Green's-function method for the O(N) spin system in 1+1 dimensions

Tetsuo Matsui*
Department of Physics, Purdue University, West Lafayette, Indiana 47907
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The Green's-function method is employed to study the two-dimensional system of classical O(N)-vector spins. The second-order double-time retarded Green's functions of the corresponding quantum Hamiltonian system on a one-dimensional lattice are decoupled approximately in a self-consistent manner. The expressions for the mass gap and the spin-correlation function thus obtained reproduce the known exact results in the large-N limit and in the strong-coupling region. Some directions are discussed to improve the present method in the region of weak coupling with N not large.

I. INTRODUCTION

The system of classical O(N)-vector spins in a two-dimensional lattice has received much attention not only as a model of magnetism but also as a convenient testing ground for new theoretical methods and as a system having considerable relevance and similarity[1,2] to the four-dimensional lattice gauge system,[3] another interesting but more complicated system. It has been analyzed by various theoretical techniques such as weak-coupling perturbation,[1] renormalization-group method,[2] strong-coupling expansion,[4,5] duality transformation and topological excitations,[6] Monte Carlo method,[7,8] the large-N limit,[9,10] variational method,[11,12] and so on.[13]

In this paper yet another method, the Green's-function method,[14,15] is employed in order to study the spin system and, at the same time, to test the applicability of the method itself, hoping to get some indications that the method will also prove its validity for some other interesting systems, such as the lattice gauge theory. The Green's-function method is one of the standard methods in quantum theory and has been used successfully in several branches of theoretical physics. In particular, in the development of its application to the Heisenberg model of ferromagnets,[14] a quantum version of our vector-spin model of $N=3$, one can expect to find some ideas and lessons useful in studying our spin system by using a similar framework and techniques. Therefore it seems appropriate to recall here the main steps of its development for the Heisenberg model briefly.

Tyablikov[16] first studied the three-dimensional Heisenberg model of spin $s=\frac{1}{2}$ in an external field by using a simple decoupling procedure to truncate the chain of Green's functions. His result of the magnetization is rather satisfactory over the entire region of temperatures, giving the exact limiting values in low and high temperatures, and has been regarded as a "first approximation." Tyablikov's decoupling approximation is known to be equivalent to the random-phase approximation[17] in the equation-of-motion approach.[18]

Extension of the method to higher-spin cases ($s>1/2$) was done by several authors.[14] In particular, Callen[19] elaborated the decoupling procedure of the higher-order Green's functions by introducing an appropriate parameter in the coefficients (mass operators) in front of the lower-order Green's functions. Among other things in his results, it is pointed out that his estimation of the Curie temperature becomes almost exact in the large-spin limit ($s\to\infty$).

The defect of Callen's theory for $s=\frac{1}{2}$ at low temperatures, i.e., appearance of an unwelcome T^3 term in the magnetization, is explained by Tahir-Kheli.[20] He showed that this unwelcome term would disappear if one introduced a suitable anomalous term in addition to the mass operator multiplying the lower-order Green's function in the expression for the higher-order one. It is this additional term, as observed by Wortis,[21] which distinguishes the Heisenberg spin system consisting of SU(2) spin variables from ordinary systems consisting of canonical boson or fermion variables, the latter satisfying the usual Dyson equations and containing no such anomalous terms.

For the Heisenberg model in one and two dimensions the spontaneous magnetization cannot be used as a mass operator because it vanishes there ($T\neq 0$).[22] So one must postpone the decoupling procedure one step (or more) further to get nonvanishing mass operators, and hence interesting results. This approach is called the second-order Green's-function approach and has been studied by several authors.[23]

Returning to our spin system, the sequence of this paper will be as follows. In Sec. II, the Green's-function method (at zero temperature) is formulated for the quantum-spin Hamiltonian system on a one-dimensional lattice. This system is obtained from the original classical system defined in a two-dimensional Euclidean rectangular lattice with spacings $a_1\equiv a, a_0$ through the transfer-matrix formalism in the limit of putting the lattice spacing a_0 in the (imaginary) time direction to zero.[24] The decoupling procedure is performed for the second-order spin Green's functions by introducing several parameters (in the mass operators), which, in turn, are determined by self-consistency requirements coming from the spectral theorem[14] and the spin kinematics. The resulting expres-

sions for the mass gap and the spin-spin correlation function are checked to reproduce the known exact results in the limit of large N[9] and in the region of strong coupling.[4] In Sec. III, some discussion is presented on possible directions of how to improve the present method in the region of weak coupling with N not large, where our result seems to become unsatisfactory.

