Variational estimate of the vacuum state of the SU(2) lattice gauge theory with a disordered trial wave function

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The variational principle is used to estimate the ground state of the Kogut-Susskind Hamiltonian of the SU(2) lattice gauge theory, with a trial wave function for which the magnetic fields on different plaquettes are uncorrelated. This trial function describes a disordered state. The energy expectation value is evaluated by a Monte Carlo method. The variational results are compared to similar results for a related Abelian gauge theory. Also, the expectation value of the Wilson loop operator is computed for the trial state, and the resulting estimate of the string tension is compared to the prediction of asymptotic freedom.

The Hamiltonian of the SU(2) lattice gauge theory in the formulation of Kogut and Susskind is[1]

$$H=\frac{g^2}{2}\sum_l E_a{}^2(l)+\frac{4}{g^2}\sum_p M(p)\ , \tag{1}$$

where l and p denote, respectively, a lattice link and plaquette; the gauge-invariant plaquette variable $M(p)$ is

$$M(p)=1-\tfrac{1}{2}\,\mathrm{Tr}U(l_1)U(l_2)U^\dagger(l_3)U^\dagger(l_4)\ , \tag{2}$$

where l_1,l_2,l_3,l_4 are the links that make up plaquette p, and $U(l)$ is an element of the group SU(2) associated with the link l. The electric field $E_a(l)$ is defined by the commutation relation

$$[E_a(l),U(l')]=-\tfrac{1}{2}\sigma_a U(l)\delta(l,l')\ , \tag{3}$$

where σ_a is the Pauli matrix.

Any group element U can be specified by three angle variables (ψ,θ,ϕ) as

$$U=\cos\psi+i\vec{\sigma}\cdot\hat{n}\sin\psi\ , \tag{4}$$

where $\hat{n}$ is a unit vector in three dimensions with polar angles (θ,ϕ); the domain of ψ and θ is $(0,\pi)$, and that of ϕ is $(0,2\pi)$. Alternatively, a gauge field A_a may be introduced by

$$U=\exp\left[\frac{i}{2}\sigma_a A_a\right]\ . \tag{5}$$

The relation between the three gauge fields A_a and the angles (ψ,θ,ϕ) is

$$A_a=2\psi\hat{n}_a\ . \tag{6}$$

The simplest explicit expression for the electric field operators E_a is in terms of the gauge fields, as

$$\begin{aligned}E_a&=if(A)\frac{\partial}{\partial A_a}-\frac{i}{A^2}[f(A)-1]A_aA_b\frac{\partial}{\partial A_b}\\&\quad-\frac{i}{2}\epsilon_{abc}A_b\frac{\partial}{\partial A_c}\ ,\\A&=(A_aA_a)^{1/2},\quad f(A)=\frac{A}{2}\cot\frac{A}{2}\ .\end{aligned} \tag{7}$$

We shall report the results of our calculations in terms of a rescaled Hamiltonian

$$H=\sum_l E_a{}^2(l)+\lambda\sum_p M(p)\ , \tag{8}$$

where $\lambda=8/g^4$. The energies calculated below are for this Hamiltonian. Also, since we work with a finite lattice to do numerical calculations on a computer, we impose periodic boundary conditions on the couplings of the fields in Eq. (2); this reduces finite-lattice-size effects.

In this paper we describe a variational estimate of the ground state of the Hamiltonian H. Our trial wave function is of the form

$$\Phi=\prod_p u(M(p))\ ; \tag{9}$$

this is manifestly gauge invariant. It describes a disordered state of the gauge fields, i.e., one in which the Wilson loop operator obeys an area law, because there is no correlation between the magnetic variables $M(p)$ on different plaquettes. Similar variational wave functions have previously been applied to the U(1) lattice gauge theory in two and three spatial dimensions,[2-4] and to the Hamiltonian form of the XY model.[5] In the two-dimensional U(1) lattice gauge theory the single-plaquette wave function $u(M)$ that minimizes the energy is the ground-state eigenfunction of a quantum pendulum.[2] In the present SU(2) calculation we do not optimize the functional form of $u(M)$, but simply take

$$u(M)=\exp(-2\alpha M)\ , \tag{10}$$

where α is the variational parameter. This should be almost as good as finding the optimal functional form of $u(M)$.

