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Boson formulation of fermion field theories

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The nonperturbative connection between a canonical Fermi field and a canonical Bose field in two dimensions is developed and its validity verified according to the tenets of quantum field theory. We advocate the point of view that a boson formulation offers a unifying theme in understanding the structure of many theories. This is illustrated by the boson formulation of a multifermion theory with chiral and internal symmetries. Many features of the massless theory, such as dynamical mass generation with asymptotic-freedom behavior, hidden chiral symmetry, and connections with models of apparently different internal symmetries, are readily transparent through such fermion-boson metamorphosis.

I. INTRODUCTION

One of the most interesting features of field theories in two dimensions is the equivalence between theories constructed with fermions and theories constructed with bosons. A theory in which the basic field operator satisfies a manifest anticommutation rule may be intrinsically related to a theory in which the field operator satisfies only a commutation relation. A well-known example is the equivalence between the quantum sine-Gordon theory and the massive Thirring model.[1] A soliton in the sine-Gordon theory corresponds to an elementary fermion in the massive Thirring model, while the topological quantum number of the soliton corresponds to the charge of the massive Thirring case. It is therefore obviously important to recognize those theories which are equivalent despite their apparently different structures. Many such equivalences have been established;[1-5] many more will undoubtedly be discovered. It is the objective of this work to develop a consistent scheme for constructing a canonical Fermi field in terms of a real canonical scalar field and verify that such a procedure is valid and consistent with the tenets of quantum field theory. We shall advocate the point of view that a boson formulation offers simplicity as well as a unifying theme in understanding the structure of many theories, and that new features are readily transparent through such a fermion-boson metamorphosis.

There are two ways of establishing equivalence between two quantum field theories. One is to compare Green's functions in the theories and adjust parameters. The other is to construct an operator solution of the field equation in one theory by means of a nonlocal expression of the operator of another. The first method presumes certain *a priori* knowledge that the two theories are related, for otherwise there is no guideline as to which two theories are to be compared. An explicit construction of the fermion operator in a theory in terms of the boson operator of another theory is known as the boson representation. In this method, a theory of fermions is transformed directly into a theory of bosons and equivalence follows as a consequence. This latter approach provides not only interesting connections between various models, but could also elucidate the features of a multicomponent theory in a nonperturbative and model-independent way.

We shall, first of all, develop the boson-representation scheme in the simple free-field case. The validity of this process is examined with special attention to the correct anticommutation structure, Green's functions, cluster properties, and Lorentz invariance of the constructed fermion theory. Such a procedure is then extended to the self-interacting fermion case, the massless Thirring model.[6] The transition of this model to a free boson theory and the subsequent metamorphosis of the massive model to the sine-Gordon theory are straightforward in the boson formulation. Of particular interest is the boson formulation of a multifermion theory with chiral and internal symmetries. It will be seen that dynamical generation of mass with asymptotic-freedom behavior occurs when the theory undergoes boson transmutation and that continuous chiral symmetry is naturally preserved in the massive case. In many situations, the dynamics of the system depends on the specific number of fermions and that different models of the same system can have quite different properties. An interesting application of the boson-representation scheme will be encountered when certain fermion theories are shown to be equivalent despite their apparently different internal symmetries. Many unusual and hidden symmetries of a fermion theory are only obvious in the boson formulation.[7] These developments have greatly extended the original scope of finding equivalences between different theories to generating new tools of investigation in two dimensions. The extension of the boson-representation scheme to four dimensions[8] is then a logical and important generalization, one which will be of potential utility in field theory.

II. ASPECTS OF BOSON REPRESENTATION

The equivalence between a fermion theory and a boson theory has so far been a unique phenomenon in two dimensions. For a simple illustration, we note that the energy-momentum tensor for a free massless Fermi field $\psi(x)$ may be written in terms of its current operators

$$j^{\mu}(x)=\tfrac{1}{4}\lim_{\epsilon\to 0}\left[\bar{\psi}\left[x+\frac{\epsilon}{2}\right],\gamma^{\mu}\psi\left[x-\frac{\epsilon}{2}\right]\right], \tag{2.1}$$

where only spatial coordinates are being explicitly displayed, in the Sommerfield-Sugawara form[9] as

$$T^{\mu\nu}=\frac{\pi}{2}(j^{\mu}j^{\nu}+j^{\nu}j^{\mu}-g^{\mu\nu}j_{\lambda}j^{\lambda}) , \quad g^{00}=-g^{11}=1 . \tag{2.2}$$

This has a familiar structure to the canonical energy-momentum tensor for a free massless scalar field

$$T^{\mu\nu}=\tfrac{1}{2}\partial^{\mu}\phi\partial^{\nu}\phi+\tfrac{1}{2}\partial^{\nu}\phi\partial^{\mu}\phi-g^{\mu\nu}\tfrac{1}{2}\partial_{\lambda}\phi\partial^{\lambda}\phi . \tag{2.3}$$

In addition, the commutator structure of the fermion currents at equal times, given by

$$[j^{\mu}(x),j^{\nu}(y)]=-\frac{i}{\pi}\epsilon^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y) , \quad \epsilon^{01}=-\epsilon^{10}=1 , \tag{2.4}$$

is also very similar to that of the canonical commutation rule for the scalar field

$$[\phi(x),\pi(y)]=i\delta(x-y) . \tag{2.5}$$

The two energy-momentum tensors in Eqs. (2.2) and (2.3) and the two commutators in Eqs. (2.4) and (2.5) are identical, respectively, if

$$j^{\mu}=\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_{\nu}\phi . \tag{2.6}$$

In this case, a single massless fermion has the same Hamiltonian as that for a single massless boson. Equation (2.6) is one of the basic ingredients in establishing many equivalences in two-dimensional cases.

