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Color screening in classical Yang-Mills theories with sources

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We show that color screening of external sources by Yang-Mills fields is gauge independent, and a sufficient condition for color screening to occur is given. A gauge-invariant conserved total color of the system is constructed.

I. INTRODUCTION

In the past few years there has been some interest in the problems of the Yang-Mills (YM) fields interacting with external sources.[1] The main motivation is that the insights and experiences gained at the classical level will illuminate our understanding of the fully quantized YM theories, particularly the nonperturbative aspects.[2] Color screening is one of those properties that can be envisaged at the classical level. Recently, questions have been raised on whether color screening is a gauge artifact[3] and whether total-screening solutions found previously[4,5] are really completely screening.[6,7] The purpose of this paper is to point out that color screening is complete and gauge independent, at least at the classical level. A sufficient condition for the YM fields to screen external sources is explicitly stated.

The crux of the color-confinement problem is the definition of the total color and what one means by color screening. For the non-Abelian gauge field interacting with an external source, the Noether charge due to the global symmetry ceases to be a conserved quantity when the symmetry becomes localized. In contrast, the total electric charge in the Abelian case is always conserved whether the U(1) symmetry is global or local. This is because the external current j_μ is an invariant irrespective of whether U(1) is global or local, so that one always has $\partial^\mu j_\mu=0$. Thus when the non-Abelian symmetry is localized, the Noether charge associated with the global symmetry can remain conserved only if we restrict the gauge transformations to a specified class, that is, the gauge transformation must be independent of x_μ at large distances. This is discussed in the next section. On the other hand, one can introduce a color direction in the internal group space at each space-time point so that a meaningful total color charge can be defined. We present this point of view in Sec. III. Once the total color of the whole system (external sources plus the YM fields) and the total color of the external source have been clarified, it is a simple matter to determine whether a YM configuration can completely screen the external source in a gauge-independent manner.

II. NOETHER COLOR

In the presence of an external source current j_μ^a, the SU(2) Yang-Mills (YM) equations are[4]

$$D_\mu F^{\mu\nu}=j^\nu\ , \tag{1a}$$

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu a^\mu+[A^\mu,A^\nu]\ , \tag{1b}$$

$$A_\mu=A_\mu^a(\sigma^a/2i)\ , \tag{1c}$$

where σ^a are the Pauli matrices and our metric is $g_{ii}=-g_{00}=1$. The external current j_μ is gauge-covariantly conserved,

$$D^\mu j_\mu=0\ , \tag{2}$$

and for static sources, $j_i^a=0$, it then follows that

$$[A^0,j_0]=0\ . \tag{3}$$

The above equations of motion can be derived from the Lagrangian density

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu}-j_\mu^a A_a^\mu\ . \tag{4}$$

Under the gauge transformations

$$U(x)=\exp[-i\omega^a(x)\sigma^a/2]\ , \tag{5}$$

we have

$$A_\mu\to UA_\mu U^{-1}-\partial_\mu UU^{-1}\ , \tag{6a}$$

$$F_{\mu\nu}\to UF_{\mu\nu}U^{-1}\ , \tag{6b}$$

$$j_\mu\to Uj_\mu U^{-1}\ . \tag{6c}$$

Although the equations of motion are gauge covariant, the Lagrangian density (4) is in general not gauge invariant. However, it is still globally invariant and one can construct the Noether current[4]

$$J^\nu=j^\nu-[A_\mu,F^{\mu\nu}]\ , \tag{7a}$$

$$\partial_\nu J^\nu=0\ . \tag{7b}$$

Expression (7a) indicates that the contributions to the total color current comes from the external color current as well as from the YM field. It can also be derived directly from the Lagrangian density (4) by

$$J^\nu=\frac{\partial\mathscr{L}}{\partial A_\nu}\ . \tag{8}$$

The total color charges arising from Eq. (7) are

$$I=I^a(\sigma^a/2i)=\int d^3x\, J^0(x) =\int_{\text{surface at }\infty} F^{i0}n_i dS\ , \tag{9}$$