In a recent paper[6] Horn and Karliner have also described the variational calculation for the trial function in Eqs. (9) and (10) on the SU(2) lattice gauge theory, along with a second calculation for a trial function with uncorrelated links projected onto the space of gauge-invariant states.

There are several motivations for our study. The most straightforward motivation is just curiosity to see whether

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this very simple trial wave function can reproduce the known properties of the vacuum of a lattice gauge theory. More precisely, the wave function Φ is a natural model of a disordered state, in that the only correlation between gauge fields is local, and is dictated by gauge invariance. Since quark confinement in lattice gauge theories is associated with a disordered vacuum state, it is important to find out whether this "minimal" model of a disordered state, in which the nature of the disorder is trivial, resembles the true vacuum state. If not, then the origin of the disorder that causes confinement must be more complicated.

We are also motivated by a desire to apply the Green's-function Monte Carlo method to the SU(2) lattice gauge theory, in the manner of our calculations on the compact U(1) lattice gauge theory.[4] A necessary first step in this program is to develop reasonable variational wave functions to be used in importance sampling in the Monte Carlo calculations.

To check whether the variational estimates based on the trial function Φ are accurate, we compare our results to similar calculations for a compact Abelian lattice gauge theory.

We evaluate the expectation value of H in the trial state Φ by a Monte Carlo method. The expectation value is the sum of the magnetic energy

$$\lambda \int d\Omega\, \Phi^2 \sum_p M(p) , \tag{11}$$

where $\lambda = 8/g^4$, and the electric energy

$$\int d\Omega \sum_l [E_a(l)\Phi]^2 ; \tag{12}$$

here $d\Omega$ is the invariant measure of the group SU(2), i.e.,

$$d\Omega = \prod_l \sin^2\psi_l d\psi_l \sin\theta_l d\theta_l d\phi_l / 2\pi^2 . \tag{13}$$

We use Creutz's heat-bath Monte Carlo method[7] to generate a set of field configurations with probability distribution Φ^2. This is equivalent to a Wilson path-integral Monte Carlo calculation in three spacetime dimensions. Then the magnetic energy is estimated as the average of $\sum_p M(p)$ over the set of configurations. The difficult part of the calculation is the evaluation of the electric energy, which is the average of

$$\sum_l [E_a(l)\Phi/\Phi]^2 ;$$

we compute this quantity from an explicit formula for $E_a(l)\Phi/\Phi$. We calculate these two energies for many values of the variational parameter α, and fit the resulting numbers to an analytic expression for the energies as functions of α. Then for a given value of the coupling constant λ we minimize the energy with respect to α.

The calculations described below are for a small three-dimensional lattice, of size $3\times3\times3$. We have checked that the results are very insensitive to lattice size by comparing the electric and magnetic energies for $3\times3\times3$ and $6\times6\times6$ lattices; on the scale of the graphs in this paper the difference is small. That is not surprising since we use periodic boundary conditions, and since there are no long-range correlations in the trial state Φ.

Figure 1 is a graph of the value of α that minimizes the energy vs coupling constant λ. The two sets of points are for the SU(2) lattice gauge theory defined above, and for comparison for a $U(1)\times U(1)\times U(1)$ lattice gauge theory. The Hamiltonian of this *Abelian* gauge theory is

$$H_{Ab} = \sum_l E_a{}^2(l) + \frac{\lambda}{4} \sum_{a,p} [1 - \cos B_a(p)] , \tag{14}$$

where $B_a(p)$ is the lattice curl of $A_a(l)$, and $E_a(l)$ is just $i\partial/\partial A_a(l)$; the index a runs from 1 to 3. The two Hamiltonians H and H_{Ab} in Eqs. (8) and (14) have the same large-λ limit: in the harmonic approximation they both describe three noninteracting free fields. The trial function for the $U(1)\times U(1)\times U(1)$ gauge theory is

$$\Phi_{Ab} = \prod_{a,p} \exp\left[\frac{\alpha}{2}\cos B_a(p)\right] . \tag{15}$$

This is gauge invariant under the Abelian group $U(1)\times U(1)\times U(1)$, and is identical to Φ in the limit of small fields.