In order to develop a consistent scheme for constructing a canonical Fermi field from a canonical Bose field, we shall regard the current operators defined in Eq. (2.1) as basic dynamical variables. The commutation relations between the current operators and the Fermi field at equal times

$$[j^{\mu}(x),\psi(y)]=-(g^{0\mu}+\epsilon^{0\mu}\gamma^{5})\psi(x)\delta(x-y) , \tag{2.7}$$

together with the Heisenberg equation of motion

$$[P_{\mu},A(x)]=-i\partial_{\mu}A(x) \tag{2.8}$$

for any operator $A(x)$ in the theory, determine the dynamics of the system completely. The momentum operators P^{μ} are the spatial integrals of the energy-momentum density, i.e.,

$$P^{\mu}=\int T^{0\mu}(x')dx' . \tag{2.9}$$

The equation of motion for the operator $\psi(x)$ is obtained by using Eqs. (2.7)–(2.9). We find

$$\partial_0\psi(x)=-i\pi[j^{0}(x)+\gamma^{5}j^{1}(x)]\psi(x) , \tag{2.10}$$

$$\partial_1\psi(x)=i\pi[j^{1}(x)+\gamma^{5}j^{0}(x)]\psi(x) , \tag{2.11}$$

which together give the Dirac equation

$$i\gamma^{\mu}\partial_{\mu}\psi(x)=0 \tag{2.12}$$

for a free massless fermion when multiplied by the γ matrices

$$\gamma^{0}=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} , \quad \gamma^{1}=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix} , \quad \gamma^{5}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} . \tag{2.13}$$

We may formally integrate Eq. (2.11) and obtain an expression for the Fermi field as

$$\psi(x)=\exp\left[i\pi\int[j^{1}(x')+\gamma^{5}j^{0}(x')]dx'\right]\psi_0 , \tag{2.14}$$

where $\psi_0=\binom{1}{1}$ is a constant in the form of a spinor.

If we write the Heisenberg equation of motion for the current operators $j^{\mu}(x)$, we find simply

$$\partial_{\mu}j^{\mu}(x)=0 . \tag{2.15}$$

The result is a statement of the current conservation condition in the theory. As a consequence, the currents may be expressed in terms of a scalar field $\phi(x)$ as in Eq. (2.6). We can now express the two-component Fermi field $\psi(x)$ directly in terms of the scalar field $\phi(x)$ in the form

$$\begin{bmatrix}\psi_+(x)\\ \psi_-(x)\end{bmatrix}=\begin{bmatrix}C_+\exp\left[i\sqrt{\pi}\left[\phi(x)-\int_{-\infty}^{x}\partial_0\phi(x')dx'\right]\right]\\ C_-\exp\left[-i\sqrt{\pi}\left[\phi(x)+\int_{-\infty}^{x}\partial_0\phi(x')dx'\right]\right]\end{bmatrix} , \tag{2.16}$$

where $C_{\pm}$ are constants to be adjusted so that the spinor components satisfy the canonical anticommutation rule for fermions,

$$\{\psi_{\rho}(x),\psi_{\sigma}^{\dagger}(y)\}=\delta(x-y)\delta_{\rho\sigma} , \quad \rho,\sigma=+,- . \tag{2.17}$$

If we introduce the chiral components of the scalar field

$$\phi_{\pm}(x)=\pm\frac{1}{\sqrt{2\pi}}\int_{0}^{\pm\infty}\frac{dk}{\sqrt{2k^{0}}}[c^{(-)}(k)e^{-ik\cdot x}+c^{(+)}(k)e^{ik\cdot x}]e^{-\alpha|k|/2} , \tag{2.18}$$

where $k^{0}=(k^{2}+\mu^{2})^{1/2}$ is the energy frequency of the boson of mass μ, $c^{(+)}(k)$ and $c^{(-)}(k)$ are the creation and annihilation operators, respectively, we may further represent the Fermi field in the compact form

$$\begin{bmatrix}\psi_+(x)\\ \psi_-(x)\end{bmatrix}=\frac{1}{(2\pi\alpha)^{1/2}}\begin{bmatrix}\exp[+i\sqrt{4\pi}\,\phi_+(x)]\\ \exp[-i\sqrt{4\pi}\,\phi_-(x)]\end{bmatrix} . \tag{2.19}$$

The quantity $1/\alpha$ is an ultraviolet cutoff and α will be taken to zero at the end of any calculation involving $\psi(x)$. According to Eq. (2.16), we also have the definition

$$\phi_{\pm}(x)=\tfrac{1}{2}\left[\phi(x)\mp\int_{-\infty}^{x}\pi(x')dx'\right] . \tag{2.20}$$

It follows that

$$\phi_+(x)+\phi_-(x)=\phi(x)\ , \tag{2.21}$$

which is consistent with the decomposition in Eq. (2.18).