II. GREEN'S-FUNCTION METHOD

A. Hamiltonian system

The Hamiltonian of our spin system is given by

$$H=\sum_x \left[\frac{1}{4\beta}\sum_{i\neq j} J_{ij}(x,t)^2+\frac{\beta}{2}\sum_i [\nabla n_i(x,t)]^2\right], \tag{1}$$

where a site in the one-dimensional lattice is specified by an integer x $(=1,2,\ldots,L)$, and the lattice spacing a is put to unity. The spin variable $n_i(x,t)$ at the site x on time t has N components $(i=1,2,\ldots,N; N\geq 2)$ and satisfies the unit-length condition

$$\sum_i n_i(x,t)n_i(x,t)=1 . \tag{2}$$

The $J_{ij}(x,t)$ stands for the generator of O(N) rotations of $n_i(x,t)$ in the i-j plane. A dimensionless parameter β is inversely proportional to the coupling constant. The ∇ is the lattice difference operator, i.e.,

$$\nabla n_i(x,t)=n_i(x+1,t)-n_i(x,t) ,$$

and the periodic boundary condtion

$$n_i(L+1,t)=n_i(1,t)$$

is used.

The equal-time commutation relations are

$$\begin{aligned}
&[n_i(x),n_j(y)]=0 ,\\
&[J_{ij}(x),n_k(y)]=-i\delta_{xy}[\delta_{ik}n_j(x)-\delta_{jk}n_i(x)] ,\\
&[J_{ij}(x),J_{kl}(y)]=-i\delta_{xy}[\delta_{ik}J_{jl}(x)-\delta_{il}J_{jk}(x)-\delta_{jk}J_{il}(x)\\
&\qquad\qquad +\delta_{jl}J_{ik}(x)] ,
\end{aligned} \tag{3}$$

where the common time t has been suppressed.

B. Green's-function method

Now let us briefly recall the basic principles of the Green's-function method.[14,15] Let us introduce the following double-time retarded Green's function for operators $A(t)$ and $B(t')$:

$$\langle\langle A(t);B(t')\rangle\rangle \equiv \theta(t-t')\langle [A(t),B(t')]\rangle , \tag{4}$$

where θ is the step function and $\langle\cdots\rangle$ stands for the expectation value with respect to the ground state of the Hamiltonian H.[25] The time development of this function is given by

$$\begin{aligned}
\frac{\partial}{\partial t}\langle\langle A(t);B(t')\rangle\rangle &= \delta(t-t')\langle [A(t),B(t)]\rangle\\
&\quad +i\langle\langle [H,A(t)];B(t')\rangle\rangle .
\end{aligned} \tag{5}$$

The time-correlation functions $\langle A(t)B(t')\rangle$ and $\langle B(t')A(t)\rangle$ can be expressed in terms of $\langle\langle A(t);B(t')\rangle\rangle$ by the spectral theorem[14,15,25]

$$\begin{aligned}
\langle A(t)B(t')\rangle &= \lim_{\epsilon\to 0}\int_0^\infty d\omega[G_{AB}(\omega+i\epsilon)-G_{AB}(\omega-i\epsilon)]e^{-i\omega(t-t')} ,\\
\langle B(t')A(t)\rangle &= -\lim_{\epsilon\to 0}\int_{-\infty}^0 d\omega[G_{AB}(\omega+i\epsilon)-G_{AB}(\omega-i\epsilon)]e^{-i\omega(t-t')} ,
\end{aligned} \tag{6}$$

where

$$G_{AB}(\omega)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty} d(t-t')\langle\langle A(t);B(t')\rangle\rangle e^{i\omega(t-t')} . \tag{7}$$

Theorem (6) can be proved by using the spectral representation.

Thus, if one can get the Green's functions by solving Eq. (5), the correlation functions can be obtained through Eq. (6). Equation (5), however, generates a chain of equations involving higher-order Green's functions, which is generally endless. Therefore, in this method, one has to replace this infinite-dimensional set of equations by some finite one approximately in order to get some concrete results. A typical procedure for this replacement, i.e., a decoupling procedure, is presented for our spin system in the next subsection.