A few words of explanation are needed at this point to explain why the two models give different results in the large-λ limit. At first sight one would guess that the results would be identical since the Hamiltonians and trial functions are identical in the limit of small fields. However, the electric energy is sensitive to a nonleading term in the small-field approximation of the Hamiltonian, and the nonleading term is different for the two models. We have checked that in our numerical results the leading, i.e., harmonic, terms of the electric and magnetic energies are equal for the two models in the limit of large α, which corresponds to small fields. The difference seen in Fig. 1 is due to a nonleading, anharmonic energy.

Figure 2 is a graph of the variational bounds on the energy per plaquette obtained for these two theories with the uncorrelated trial wave function Φ. The curve on that graph is the ground-state energy per plaquette of the free-field harmonic approximation of H_{Ab}, given by

$$\frac{E_0}{N_p} = c(n)\sqrt{2\lambda} - c^2(n)/3 + O(\lambda^{-1/2}) \tag{16}$$

for an $n\times n\times n$ lattice, where N_p is the number of plaquettes; the constant $c(n)$ is weakly dependent on n, e.g.,

$$c(3) = 1.181, \quad c(\infty) = 1.194 . \tag{17}$$

In Eq. (16), the term independent of λ derives from the four-field coupling in the small-field approximation of the theory H_{Ab}; this term would be different for the non-Abelian Hamiltonian H. On the other hand, the term proportional to $\lambda^{1/2}$ is the same for H and H_{Ab} since the harmonic approximations of these theories are the same.

The variational bound on the vacuum energy of the U(1) lattice gauge theory, based on the uncorrelated wave function in Eq. (15), lies well above the harmonic limit of the energy. This is an indication of the phase transition of the U(1) lattice gauge theory; it is known that for $\lambda \lesssim 4.5$ the disordered state does approximate the ground state,

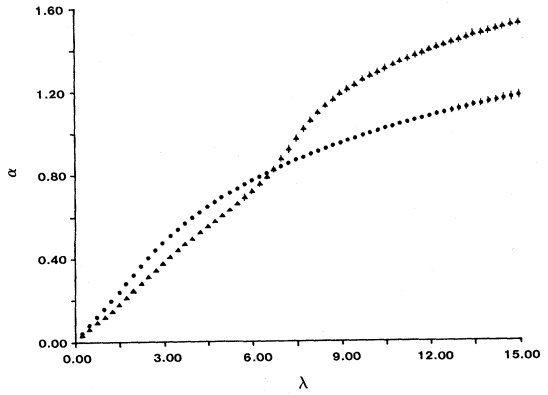

FIG. 1. Variational parameter α vs coupling constant λ. The circles (●) are the SU(2) gauge model defined in Eq. (8); the triangles (▲) are for the U(1)×U(1)×U(1) model defined in Eq. (14).

but that for $\lambda \gtrsim 4.5$ it is quite different from the harmonic free-field vacuum, and so has a significantly higher energy. It is possible to construct a wave function for which the energy approaches the harmonic limit, i.e., the curve in Fig. 2, for large λ. Thus the phase transition of the compact U(1) gauge theory can be visualized as a level crossing: for $\lambda < \lambda_c$ the disordered state, similar to that of Eq. (15), lies lower in energy, but for $\lambda > \lambda_c$ the harmonic free-field state lies lower.