A. Anticommutativity

The chiral components of the scalar field do not satisfy the usual commutation rules for the scalar field. If ϕ and π satisfy the commutation rule in Eq. (2.5), then it follows from Eq. (2.20) that

$$[\phi_\pm(x),\phi_\pm(y)]=\pm\frac{i}{4}\epsilon(x-y)\ , \tag{2.22}$$

$$[\phi_+(x),\phi_-(y)]=\frac{i}{4}\ . \tag{2.23}$$

The term $\epsilon(x-y)=(x-y)/|x-y|$ is an odd step function and has an absolute value of unity. These commutator structures are important for establishing the correct anticommutation rules for the spinor components of the Fermi field. For example, using the operator identity

$$e^Ae^B=e^Be^Ae^{[A,B]} \tag{2.24}$$

when $[A,[A,B]]=0$, we find

$$\begin{aligned}\psi_+(x)\psi_+(y)&=\frac{1}{2\pi\alpha}e^{i\sqrt{4\pi}\phi_+(x)}e^{i\sqrt{4\pi}\phi_+(y)}\\&=-\frac{1}{2\pi\alpha}e^{i\sqrt{4\pi}\phi_+(y)}e^{i\sqrt{4\pi}\phi_+(x)}\\&=-\psi_+(y)\psi_+(x)\ ,\end{aligned} \tag{2.25}$$

with similar results holding for other combinations of spinor components. The homogeneous anticommutation rules for the Fermi field

$$\{\psi_\rho(x),\psi_\sigma(y)\}=0 \tag{2.26}$$

are therefore established.

To obtain the inhomogeneous anticommutation relation, it is necessary to examine the short-distance behavior of the fermion two-point function. With the use of the identity

$$e^Ae^B=:e^{A+B}:\exp[\langle 0|AB|0\rangle+\tfrac{1}{2}\langle 0|A^2|0\rangle+\tfrac{1}{2}\langle 0|B^2|0\rangle]\ , \tag{2.27}$$

where $:A:$ denotes the normal ordering of the operator A, and $|0\rangle$ the free Bose vacuum, we find

$$\psi_+(x)\psi_+^\dagger(y)=\frac{1}{2\pi\alpha}:e^{i\sqrt{4\pi}\phi_+(x)-i\sqrt{4\pi}\phi_+(y)}:\exp\{4\pi[\langle 0|\phi_+(x)\phi_+(y)|0\rangle-\tfrac{1}{2}\langle 0|\phi_+(x)\phi_+(x)|0\rangle-\tfrac{1}{2}\langle 0|\phi_+(y)\phi_+(y)|0\rangle]\}\ . \tag{2.28}$$

The exponential term in Eq. (2.28) is evaluated by considering the momentum-space expansion of the scalar two-point function according to Eq. (2.18). The result is

$$\langle 0|\phi_+(x)\phi_+(y)-\phi_+{}^2(x)|0\rangle=\frac{1}{4\pi}\ln\frac{i\alpha}{x-y+i\alpha}\ . \tag{2.29}$$

It follows that for $x\sim y$,

$$\psi_+(x)\psi_+^\dagger(y)=\frac{1}{2\pi\alpha}:e^{i\sqrt{4\pi}[\phi_+(x)-\phi_+(y)]}:\frac{i\alpha}{x-y+i\alpha}\ . \tag{2.30}$$

Similarly, $\psi_+^\dagger(y)\psi_+(x)$ is obtained as the same expression in Eq. (2.30) with α replaced by $-\alpha$. The resulting combination

$$\begin{aligned}\psi_+(x)\psi_+^\dagger(y)+\psi_+^\dagger(y)\psi_+(x)&=\lim_{\alpha\to 0}\frac{1}{2\pi\alpha}\left[\frac{i\alpha}{x-y+i\alpha}-\frac{i\alpha}{x-y-i\alpha}\right]\\&=\frac{1}{\pi}\lim_{\alpha\to 0}\frac{\alpha}{\alpha^2+(x-y)^2}\end{aligned} \tag{2.31}$$

then has the desired inhomogeneous anticommutation structure of Eq. (2.17).

B. Green's functions

It is important that the fermion two-point functions $\Gamma_\pm(x,y)$ obtained through the scalar two-point functions are identical to those which would be obtained by using a momentum-space expansion of the Fermi field itself. A direct calculation in the representation

$$\psi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{dp}{\sqrt{2p^0}}[a^{(-)}(p)u(p)e^{-ip\cdot x}+b^{(+)}(p)v(p)e^{ip\cdot x}]e^{-\alpha|p|/2}\ , \tag{2.32}$$

where $a^{(-)}(p)$ and $b^{(+)}(p)$ are the annihilation and creation operators for the spinors $u(p)$ and $v(p)$, respectively, shows that this is indeed the case, since

$$\begin{aligned}\Gamma_\pm(x,y)&=\langle 0|\psi_\pm(x)\psi_\pm^\dagger(y)|0\rangle\\&=\mp\frac{1}{2\pi i}\frac{1}{x-y\pm i\alpha}\ .\end{aligned} \tag{2.33}$$