C. Decoupling procedure

Let us consider the function $\langle\langle n_i;n_i'\rangle\rangle$, where abbreviations $n_i\equiv n_i(x,t)$ and $n_i'\equiv n_i(x',t')$ are used, and the summation convention over the repeated index (i) is also employed. Its first-order equation of motion is

$$\begin{aligned}
\frac{\partial}{\partial t}\langle\langle n_i;n_i'\rangle\rangle &= \frac{1}{\beta}\langle\langle J_{ij}n_j;n_i'\rangle\rangle\\
&\quad -\frac{i}{2\beta}(N-1)\langle\langle n_i;n_i'\rangle\rangle ,
\end{aligned} \tag{8}$$

where $J_{ij}\equiv J_{ij}(x,t)$. Because our system is known to have the ground-state singlet under the global O(N) rotation,[22]

i.e., $\langle n_i \rangle = \langle J_{ij} \rangle = 0$, any decoupling procedure to truncate the sequence of the Green's functions should be postponed one step further. The equation of motion for $\langle\langle J_{ij} n_j ; n_i' \rangle\rangle$ appearing in Eq. (8) is

$$\frac{\partial}{\partial t}\langle\langle J_{ij} n_j ; n_i' \rangle\rangle = -i(N-1)\delta_{xx'}\delta(t-t') + \frac{1}{\beta}\langle\langle J_{ij} J_{jk} n_k ; n_i' \rangle\rangle - \frac{i}{2\beta}(N-1)\langle\langle J_{ij} n_j ; n_i' \rangle\rangle + \beta\langle\langle \bar{n}_i - n_i \bar{n}_j n_j ; n_i' \rangle\rangle , \tag{9}$$

where $\bar{n}_i \equiv n_i^+ + n_i^-$ and $n_i^\pm \equiv n_i(x \pm 1, t)$.

Now let us describe $\langle\langle J_{ij} J_{jk} n_k ; n_i' \rangle\rangle$ and $\langle\langle n_i \bar{n}_j n_j ; n_i' \rangle\rangle$, both appearing in Eq. (9), in terms of the lower-order functions $\langle\langle n_i ; n_i' \rangle\rangle$ and $\langle\langle J_{ij} n_j ; n_i' \rangle\rangle$, approximately. For $\langle\langle J_{ij} J_{jk} n_k ; n_i' \rangle\rangle$, the following substitution is proposed:

$$\langle\langle J_{ij} J_{jk} n_k ; n_i' \rangle\rangle \rightarrow C_1 \langle\langle J_{ij} n_j ; n_i' \rangle\rangle + C_2 \langle\langle n_i ; n_i' \rangle\rangle \tag{10}$$

with the introduction of yet unknown parameters C_1, C_2 which might have β and N dependence but not have space-time dependence. Equation (10) simply states that our approximate space of the Green's functions is a linear one. By substituting Eq. (10) into the spectral theorem one gets

$$\langle [J_{ij} J_{jk} n_k, n_i'] \rangle = C_1 \langle [J_{ij} n_j, n_i'] \rangle + C_2 \langle [n_i, n_i'] \rangle . \tag{11}$$

At equal time, by using the commutation relations, this equation reduces to determine C_1 as

$$C_1 = i(N-1) . \tag{12}$$

The parameter C_2, on the other hand, will be determined later self-consistently [see the argument below Eq. (24)].

For $\langle\langle n_i \bar{n}_j n_j ; n_i' \rangle\rangle$, let us propose the following substitution:

$$\langle\langle n_i n_j^\pm n_j ; n_i' \rangle\rangle \rightarrow D_1 \langle n_j^\pm n_j \rangle \langle\langle n_i ; n_i' \rangle\rangle + D_2 \langle\langle n_i^\pm ; n_i' \rangle\rangle , \tag{13}$$

where the parameters D_1 and D_2 may depend on N but are assumed not on β and space-time. The β independence of $D_{1,2}$ is suggested by the appearance of the β-dependent factor $\langle n_j^\pm n_j \rangle$ on the right-hand side of (13). Again the use of the spectral theorem with Eq. (13) gives

$$\langle n_i n_j^\pm n_j n_i' \rangle = D_1 \langle n_j^\pm n_j \rangle \langle n_i n_i' \rangle + D_2 \langle n_i^\pm n_i' \rangle . \tag{14}$$