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which can also be written as

$$I=I_{\text{source}}+I_{\text{field}} \ , \tag{10a}$$

$$I_{\text{source}}=\int d^3x\, j^0 \ , \tag{10b}$$

$$I_{\text{field}}=\int d^3x[A_i,F^{i0}] \ . \tag{10c}$$

Under global SU(2) transformations, the total Noether charge is conserved provided J_i vanishes fast enough at large distances. However, under local gauge transformations, I_a is not gauge covariant unless the gauge transformations are restricted such that for $r=(x_ix^i)^{1/2}\to\infty$,

$$U(x_i,x_0)\to U(x_0) \ . \tag{11}$$

Condition (11) is not enough to ensure that the conservation of I be gauge independent. If we furthermore impose the strong boundary condition[8] on $U(x_\mu)$ such that, as $r\to\infty$,

$$U(x_i,x^0)\to U_c \text{ (constant matrix)} \ , \tag{12}$$

it follows that the total color is conserved in any gauge choice. Condition (12) essentially compactifies R^3 to S^3 and it does not permit monopole solution. It also means at large r, A_μ transforms covariantly and hence so does J_μ.

Thus under the gauge transformations which satisfy the strong boundary condition (12), I_a is gauge covariant and a gauge-invariant conserved charge can be defined by

$$\bar{Q}=(I_aI_a)^{1/2} \ , \tag{13}$$

which is the total color of the system consisting of the external source and the YM fields. In order to render the concept of "color screening" more precise, we need to have a gauge-invariant specification of the external source. Let $\eta^a(x)$ be a unit vector in the internal group space; for instance, when the source is static, one can choose $\eta^a(x)$ to be along the common direction of A_0 and j_0 by virtue of Eq. (3). Then a gauge-invariant specification of the total external charge is

$$\bar{Q}_s=\int d^3x\, j_0^a\eta^a \ , \tag{14}$$

which is different from I_{source}, the latter being gauge dependent and can be gauge transformed to zero.[7]

We say that the external source is completely screened by the YM fields if $\bar{Q}=0$ and $\bar{Q}_s\neq 0$. This is a gauge-independent statement and the unnecessary confusion[3,6] in discussing color-screening problems is avoided. By virtue of expression (9), we see that color screening will result if at large distances the electric field $E^i=F^{i0}$ behaves as

$$\lim_{r\to\infty} r^{2+\epsilon}E^i=0 \ . \tag{15}$$

However, this result is valid only under the restricted gauge transformations of Eq. (12).

We now discuss the total-screening solution of Sikivie and Weiss.[4] In the Abelian gauge frame,

$$A_0^a=0 \ , \tag{16a}$$

$$A_i^a=E_i^at-\delta^{a1}\partial_i[2\pi nh(r)] \ , \tag{16b}$$

$$E_i^a=\frac{Q}{4\pi}\frac{x_i}{r^3}\frac{1}{2\pi n}\{\delta^{a2}[1-\cos(2\pi nh(r))] -\delta^{a3}\sin(2\pi nh(r))\} \ , \tag{16c}$$

$$j_\mu^a=\delta_\mu^0 q^a \ , \quad q^a=q\delta^{a3}=\frac{Q}{4\pi}\left[\frac{-1}{r^2}\frac{dh(r)}{dr}\right]\delta^{a3} \ , \tag{16d}$$

where

$$h(r)=\frac{1}{Q}\int_r^\infty r\pi r'^2 q(r')dr' \ . \tag{16e}$$

If we set $\eta^a=\delta^{a3}$, then the gauge-invariant measure of this external source is

$$\bar{Q}_s=\frac{Q}{4\pi} \ , \tag{17}$$

whereas the total non-Abelian charge $\bar{Q}$ vanishes. Hence color screening definitely takes place and is gauge independent under the restricted gauge transformations. In Ref. 3, a gauge transformation which satisfies condition (12),