Higher Green's functions for the Fermi field can all be obtained through the corresponding scalar functions. The four-point function

$$\Gamma_+(x,y,w,z)=\langle 0|\psi_+(x)\psi_+^\dagger(y)\psi_+(w)\psi_+^\dagger(z)|0\rangle \tag{2.34}$$

in the boson representation is

$$\frac{1}{(2\pi\alpha)^2}\langle 0|e^{i\sqrt{4\pi}\phi_+(x)}e^{-i\sqrt{4\pi}\phi_+(y)}\times e^{i\sqrt{4\pi}\phi_+(w)}e^{-i\sqrt{4\pi}\phi_+(2)}|0\rangle\ . \tag{2.35}$$

Through an identity similar to that in Eq. (2.27), the expression in (2.35) becomes

$$\frac{1}{(2\pi\alpha)^2}\exp\{4\pi[\langle 0|\phi_+(x)\phi_+(y)-\phi_+{}^2|0\rangle-\langle 0|\phi_+(x)\phi_+(w)-\phi_+{}^2|0\rangle$$
$$+\langle 0|\phi_+(x)\phi_+(z)-\phi_+{}^2|0\rangle+\langle 0|\phi_+(y)\phi_+(w)-\phi_+{}^2|0\rangle-\langle 0|\phi_+(y)\phi_+(z)-\phi_+{}^2|0\rangle$$
$$+\langle 0|\phi_+(w)\phi_+(z)-\phi_+{}^2|0\rangle]\}\ , \tag{2.36}$$

where the quantity $\langle 0|\phi_+{}^2|0\rangle$ without explicit spatial coordinates denotes the vacuum expectation value of a two-point function at the same point. The above expression takes the following form when substituted with the appropriate scalar two-point functions:

$$\frac{1}{(2\pi i)^2}\frac{(x-w+i\alpha)(y-z+i\alpha)}{(x-y+i\alpha)(x-z+i\alpha)(y-w+i\alpha)(w-z+i\alpha)}=\frac{1}{(2\pi i)^2}\left[\frac{1}{(x-y+i\alpha)}\frac{1}{(w-z+i\alpha)}-\frac{1}{(x-z+i\alpha)}\frac{1}{(w-y+i\alpha)}\right]\ , \tag{2.37}$$

in the limit $\alpha\to 0$. It can be seen that the last expression is simply the sum of products of the fermion two-point functions. It follows that the decomposition

$$\langle 0|\psi_+(x)\psi_+^\dagger(y)\psi_+(w)\psi_+^\dagger(z)|0\rangle=\langle 0|\psi_+(x)\psi_+^\dagger(y)|0\rangle\langle 0|\psi_+(w)\psi_+^\dagger(z)|0\rangle$$
$$-\langle 0|\psi_+(x)\psi_+^\dagger(z)|0\rangle\langle 0|\psi_+(w)\psi_+^\dagger(y)|0\rangle \tag{2.38}$$

holds in the boson representation. A similar result is valid for $\Gamma_-(x,y,w,z)$ when it is evaluated in the same way.

C. Cluster properties

A general n-point function $\Gamma_+(x_1,\dots,x_n)$ has the following decomposition into sums of products of all possible two-point functions in the boson representation:

$$\langle 0|\psi_+(x_1)\psi_+^\dagger(x_2)\cdots\psi_+(x_{n-1})\psi_+^\dagger(x_n)|0\rangle=\frac{1}{(2\pi\alpha)^{n/2}}\langle 0|e^{i\sqrt{4\pi}\phi_+(x_1)}e^{-i\sqrt{4\pi}\phi_+(x_2)}\cdots e^{i\sqrt{4\pi}\phi_+(x_{n-1})}e^{-i\sqrt{4\pi}\phi_+(x_n)}|0\rangle$$
$$=\frac{1}{(2\pi\alpha)^{n/2}}\exp\left[4\pi\sum_{i<j}^{n}(-1)^{j-i+1}[\langle 0|\phi_+(x_i)\phi_+(x_j)-\phi_+{}^2|0\rangle]\right]$$
$$=\frac{1}{(2\pi\alpha)^{n/2}}\exp\left[\sum_{i<j}(-1)^{j-i+1}\ln 2\pi\alpha\langle 0|\psi_+(x_i)\psi_+^\dagger(x_j)|0\rangle\right]\ . \tag{2.39}$$

Accordingly, we have obtained the cluster decomposition property of the fermion Green's functions. The cluster property is an important property for establishing the uniqueness of the vacuum in a field theory. In the present case, we have demonstrated that the Bose vacuum, as well as the Fermi vacuum, are unique in their own Fock spaces. The two vacuums are nonetheless mapped into one another since under a chiral transformation, the Fermi vacuum will be shifted by a constant phase, while the Bose vacuum by a constant parameter.

