)}X^J+\omega Y^J,$$
$$G^J=\left[\frac{J(J+2)(2J-1)}{6(2J+1)(2J+3)^2}\right]^{1/2}\frac{2(J-2)}{2J-1}X^J+\left[\frac{(J-1)(J+1)(2J+3)}{6(2J+1)(2J-1)^2}\right]^{1/2}\frac{2(J+3)}{2J+3}Y^J, \tag{A21}$$
$$I^J=\omega X^J+\frac{J^2+J+1}{(2J+1)(2J-1)}Y^J\,.$$

When $J=0$, the Y^J amplitudes vanish and (A19) reduce to two coupled equations.

A second set of even-J equations are

$$(E-2k)B_1^J(k)=-I'\{K_J(k,p)[\rho_+B_1^J(p)+\rho_-B_2^J(p)]\}\,,$$
$$(E+2k)B_2^J(k)=I'\{K_J(k,p)[\rho_+B_2^J(p)+\rho_-B_1^J(p)]\}\,, \tag{A22}$$

and $K_J(k,p)$ is defined by (A13a). Finally there is a set of odd-J equations, starting with $J=3$:

$$(E-2k)H_1^J(k)=-I'\{K_J'(k,p)[\rho_+H_1^J(p)+\rho_-H_2^J(p)]\}\,,$$
$$(E+2k)H_2^J(k)=I'\{K_J'(k,p)[\rho_+H_2^J(p)+\rho_-H_1^J(p)]\}\,, \tag{A23}$$

where

$$K_J'(k,p)=\frac{(J-1)\bar{F}_{J+1}+(J+2)\bar{F}_{J-1}}{2J+1}\,.$$

[1]A. H. Mueller, Phys. Rep. 73, 237 (1981).

[2]B. Berg and A. Billoire, Phys. Lett. 114B, 324 (1982); K. Ishikawa, G. Schierholz, and M. Teper, *ibid.* 110B, 399 (1982); G. Munster, Nucl. Phys. B205, 648 (1982); M. Creutz, Phys. Rev. D 21, 2308 (1980).

[3]G. 't Hooft, Nucl. Phys. B138, 1 (1978).

[4]H. Pagels and E. Tomboulis, Nucl. Phys. B143, 485 (1978); S. Adler, Phys. Rev. D 23, 2905 (1981); J. Ambjorn and P. Olesen, Nucl. Phys. B170, 60 (1980); H. Nielsen and P. Olesen, *ibid.* B160, 380 (1979); T. Hansson, K. Johnson, and C. Peterson, Phys. Rev. D 26, 2069 (1982).

[5]I. Tamm, J. Phys. (Moscow) 9, 449 (1945); S. M. Dancoff, Phys. Rev. 78, 382 (1950).

[6]F. Dyson, Phys. Rev. 90, 994 (1953); 91, 1543 (1953).

[7]E. Haqq and H. Suura, Phys. Rev. D 24, 2952 (1981); H. Suura, Phys. Rev. Lett. 44, 1319 (1980). These authors approach the quark problem in the temporal gauge, in configuration space, and automatically get confinement from a flux string that connects the two particles.

[8]F. L. Feinberg, Phys. Rev. D 17, 2659 (1978).

[9]A. R. Swift and F. Roig, Phys. Rev. D 18, 1306 (1978); J. I. Donna and A. R. Swift, *ibid.* 18, 1320 (1978).

[10]Renormalization is discussed by J. C. Taylor, Phys. Rev. 95, 1313 (1953).

[11]J. F. Donoghue, K. Johnson, and B. A. Li, Phys. Lett. 99B, 416 (1980); J. F. Donoghue, in *Particles and Fields—1981: Testing the Standard Model,* proceedings of the meeting of the division of Particles and Fields of the APS, Santa Cruz, California, edited by C. A. Heusch and W. T. Kirk (AIP, New York, 1982), p. 97.

[12]B. L. Moiseiwitsch, *Variational Principles* (Interscience, New York, 1966).

[13]P. Cea, G. Nardulli, and G. Paisano, Phys. Rev. D 28, 2291 (1983).

[14]E. Eichten, K. Gottfried, T. Kinoshita, K. Lane, and T. Yan, Phys. Rev. D 21, 203 (1980).

[15]A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).