At equal time, by setting $x = x'$, this reduces to give a condition (recall $n_i n_i = 1$)

$$1 = D_1 + D_2 . \tag{15}$$

Next, at equal time, by setting $x' = x \pm 1$, Eq. (14) reduces to

$$\langle n_i n_j n_j^\pm n_i^\pm \rangle = D_1 \langle n_j^\pm n_j \rangle \langle n_i^\pm n_i \rangle + D_2 . \tag{16}$$

Then, in this equation, by considering the limit of strong coupling ($\beta \rightarrow 0$) where the nearest-neighbor correlations obviously become vanishing ($\langle n_j^\pm n_j \rangle \rightarrow 0, \langle n_i n_j n_j^\pm n_i^\pm \rangle \rightarrow N^{-1}$), D_2 can be determined as

$$D_2 = \frac{1}{N} . \tag{17}$$

It is noticed that, in the weak-coupling limit $\beta \rightarrow \infty$, Eq. (16) gives the same information as Eq. (15), because the nearest-neighbor correlations become maximum ($\langle n_i n_j n_j^\pm n_i^\pm \rangle \rightarrow 1$, $\langle n_i^\pm n_i \rangle \rightarrow 1$). Hence the expressions (15) and (17) for $D_{1,2}$ can be used consistently throughout the entire region of β.

D. Mass gap and spin-spin correlation

Once the decoupling procedures (10) and (13) are introduced, $\langle\langle J_{ij} n_j ; n_i' \rangle\rangle$ and hence $\langle\langle n_i ; n_i' \rangle\rangle$ can be solved in closed forms. Let us denote the space-time Fourier transform of these functions as $G_{Jn}(k,\omega), G_n(k,\omega)$:

$$\langle\langle J_{ij} n_j ; n_i' \rangle\rangle = L^{-1} \int_{-\infty}^{\infty} d\omega \sum_k G_{Jn}(k,\omega) e^{ik(x-x') - i\omega(t-t')} ,$$
$$G_{Jn}(k,\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} d(t-t') \sum_{x-x'=0}^{L-1} \langle\langle J_{ij} n_j ; n_i' \rangle\rangle e^{-ik(x-x') + i\omega(t-t')} , \tag{18}$$

where the momentum k takes values $k = 2\pi m/L$, $m = -L/2-1, \ldots, -1, 0, 1, \ldots, L/2$, and $G_n(k,\omega)$ is given by the replacement $J_{ij} n_j \rightarrow n_i$ in Eq. (18). Equation (9) is solved for $G_{Jn}(k,\omega)$ as

$$G_{Jn}(k,\omega) = \frac{(N-1)/2\pi + i\gamma(k) G_n(k,\omega)}{\omega + (N-1)/2\beta} , \tag{19a}$$

$$\gamma(k) \equiv C_2/\beta + 2\beta(1 - N^{-1})(\cos k - f) , \tag{19b}$$

$$f \equiv \langle n_j^\pm n_j \rangle . \tag{19c}$$

Then this expression of G_{Jn} is substituted in Eq. (8) to give the following simple structures of G_n;

$$G_n(k,\omega) = \frac{R}{\omega^2 - E(k)^2} , \tag{20a}$$

$$E(k)^2 \equiv [(N-1)/2\beta]^2 - \gamma(k)/\beta , \tag{20b}$$

$$R \equiv -(N-1)/2\pi\beta i . \tag{20c}$$

Now there are two yet undetermined parameters, C_2 and $f = \langle n_j^\pm n_j \rangle$. In the following they are determined self-consistently by invoking the spectral theorem (6) again. For $\langle\langle n_i ; n_i' \rangle\rangle$ it gives an expression for the spin-spin correlation function[26]

$$\langle n_i n_i' \rangle = -\frac{i\pi}{L} \sum_k \frac{R}{E(k)} e^{ik(x-x') - iE(k)(t-t')} . \tag{21}$$