D. Lorentz invariance

Another point which deserves attention is whether the construction outlined so far gives the Fermi field a genuine spinor character, as required by its transformation property under proper Lorentz transformations. Such a criterion is given by the commutator

$$i[M^{\mu\nu},\psi]=x^\mu\partial^\nu\psi-x^\nu\partial^\mu\psi+\tfrac{1}{4}[\gamma^\mu,\gamma^\nu]\psi\ , \tag{2.40}$$

where

$$M^{\mu\nu}=x^\mu P^\nu-x^\nu P^\mu \tag{2.41}$$

is the angular momentum density. The only nonvanishing component of the angular momentum tensor in two dimensions is

$$M^{01}=x^0\int T^{01}(y)dy-\int yT^{00}(y)dy\ , \tag{2.42}$$

where

$$T^{0\mu}=(\partial_1\phi_+)^2+g^{0\mu}(\partial_1\phi_-)^2 \tag{2.43}$$

are the energy-momentum tensor components expressed in terms of the chiral components of the scalar field. To establish the condition stated in Eq. (2.40), it is necessary to consider the normal-ordered form of the exponential connection in Eq. (2.19). For the spinor component ψ_+, Eq. (2.40) takes the form

$$i[M^{01},\psi_+]=\frac{ix^0}{(2\pi\alpha)^{1/2}}\int[(\partial_1\phi_+(y))^2-(\partial_1\phi_-(y))^2,:e^{i\sqrt{4\pi}\phi_+(x)}:]dy$$
$$-\frac{i}{(2\pi\alpha)^{1/2}}\int y[(\partial_1\phi_+(y))^2+(\partial_1\phi_-(y))^2,:e^{i\sqrt{4\pi}\phi_+(x)}:]dy\;. \quad (2.44)$$

The commutators in the above integrals are evaluated with the use of the operator identities

$$[A^2,:e^B:]=[A,B](A:e^B:+:e^B:A)\;, \quad (2.45)$$

$$A:e^B:=:Ae^B:+\langle 0|AB|0\rangle:e^B:\;, \quad (2.46)$$

and the once-differentiated commutation relations from Eqs. (2.22) and (2.23). The first integral in Eq. (2.44) may be written as

$$-\sqrt{\pi}\int\delta(x-y)[:\partial_1\phi_+(y)e^{i\sqrt{4\pi}\phi_+(x)}:+:e^{i\sqrt{4\pi}\phi_+(x)}\partial_1\phi_+(y):]dy$$
$$-2\pi i\int\delta(x-y)[\langle 0|\partial_1\phi_+(y)\phi_+(x)|0\rangle+\langle 0|\phi_+(x)\partial_1\phi_+(y)|0\rangle]:e^{i\sqrt{4\pi}\phi_+(x)}:dy\;, \quad (2.47)$$

where

$$\langle 0|\partial_1\phi_+(y)\phi_+(x)|0\rangle=\frac{1}{4\pi(x-y-i\alpha)}\;, \quad (2.48)$$

$$\langle 0|\phi_+(x)\partial_1\phi_+(y)|0\rangle=\frac{1}{4\pi(x-y+i\alpha)}\;. \quad (2.49)$$

The first integral in expression (2.47), when multiplied by the factor $ix^0/(2\pi\alpha)^{1/2}$, is simply the term

$$-\frac{x^0}{(2\pi\alpha)^{1/2}}\partial_1:e^{i\sqrt{4\pi}\phi_+(x)}:=x^0\partial^1\psi_+(x)\;. \quad (2.50)$$

The second integral in (2.47) may be evaluated by choosing an explicit representation for the δ function, such as the one in Eq. (2.31). This brings the integral to the form

$$\frac{1}{4\pi}\int\left[\frac{1}{(x-y+i\alpha)^2}-\frac{1}{(x-y-i\alpha)^2}\right]:e^{i\sqrt{4\pi}\phi_+(x)}:dy\;. \quad (2.51)$$

By a contour integration, we find the integrand in the above develops a vanishing residue, and accordingly, the integral itself vanishes. A similar evaluation of the commutator in the second integral in Eq. (2.44) gives the same expression as in Eq. (2.47), except this time the integrands are multiplied by the extra spatial coordinate y. The integrand in (2.51), when multiplied by y, develops a residue of unity. Accordingly, the second integral in Eq. (2.44), when multiplied by the factor $-i/(2\pi\alpha)^{1/2}$, gives the terms

$$x\partial_1\psi_+(x)+\tfrac{1}{2}\psi_+(x)=-x\partial^0\psi_+(x)+\tfrac{1}{2}\psi_+(x)\;. \quad (2.52)$$

The results of Eqs. (2.50) and (2.52) show that for ψ_+,

$$i[M^{01},\psi_+]=x^0\partial^1\psi_+-x\partial^0\psi_++\tfrac{1}{2}\psi_+\;, \quad (2.53)$$

while an analogous calculation with ψ_- gives

$$i[M^{01},\psi_-]=x^0\partial^1\psi_--x\partial^0\psi_--\tfrac{1}{2}\psi_-\;. \quad (2.54)$$

Equations (2.53) and (2.54) are then the two components of Eq. (2.40). They show that the Fermi field indeed possesses a spinor structure in the boson construction.