It is interesting to note that the variational bound of the vacuum energy of the SU(2) gauge theory lies lower than that of the U(1) gauge theory,[8] and only slightly higher than the harmonic limit of the ground-state energy of H_{Ab}.

Figure 3 shows the mean magnetic field on a plaquette for the two theories, estimated from the variational wave function Φ; to be precise the quantity V defined by

$$V = \begin{cases} 1 - \frac{1}{2}\,\mathrm{Tr} U_1 U_2 U_3^\dagger U_4^\dagger & \text{for SU(2)} \\ 1 - \cos B & \text{for U(1)} \end{cases} \tag{18}$$

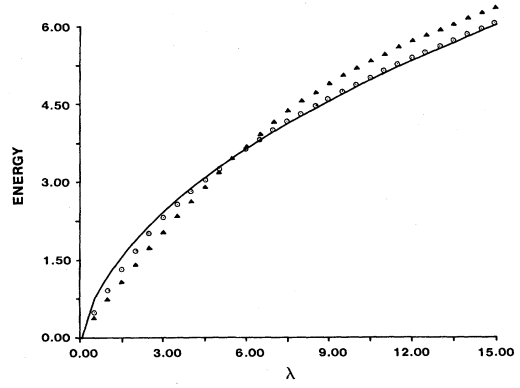

FIG. 2. The variational energy bounds vs λ. The circles (●) are for the SU(2) model and the triangles (▲) are for the U(1)×U(1)×U(1) model. The curve is the large-λ limit of the vacuum energy of H_{Ab}, given in Eq. (16).

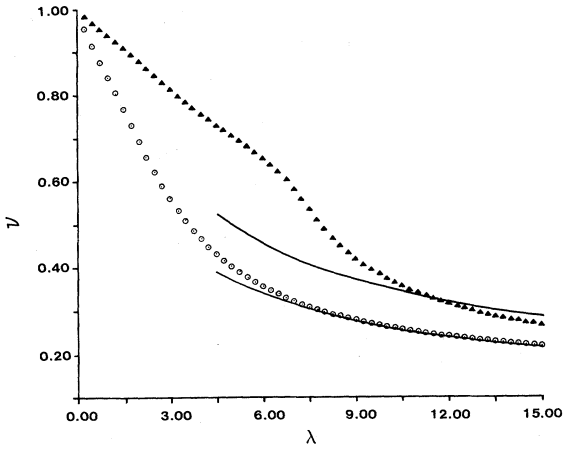

FIG. 3. The mean plaquette field V in the variational state Φ; the circles (●) are for the SU(2) model, and the triangles (▲) are for the U(1)×U(1)×U(1) model. The curve is the harmonic limit of the vacuum expectation value of V, given in Eq. (19).

on any plaquette. Again the curves are the harmonic limits

$$\langle V \rangle_0 = fc(n)/\sqrt{2\lambda}\,, \tag{19}$$

where $f=1$ for the SU(2) gauge theory, and $f=\frac{4}{3}$ for the U(1) gauge theory.

In the SU(2) gauge theory, the mean plaquette field V decreases monotonically as λ increases, down to values that are slightly larger than the harmonic free-field value given in Eq. (19). Since Eq. (19) is the large-λ limit of the vacuum expectation value of V, this discrepancy at large λ is a measure of the inadequacy of the uncorrelated trial function Φ as a model of the vacuum state. In the U(1) gauge theory there is a large difference between the variational estimate of V and the harmonic limit. Other calculations, such as those of Ref. 4, indicate that there is a phase transition in this model at $\lambda \cong 4.5$, to a state with mean plaquette field V nearly equal to the harmonic limit.