At equal times, by setting $x = x'$, Eq. (21) reduces to give

$$1=\frac{N-1}{2\beta L}\sum_k \frac{1}{E(k)} . \tag{22}$$

Next, by setting $x'=x\pm 1$, it gives

$$f=\frac{N-1}{2\beta L}\sum_k \frac{\cos k}{E(k)} . \tag{23}$$

To see the meaning of these equations, the mass gap m is introduced through

$$E(k)^2=(1-N^{-1})(2-2\cos k)+m^2 . \tag{24}$$

The reason why m is called the mass gap, i.e., the energy difference between the ground state and the first-excited state, comes from the expression (20a), because the poles of the Green's functions in the ω plane generally represent the energies of the excited states of the Hamiltonian.[14,15] Then Eq. (22) serves as the gap equation which determines m as a function of β and N, $m(\beta,N)$. Next, Eq. (23) determines f as a function of $m(\beta,N)$. Finally, the relations (24), (20b), and (19b) determine C_2 as a function of $m(\beta,N)$.

For later convenience the expressions for the consistency equations (22) and (23) in the infinite-volume limit $L\to\infty$ are given by

$$1=\frac{[N(N-1)]^{1/2}}{2\pi\beta}\sqrt{\alpha}\,K(\sqrt{\alpha}) , \tag{25}$$

$$f=-\frac{[N(N-1)]^{1/2}}{\pi\beta\sqrt{\alpha}}E(\sqrt{\alpha})+1+\frac{\mu^2}{2} , \tag{26}$$

where

$$\mu^2\equiv m^2/(1-N^{-1}),\quad \alpha\equiv(1+\mu^2/4)^{-1} , \tag{27}$$

and $K(\sqrt{\alpha})$ and $E(\sqrt{\alpha})$ are the elliptic integrals of the first kind and the second kind, respectively.[27]

Before investigating the behavior of $m(\beta,N)$, two comments are in order. First, there appears in the dispersion relation (24) an extra factor $(1-N^{-1})$ in front of the usual expression $(2-2\cos k)$ of spin-wave propagations in the crystal. This effect of wave-function renormalization has been omitted in the study of the variational method.[12] Second, Eq. (21) gives a simple expression for the spin-spin correlation $\langle n_i(x,y)n_i(x',y')\rangle$ in the original spin system (where x and y are integers labeling sites in the rectangular lattice) in the limit of $a_0\to 0$ (Ref. 24) not only for spins on the x axis but also for spins in general positions $(y\neq y')$. It is given by the replacement $t-t'\to\mp i(y-y')a_0$ for $y-y'\gtrless 0$ in Eq. (21).

E. Large-N limit

Let us take the limit of $N\to\infty$ with N/β fixed in our gap equation (25). It reads

$$1=\frac{N\sqrt{\alpha}}{2\pi\beta}K(\sqrt{\alpha}) , \tag{28}$$

which agrees with the gap equations (19) and (23) of Guha and Sakita[9] obtained in the collective field theory (with $\beta\equiv g^{-1}$). It can also be checked that our spin-spin correlation (21) agrees with their result (17) at equal times.

F. Strong-coupling region

In the region of strong coupling $(\beta\sim 0)$, by using the asymptotic form[27] of $K(\sqrt{\alpha})$ in Eq. (25) for large values of mass gap $(m\gg 1)$, m is expanded in powers of β as

$$m=\frac{N-1}{2\beta}-\frac{2}{N}\beta+\frac{2}{N^2(N-1)}\beta^3+O(\beta^5) , \tag{29}$$

the first two terms of which agree with the result of the strong-coupling expansion.[4,28] This agreement is an advantage of the present method over the variational method of Ref. 12 which fails to give the first term.

G. Weak-coupling region

This region $(\beta\sim\infty)$ can be examined by taking the mass gap small $(m\ll 1)$. The leading expression for m is given by

$$m\sim\exp\{-2\pi\beta/[N(N-1)]^{1/2}\} . \tag{30}$$

Here one may recall the analysis of the nonlinear O(N) σ model defined in a two-dimensional continuum by the weak-coupling renormalization-group method,[1] which gives the following expression for a mass in the model:

$$m_{\rm WC}\sim\exp[-2\pi\beta/(N-2)] . \tag{31}$$

The discrepancy between the above two expressions (30) and (31) is not appreciable for large values of N $(N\gg 1)$, but is remarkable for N not large. Some discussion on this point is given in Sec. III.