E. Bilinear forms

Certain bilinear quantities of the Fermi field transform, however, as simple local densities of the Bose field, even though the connection between the Fermi field and the Bose field is nonlocal in character. For consistency, we find that the canonical Hamiltonian for the Fermi field transforms directly into the canonical Hamiltonian for the free scalar field through the connection in Eq. (2.19):

$$-i\bar{\psi}\gamma^1\partial_1\psi\leftrightarrow\tfrac{1}{2}(\partial_0\phi)^2+\tfrac{1}{2}(\partial_1\phi)^2\;. \quad (2.55)$$

We shall regard these two Hamiltonians as being separately derived from their corresponding free-field Lagrangians via the canonical method. In this respect, we may then state the formal equivalence of the Lagrangians for the Fermi field and the Bose field:

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi\leftrightarrow\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi\;. \quad (2.56)$$

It should be pointed out that a direct application of the connection formula at the Lagrangian level would not affirm such an equivalence. Other bilinear expressions in two dimensions are

$$\bar{\psi}\psi\leftrightarrow-\frac{1}{\pi\alpha}\cos\sqrt{4\pi}\,\phi\;, \quad (2.57a)$$

$$i\bar{\psi}\gamma^5\psi\leftrightarrow\frac{1}{\pi\alpha}\sin\sqrt{4\pi}\,\phi\;, \quad (2.57b)$$

$$\bar{\psi}\gamma^\mu\psi\leftrightarrow\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi\;, \quad (2.57c)$$

$$\bar{\psi}\gamma^5\gamma^\mu\psi\leftrightarrow\frac{1}{\sqrt{\pi}}\partial^\mu\phi\;. \quad (2.57d)$$

These equivalences have important consequences. Bilinear terms appearing in the Lagrangian of a fermion theory may now be written in terms of the corresponding expressions involving the Bose field so that the fermion theory may be considered as an equivalent boson theory. This provides not only interesting equivalences between theories in two dimensions, but could also lead to a simpler version of the same theory, such as the well-known equivalence between the massless Thirring model and the free massless pseudoscalar theory,[10,11] and the equivalence between massless quantum electrodynamics in two dimensions and the massive scalar free-field theory.[2]

III. THE THIRRING MODEL

We extend the construction of the Fermi field in the free-field formulation to interacting theories and to theories with multifermions. The only self-interacting theory in two dimensions involving a single fermion is the massless Thirring model, described by the formal Lagrangian

$$\mathscr{L}=i\bar{\psi}\gamma^\mu\partial_\mu\psi-\frac{g}{2}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)\ . \tag{3.1}$$

It is known in this model that a consistent definition of the interacting currents j^μ which are compatible with the relativistic requirement is given by[10]

$$j^\mu(x)=\left[g^{0\mu}+\frac{\epsilon^{0\mu}}{1+g/\pi}\right]\times\frac{1}{4}\lim_{\epsilon\to 0}\left[\bar{\psi}\left[x+\frac{\epsilon}{2}\right],\gamma^\mu\psi\left[x-\frac{\epsilon}{2}\right]\right]\ , \tag{3.2}$$

where there is no summation over the index μ. To be more precise, one should regard the coupling between the fermions as a nonlocal function of the spatial coordinates, whose Fourier momentum increases at a slower rate than the ultraviolet cutoff of the theory so that in the cutoff limit, the current operator behaves as if it is a simple juxtaposition of the canonical Fermi fields. The equal-time commutation rules between the currents and the Fermi field in the Thirring model are

$$[j^\mu(x),\psi(y)]=-\left[g^{0\mu}+\frac{\epsilon^{0\mu}}{1+g/\pi}\gamma^5\right]\psi(x)\delta(x-y)\ , \tag{3.3}$$

$$[j^\mu(x),j^\nu(y)]=-\frac{i}{\pi}\left[\frac{\epsilon^{\mu\nu}}{1+g/\pi}\right]\frac{\partial}{\partial x}\delta(x-y)\ , \tag{3.4}$$

while the energy-momentum tensor expressed in terms of the interacting currents is

$$T^{\mu\nu}=\left[1+\frac{g}{\pi}\right]\frac{\pi}{2}(j^\mu j^\nu+j^\nu j^\mu-g^{\mu\nu}j_\lambda j^\lambda)\ . \tag{3.5}$$

The time and space derivatives of the Fermi field follow from the Heisenberg equation (2.8). They are

$$\partial_0\psi(x)=-i\pi\left[1+\frac{g}{\pi}\right]\left[j^0+\frac{\gamma^5}{1+g/\pi}j^1\right]\psi(x)\ , \tag{3.6}$$

$$\partial_1\psi(x)=i\pi\left[1+\frac{g}{\pi}\right]\left[j^1+\frac{\gamma^5}{1+g/\pi}j^0\right]\psi(x)\ . \tag{3.7}$$

Together they give the equation of motion for the massless Thirring model

$$i\gamma^\mu\partial_\mu\psi(x)=g\gamma^\mu j_\mu\psi(x)\ . \tag{3.8}$$

The currents in Eq. (3.2) are again conserved, and may therefore be expressed in terms of a scalar field $\tilde{\phi}(x)$ in the form

$$j^\mu(x)=\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\tilde{\phi}(x)\ . \tag{3.9}$$

For Eq. (3.9) to be compatible with the commutator in Eq. (3.4), we find that

$$[\tilde{\phi}(x),\partial_0\tilde{\phi}(y)]=\frac{1}{1+g/\pi}i\delta(x-y)\ . \tag{3.10}$$

This would only coincide with the canonical commutation relation of Eq. (2.5) for a free scalar field $\phi(x)$ if

$$\tilde{\phi}(x)=\frac{\phi(x)}{(1+g/\pi)^{1/2}}\ . \tag{3.11}$$

The canonical conjugate for $\tilde{\phi}(x)$ is, however,

$$\tilde{\pi}(x)=(1+g/\pi)^{1/2}\partial_0\phi(x)\ . \tag{3.12}$$

With this definition, $\tilde{\phi},\tilde{\pi}$ have the canonical commutation structure of Eq. (2.5).