The vacuum expectation value of V is related to the

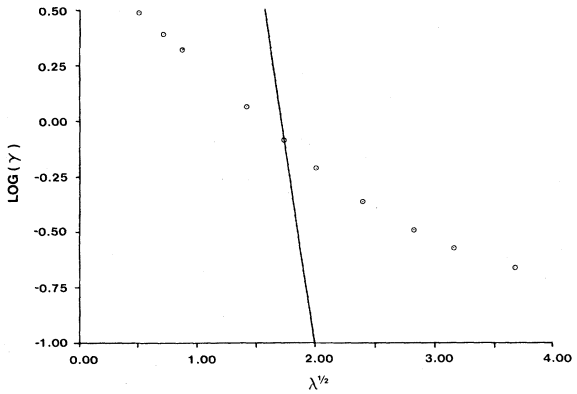

FIG. 4. The log (base 10) of the string tension, γ, estimated from the variational wave function Φ, vs $\lambda^{1/2}$. The solid line is the asymptotic-freedom prediction of Eq. (23) with an arbitrary intercept with the $\lambda^{1/2}$ axis.

vacuum energy E in an interesting way. First-order perturbation theory implies that

$$\langle V\rangle=\frac{f}{N_p}\frac{dE}{d\lambda}\ , \tag{20}$$

where the angle brackets stand for vacuum expectation value; of course this relation holds only for the exact vacuum. The fact that $\langle\Phi\,|\,V\,|\,\Phi\rangle$ is too large at large λ indicates that the difference between the trial state Φ and the exact vacuum grows as λ increases.

Perhaps the quantity that reveals the most concerning the nature of the vacuum state is the expectation value of the Wilson loop operator.[9] We can easily calculate this correlation function for the variational wave function Φ, i.e., the quantity

$$W=\langle\Phi\,|\,\tfrac{1}{2}\,\mathrm{Tr}U_1U_2U_3\cdots U_N\,|\,\Phi\rangle\ , \tag{21}$$

where $U_1U_2U_3\cdots U_N$ are the group elements around a loop. Because Φ describes a disordered state, W obeys the area law

$$W\sim\exp(-\gamma A)\ , \tag{22}$$

where A is the area enclosed by the loop. The quantity γ is an *estimate* of the string tension, insofar as the trial function Φ is a reasonable approximation of the vacuum state. Figure 4 shows the value of γ obtained from the variational wave function as a function of the coupling constant λ, for the SU(2) lattice gauge theory; the solid line on the graph indicates the large-λ limit of the string tension derived from asymptotic freedom. In the limit of large λ the dependence of the string tension on the coupling constant is given by the relation[7]

$$\gamma=K\exp\left[\frac{-6\pi^2}{11}\sqrt{2\lambda}\right]\ ; \tag{23}$$

here the constant K is unknown, but the exponent is calculated from asymptotic freedom. The solid line in Fig. 4 shows the slope of $\log(\gamma)$ vs $\lambda^{1/2}$ implied by Eq. (23); the intercept with the $\lambda^{1/2}$ axis depends on the unknown quantity K, and is chosen arbitrarily on this figure.

The results shown in Fig. 4 were obtained for a $6\times6\times6$ lattice, from calculations of W for loops with dimensions between 1×1 and 3×3, both square and rectangular.

Figure 4 shows that the string tension produced by the uncorrelated variational wave function Φ remains significantly too large for large λ. That is, this wave function is excessively disordered compared to the vacuum state in the weak-coupling limit.

Quark confinement in QCD does require that the vacuum state be disordered, i.e., that W obey the area law (22), even in the limit $\lambda\to\infty$, for a non-Abelian gauge theory. Figure 4 shows that the disorder of the weak-coupling vacuum state is more subtle than the trivial lack of correlation between magnetic fields on different plaquettes that defines the trial function Φ. What kind of wave function *does* describe the weak-coupling vacuum? In the case of the U(1) gauge theory the answer is known: a wave function derived from the harmonic free-field approximation can be constructed that approximates the vacuum in the large-λ limit. Of course that state is nonconfining. In the SU(2) gauge theory there might be disordering topological configurations[10] superimposed on the harmonic vacuum that maintain confinement in the limit $\lambda\to\infty$ but that do not affect average quantities such as the vacuum energy and the mean plaquette field V.

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