In connection with the mass gap $m_{\rm VM}$ given by the variational method of Ref. 12 [Eqs. (3.27) and (3.30)], let us point out here that Eqs. (30) and (31) satisfy the inequaltiy $m>m_{\rm VM}>m_{\rm WC}$ at $\beta\sim\infty$.

H. Intermediate region

Fox *et al.*[8] studied the present system of $N=2$ and 3 by a Monte Carlo method. Among other things, they estimated the mass gap $m_{\rm MC}$ through the falloff of spin-spin correlation using rectangular lattices $(a_1/a_0>1)$. Below our values of m given by Eq. (25) are listed together with some of the values of $m_{\rm MC}$ in Tables 4 and 2 of Ref. 8 and the values of $m_{\rm VM}$ of Ref. 12 [Eqs. (3.34) and (3.38)] for comparison:

$N=3$

β	$m_{\rm MC}$	$m_{\rm VM}$	m
0.6	1.422	1.79	1.30
1.0	0.49	0.70	0.49
1.45	0.107	0.23	0.16

(32)

where the asymmetries of lattices used for $m_{\rm MC}(N=3)$ are $a_1/a_0=(16,8,4)$ from the above, respectively.[29] The values of m and $m_{\rm VM}$ correspond to the limit $a_0\to 0$, of course. For the discussion of the asymmetry dependence of $m_{\rm MC}$ see Ref. 8:

$N=2$

β	m_{MC}	m_{VM}	m
0.4	0.925	1.52	0.89
0.65	0.311	0.58	0.31
0.8	0.114	0.31	0.16

(33)

Equations (32) and (33) show that the values of m are closer to m_{MC} than those of m_{VM} are for every value of β at which comparison is made. This shows that the present method has an advantage over the method of Ref. 12 in these intermediate regions.

III. DISCUSSION

Let us begin by considering the validity of the decoupling approximation of Sec. II C in the weak-coupling region ($\beta\sim\infty$). In Sec. II G the discrepancy is observed between two mass gaps m and m_{WC} at $\beta\sim\infty$ with N not large. Because the mass gap m, if given without any approximation, is expected at $\beta\sim\infty$ to have an expression very similar to m_{WC},[30] that discrepancy seems to indicate that the present approximation is not satisfactory at $\beta\sim\infty$ with N not large. Two possible directions are pointed out below to improve our approximation there by recalling the development in the Heisenberg model (see Sec. I).

(1) One way is to introduce additional "anomalous" terms in the expressions of higher-order Green's functions. This procedure is inspired by the works of Wortis,[21] Tahir-Kheli,[20] and Haas and Jarrett.[31] Putting it explicitly, suppose we add the terms $Q_1\delta_{xx'}\delta(t-t')$ and $Q_2^{\pm}\delta_{xx'}\delta(t-t')$ to the right-hand sides of Eqs. (10) and (13), respectively, where $Q_{1,2}$ are assumed to be functions of β and N only. The space-time structure $\delta_{xx'}\delta(t-t')$ is chosen in accord with those considered in the above works, and implies including the unit operator into our approximate ring of the Green's functions. Note that this modification does not affect the values of C_1,D_1,D_2 given in Sec. II because the added terms have no discontinuities on the real ω axis.

The effect of these terms is described completely by the following replacement of R appearing in Eqs. (20) and (21) by $\hat{R}$:

$$R\rightarrow\hat{R}\equiv R+(2\pi)^{-1}(-\beta^{-2}Q_1+Q_2^{+}+Q_2^{-})\,. \tag{34}$$

Therefore, for example, one can reproduce the expression (31) of M_{WC} by choosing $Q_1,Q_2^{\pm}$ so that they give

$$\hat{R}=(N-2)[N(N-1)]^{-1/2}R \tag{35}$$

at $\beta\sim\infty$. Of course the introduction of $Q_1,Q_2^{\pm}$ should preserve the satisfactory features of the method in Sec. II in the large-N limit and in the strong-coupling region (see Secs. II E and II F). It requires, therefore, Q_1 and $Q_2^{\pm}$ to behave so that $\hat{R}$ approaches R in the limits of $N\rightarrow\infty$ and of $\beta\rightarrow 0$.