If we integrate Eq. (3.7) and express the currents in terms of $\tilde{\phi}(x)$ according to Eq. (3.9), we arrive at the following representation for the Fermi field:

$$\psi_\rho(x)=\begin{bmatrix}\psi_+(x)\\ \psi_-(x)\end{bmatrix}=\frac{1}{(2\pi\alpha)^{1/2}}\begin{bmatrix}\exp[+i\sqrt{4\pi}\tilde{\phi}_+(x)]\\ \exp[-i\sqrt{4\pi}\tilde{\phi}_-(x)]\end{bmatrix}\ , \tag{3.13}$$

where

$$\tilde{\phi}_\pm(x)=\tfrac{1}{2}\left[\tilde{\phi}(x)\mp\int_{-\infty}^{x}\tilde{\pi}(x')dx'\right]\ . \tag{3.14}$$

The chiral components $\tilde{\phi}_\pm(x)$ satisfy the same commutation rules as those of $\phi_\pm(x)$ given in Eqs. (2.22) and (2.23), therefore the Fermi field has again the correct homogeneous anticommutation structure of Eq. (2.26).

The two-point functions in the interacting case are very similar to those in the free fermion theory. Analogous to Eq. (2.28), we find, for ψ_+,

$$\psi_+(x)\psi_+^\dagger(y)=\frac{1}{2\pi\alpha}:e^{i\sqrt{4\pi}\tilde{\phi}(x)-i\sqrt{4\pi}\tilde{\phi}(y)}:\exp\{4\pi[\langle 0|\tilde{\phi}_+(x)\tilde{\phi}_+(y)|0\rangle-\langle 0|\tilde{\phi}_+{}^2|0\rangle]\}\ . \tag{3.15}$$

Here the scalar two-point function is given by

$$\langle 0|\tilde{\phi}_+(x)\tilde{\phi}_+(y)|0\rangle-\langle 0|\tilde{\phi}_+{}^2|0\rangle=\frac{1}{4}\left[\left\langle 0\left|\left[\frac{\phi(x)}{\gamma}-\gamma\int_{-\infty}^{x}\pi(x')dx'\right]\left[\frac{\phi(y)}{\gamma}-\gamma\int_{-\infty}^{y}\pi(y')dy'\right]\right|0\right\rangle-\left\langle 0\left|\left[\frac{\phi(x)}{\gamma}-\gamma\int_{-\infty}^{x}\pi(x')dx'\right]^2\right|0\right\rangle\right]\ , \tag{3.16}$$

where $\gamma=(1+g/\pi)^{1/2}$. The corresponding scalar function in the free-field case is obtained from above with $\gamma=1$. By comparing the interacting and the free scalar two-point functions, we find

$$\langle 0|\tilde{\phi}_+(x)\tilde{\phi}_+(y)-\tilde{\phi}_+{}^2|0\rangle=\langle 0|\phi_+(x)\phi_+(y)-\phi_+{}^2|0\rangle+\frac{1}{4}\left[\frac{g^2/\pi^2}{1+g/\pi}\right]\ln\frac{\alpha^2}{\alpha^2+(x-y)^2}\ . \quad (3.17)$$

Thus if $\langle 0|\psi_+(x)\psi_+^\dagger(y)|0\rangle_0$ denotes the free-fermion two-point function, then the corresponding two-point function in the massless Thirring model is given by

$$\begin{aligned}\langle 0|\psi_+(x)\psi_+^\dagger(y)|0\rangle &=\langle 0|\psi_+(x)\psi_+^\dagger(y)|0\rangle_0\left[\frac{\alpha^2}{\alpha^2+(x-y)^2}\right]^\lambda\\ &=\frac{1}{2\pi\alpha}\left[\frac{i\alpha}{x-y+i\alpha}\right]\left[\frac{\alpha^2}{\alpha^2+(x-y)^2}\right]^\lambda\end{aligned} \quad (3.18)$$

with $\lambda=(g^2/\pi)/(1+g/\pi)$. Similarly,

$$\langle 0|\psi_+^\dagger(y)\psi_+(x)|0\rangle=\frac{1}{2\pi\alpha}\left[\frac{-i\alpha}{x-y-i\alpha}\right]\times\left[\frac{\alpha^2}{\alpha^2+(x-y)^2}\right]^\lambda . \quad (3.19)$$

The requirement that the anticommutator obtained from these two-point functions be proportional to the δ function implies

$$\begin{aligned}\int_{-\infty}^{\infty}\{\psi_+(x),\psi_+^\dagger(y)\}dx &=\int_{-\infty}^{\infty}\frac{1}{\pi\alpha}\left[\frac{\alpha^2}{\alpha^2+(x-y)^2}\right]^{\lambda+1}dx\\ &=Z\int_{-\infty}^{\infty}\delta(x-y)dx=Z\ .\end{aligned} \quad (3.20)$$

On evaluating Z, we find

$$Z=\frac{\Gamma(2\lambda+1)}{2^{2\lambda}[\Gamma(\lambda+1)]^2}\ , \quad (3.21)$$

where $\Gamma(n)$ is the Γ function of positive argument n. The inhomogeneous anticommuation relations in the Thirring model can be summarized by

$$\{\psi_\rho(x),\psi_\sigma^\dagger(y)\}=Z\delta(x-y)\delta_{\rho\sigma}\ , \quad (3.22)$$

in which Z denotes the effect of rescaling as a result of fermion self-interaction.