$$U=\exp\left[2\pi nh(r)\frac{\sigma^1}{2i}\right] \ , \tag{18}$$

is employed to convert solution (16) into the "physical gauge" frame

$$A_0'^a=0 \ , \tag{19a}$$

$$A_i'^a=E_i'^at \ , \tag{19b}$$

$$E_i'^a=\frac{Q}{4\pi}\frac{x_i}{r^3}\frac{1}{2\pi n}\{\delta^{a2}[\cos(2\pi nh(r))-1] -\delta^{a3}\sin(2\pi nh(r))\} \ , \tag{19c}$$

$$q'^a=\frac{Q}{4\pi}\left[-\frac{1}{r^2}\frac{dh(r)}{dr}\right][\delta^{a2}\sin(2\pi nh(r)) +\delta^{a3}\cos(2\pi nh(r))] \ . \tag{19d}$$

In this physical gauge frame,

$$I_{\text{source}}^a=\int d^3x\, q'^a \tag{20}$$

vanishes and the author of Ref. 3 then claimed that the total-screening solution is a solution to the color-singlet source problem and hence questioned the meaning of color screening. In our opinion this is merely an interpretation of the screening solution in a particular gauge frame. If we follow definition (14), in the physical gauge frame we still have $\bar{Q}_s=Q/4\pi$ and $\bar{Q}=0$ and hence it is perfectly legitimate to talk about color screening in the physical gauge frame. In all the above discussions, we have tacitly assumed $q>0$ for all r.

In passing we note that the argument of Ref. 7 will not apply here as long as the gauge transformations are in the restricted class.

III. CONSERVED GAUGE-INVARIANT CHARGE

In the above discussions, although characterization of the external source by Eq. (14) is gauge invariant, the total color is gauge dependent unless gauge transformations are restricted to those satisfying condition (12). The restriction (12) is quite severe as it requires the YM connection to vanish faster than $1/r$ at large r. Consequently we raise the question of whether it is possible to discuss color screening in a gauge-independent manner without imposing the condition (12). The answer is yes as we shall see below. The essential idea is to construct a gauge-invariant conserved current.

The Noether current (7a) is not gauge covariant because the YM field A^a_μ transforms inhomogeneously. In contrast the electromagnetic current and field strength are gauge invariant. The YM field strength $F^a_{\mu\nu}$, however, transforms covariantly and from which a gauge-invariant entity can be obtained if a color direction is chosen suitably at each space-time point. To this end we follow Ref. 9 and introduce an adjoint-representation scalar field,[9]

$$\eta^a(x)\eta^a(x)=1 \ , \tag{21a}$$

$$\eta(x)=\eta^a(x)\sigma^a/2i \ , \tag{21b}$$

and under a gauge transformation

$$\eta \rightarrow U\eta U^{-1} \ .$$

We now define the generalized electromagnetic field strength $\mathcal{F}_{\mu\nu}$ by

$$\mathcal{F}_{\mu\nu}=F^a_{\mu\nu}\eta^a \tag{22}$$

which is of course a gauge singlet. Once this is done a current which is conserved and gauge invariant can be easily written as

$$\mathcal{J}^\nu=\partial_\mu\mathcal{F}^{\mu\nu}=D_\mu\mathcal{F}^{\mu\nu} \ . \tag{23}$$

Expression (23) can also be written as

$$\mathcal{J}^\nu=D_\mu(\eta^aF^{a\mu\nu})=\eta^aj^{a\nu}+(D_\mu\eta)^aF^{a\mu\nu} \ . \tag{24}$$

The field $\eta^a(x)$ selects a direction in the internal group space at each space-time x_μ. With respect to it, the YM connection can be resolved into two parts:[9]

$$A_\mu=h_\mu+K_\mu \ , \tag{25}$$

$$h_\mu=(A^a_\mu\eta^a)\eta-[\eta,\partial_\mu\eta] \ , \tag{26}$$

$$K_\mu=[\eta,D_\mu\eta] \ . \tag{27}$$

When a gauge transformation is performed along $\eta^a(x)$, $h_\mu(x)$ remains invariant while $K_\mu(x)$ transforms covariantly. $h_\mu(x)$ is known as the restricted connection. For our purpose we note that Eq. (26) yields[10]