It is of interest to point out the similarity in the required behaviors of our additional terms ($Q_1,Q_2^{\pm}$) and of the additional term in the three-dimensional Heisenberg model. In fact, for the latter, Tahir-Kheli[20] estimated it, referring to the results of other techniques (Dyson's spin-wave method, high-temperature expansions, etc.), as follows: at high temperatures and for large spins it is negligible; and at low temperatures it is significant for $s=\frac{1}{2}$ and relatively unimportant for higher spins; near the Curie temperature T_C it is important except for large spins.

The similarity becomes obvious if one recalls the following plausible correspondences[18,25] between these two spin systems: $N\leftrightarrow s$; $\beta\sim 0\leftrightarrow T\sim\infty$; $\beta\sim\infty\leftrightarrow T\sim T_C$.[32] These similar behaviors may be taken as one of the indications of mutual relevance between these two systems, and of the usefulness, to some extent, of the Green's-function method to our system as well as to the Heisenberg model.

One unsatisfactory point in this direction of improvement is that it seems rather hard to determine the additional terms completely within the framework of the Green's-function method, namely, to determine them through some (approximate) principles over the entire region of β and N without recourse to the results of other independent analyses valid for some particular regions of β and N.

(2) Another direction is to postpone the decoupling procedure one step (or more) further. It again means to enlarge our ring of the Green's functions, and will introduce extra parameters which will serve to express more accurately the structures embodied in the Green's functions. For the Heisenberg model, particularly in low dimensions, some studies[23] have been made in this direction as mentioned in Sec. I. Each of them employs its own approach to get some interesting results, yet involving more or less unsatisfactory points. Generally speaking, higher-order decoupling schemes require more sophisticated principles and elaborate manipulations to introduce and determine decoupling parameters and the physical reasoning to employ them becomes less simple and intuitive. For this reason and others, no explicit attempt in this direction is discussed in this paper.

For further possible directions, to get some ideas of them, it may be useful to refer to recent developments of the Green's-function approach to the Heisenberg model[33] again.

Now let us discuss some indications of the usefulness of the Green's-function method in studying another interesting system, the four-dimensional SU(N) lattice gauge theory.[3] Thanks to its great flexibility, owing mainly to associated decoupling procedures, it seems quite probable to get closed expressions for, say, glueball masses over the entire region of N and gauge-coupling constant in a simple self-consistent manner. Because in the large-N limit gauge-invariant correlation functions are known to obey the factorization rule,[34] a simple decoupling procedure is expected to give rise to good results in this limit. However, at the same time, such a simple decoupling may probably give unsatisfactory results at weak gauge couplings with N not large, because the lattice gauge theory is described again by the variables having compact domains

and satisfying SU(N) algebra. In considering such a problem, if any, the discussion given above for the present spin system is expected to be helpful. Some investigation of lattice gauge theories by the Green's-function method is undertaken and expected to be reported in the future.

Note added in proof. After the completion of this work the author received a private communication from Dr. A Patkós on a possibility that some values of m_{MC} given in Ref. 8 may need some corrections.

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*Present address: Department of Physics, Freie University Berlin, D-1000 Berlin 33, Federal Republic of Germany.

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[25]Recall that our *Hamiltonian* system is put at zero temperature, although the parameter β could be regarded as the inverse temperature in the original classical spin system in a two-dimensional lattice.

[26]Below, $E(k)$ of (20.2) is treated to be real and positive. This assumption can be verified easily for the entire region of N (≥ 2) and β ($0 < \beta < \infty$) by inspecting Eqs. (24), (25), and (27) below.

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[28]For the cases of general N (≥ 2), see also Appendix B of Ref. 12. There the expressions for H in Eqs. (B2) and (B4) should be corrected by multiplying a factor $\frac{1}{2}$ to their right-hand sides.

[29]For $N = 2$, the values of a_1/a_0 for m_{MC} are not listed explicitly in Ref. 8, but some discussion for them can be found there.

[30]The argument for it comes as follows. For $\beta \sim \infty$, slowly-varying spin configurations (i.e., long-wavelength modes) are supposed to play the dominant role both on a lattice and in the continuum. And these configurations hardly notice the difference between lattice and continuum.

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