$$\partial_\mu\eta+[h_\mu,\eta]=0 \tag{28}$$

and consequently Eq. (24) can be rewritten as

$$\begin{aligned}\mathcal{J}^\nu&=\eta^aj^{a\nu}+[K_\mu,F^{\mu\nu}]^a\eta^a\\&=\eta^a\mathcal{J}^{a\nu} \ ,\end{aligned} \tag{29a}$$

where

$$[K_\mu,F^{\mu\nu}]^a=i\epsilon^{abc}K^b_\mu F^{c\mu\nu} \ . \tag{29b}$$

Comparing with the Noether current (7a), clearly $\mathcal{J}^{a\nu}$ transforms covariantly so that $\mathcal{J}^\nu=\eta^a\mathcal{J}^{a\nu}$ is a gauge invariant; and as in the Noether-current case, $\mathcal{J}^\nu$ consists of two parts: the external current j^ν and the current carried by the YM field.

With $\mathcal{J}^\nu(x)$ we now define a gauge-invariant total color charge as

$$\begin{aligned}Q_A&=\int d^3x\, \mathcal{J}^0(x)\\&=\int_{\text{surface at }\infty}(\eta^aF^{ai0})n_i dS \ ,\end{aligned} \tag{30}$$

which is conserved as long as $\mathcal{J}^i$ vanishes fast enough at large r. Color screening occurs if $Q_A=0$ and $\overline{Q}_s\neq 0$. This is a gauge-invariant statement. As before, a sufficient condition for Q_A to vanish is

$$\lim_{r\rightarrow\infty}(r^{2+\epsilon}\eta^aF^{ai0})=0 \ . \tag{31}$$

We note that the long-distance behavior of $F^{ai0}(x)$ can be affected by gauge transformations but not $\eta^aF^{ai0}(x)$, hence condition (31) is independent of gauge choice, in contrast with condition (15).

We have checked all the known screening solutions and found that for all cases $Q_A=0$ and $\overline{Q}_s\neq 0$, thus justifying the claim that color screening is a gauge-independent effect, at least at the classical level.

IV. COMMENTS

We make some remarks.

(1) The generalized electromagnetic field strength (22) is different from 't Hooft's definition,[11] although at large distances they agree. Definition (22) is also advocated in Ref. 12.

(2) A gauge-invariant conserved current is also given by Ref. 13. It makes use of a background field to derive Killing vectors and hence the construction is different from Sec. III here.

(3) The conservation of $\mathcal{J}^\nu$ can be shown to be associated with the invariance of the Lagrangian density (4) under gauge transformations along $\eta^a(x)$ as defined by Eq. (21). To see this, consider infinitesimal transformations of the form

$$U(x)=\exp(-i\delta\theta\eta^a\sigma^a/2) \ , \quad \delta\theta=\text{constant} \ . \tag{32}$$

Suppose $\eta^a(x)$ is chosen such that in the decomposition of the Yang-Mills field (25)—(27), the condition

$$[h^\mu,j_\mu]=0 \tag{33}$$

is satisfied. Then the variation in the Lagrangian density is

$$0=\frac{\delta\mathscr{L}}{\delta\theta}=\frac{\partial\mathscr{L}}{\partial A_\mu^a}\frac{\delta A_\mu^a}{\delta\theta}+\frac{\partial\mathscr{L}}{\partial(\partial_\mu A_\nu^a)}\frac{\delta(\partial_\mu A_\nu^a)}{\delta\theta}+\frac{\partial\mathscr{L}}{\partial j_\mu^a}\frac{\delta j_\mu^a}{\delta\theta}$$

$$=\eta^a[j_\mu,A^\mu]^a-\partial^\mu[F_{\mu\nu}^a(D^\nu\eta)^a]$$

$$=\partial^\mu[\eta^a j_\mu^a-F_{\mu\nu}^a(D^\nu\eta)^a]$$

$$=\partial^\mu\mathscr{J}_\mu ,$$

using the equations of motion (1a) and the fact that j_μ is covariantly conserved. However, condition (33) is gauge dependent.

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