If we substitute the expressions for the currents in terms of the field $\tilde{\phi}(x)$, we find the Hamiltonian from Eq. (3.5) is simply that for a free massless boson

$$T^{00}=\tfrac{1}{2}(\partial_0\phi)^2+\tfrac{1}{2}(\partial_1\phi)^2\ . \quad (3.23)$$

This equivalence between the massless Thirring model and the free massless scalar field may further be confirmed by examining the form of the n-point functions, cluster decompositions, and the Lorentz structure in the Thirring model. They follow in a straightforward way and are consistent with the free fermion results.

When a Hamiltonian density representing a fermion mass term $m\bar{\psi}\psi$ is added to the massless Hamiltonian from Eq. (3.5), the resulting theory is the massive Thirring model. Such a mass term has the effect of introducing an interacting term to the free boson Hamiltonian in Eq. (3.23), since according to Eq. (2.57a),

$$m\bar{\psi}\psi\leftrightarrow-\frac{m}{\pi\alpha}\cos\sqrt{4\pi}\tilde{\phi}\ . \quad (3.24)$$

The Hamiltonian in Bose form is therefore given by

$$T^{00}=\tfrac{1}{2}(\partial_0\phi)^2+\tfrac{1}{2}(\partial_1\phi)^2-\frac{m}{\pi\alpha}\cos\left[\frac{4\pi}{1+g/\pi}\right]^{1/2}\phi, \quad (3.25)$$

which is derivable from the sine-Gordon Lagrangian

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi+\frac{m}{\pi\alpha}\cos\left[\frac{4\pi}{1+g/\pi}\right]^{1/2}\phi\ . \quad (3.26)$$

It is well known that the sine-Gordon theory possesses time-independent solutions of finite energy called solitons.[12] In the semiclassical analysis[13] these solutions correspond to a particle in the quantum theory. Such a duality between the massive Thirring model and the quantum sine-Gordon theory is readily made transparent by the boson-representation method. In particular, the method shows that the massless and the massive Thirring models have quite distinct structures.

IV. A U(N) CHIRAL-INVARIANT THEORY

We are ready to consider a multifermion theory in two dimensions with chiral and internal symmetries. A theory which is invariant under both the U(N) internal-symmetry transformation and the continuous chiral transformation is the chiral Gross-Neveu model,[14] with Lagrangian

$$\mathscr{L}=\sum_{a=1}^{N}i\bar{\psi}^a\gamma^\mu\partial_\mu\psi^a+\frac{g}{2}\left[\left[\sum_{a=1}^{N}\bar{\psi}^a\psi^a\right]^2-\left[\sum_{a=1}^{N}\bar{\psi}^a\gamma^5\psi^a\right]^2\right]\ . \quad (4.1)$$

This Lagrangian is formally identical to the U(N) Thirring model[15]

$$\mathscr{L}=\sum_{a=1}^{N}i\bar{\psi}^a\gamma^\mu\partial_\mu\psi^a-g\sum_{A=0}^{N^2-1}\left[\bar{\psi}\gamma^\mu\frac{\lambda^A}{2}\psi\right]\left[\bar{\psi}\gamma_\mu\frac{\lambda^A}{2}\psi\right] \quad (4.2)$$

through the following Fierz identity generalized to the symmetry group U(N):

$$\left[\sum_{a=1}^{N}\bar{\psi}^a\chi^a\right]\left[\sum_{b=1}^{N}\bar{\xi}^b\eta^b\right]-\left[\sum_{a=1}^{N}\bar{\psi}^a\gamma^5\chi^a\right]\left[\sum_{b=1}^{N}\bar{\xi}^b\gamma^5\eta^b\right]=-2\sum_{A=0}^{N^2-1}\left[\bar{\psi}\gamma^\mu\frac{\lambda^A}{2}\eta\right]\left[\bar{\xi}\gamma_\mu\frac{\lambda^A}{2}\chi\right]. \quad (4.3)$$

Here ψ,χ,ξ,η are anticommuting spinors; λ^A are the SU(N) matrices with normalization $\mathrm{Tr}(\lambda^A\lambda^B)=2\delta^{AB}$, and $\lambda^0=(2/N)^{1/2}I$. From the above identity, we can see that the chiral Gross-Neveu model has an intrinsic U(N) structure, and that a minus sign should necessarily appear in the interacting part of the Lagrangian (4.1).

We shall refer to the two versions in Lagrangians (4.1) and (4.2) as the U(N) chiral-invariant model. Both versions have been separately investigated.[16] The model is asymptotically free and exhibits dynamical mass generation without the breaking of its continuous chiral invariance. The spectrum of the model has been studied nonperturbatively by diagonalizing its Hamiltonian; it is found that there are both massive and massless excitations, with the latter being a decoupled scalar boson. Many of these features can be simply understood by examining the Bose form of the theory. Many new and unexpected properties, such as hidden duality and chiral symmetries, and the interconnections between some U(N) model and the O($2N$) Gross-Neveu model, are only transparent by considering the equivalent Bose theory. We illustrate the main aspect of the boson formulation of this multicomponent theory and investigate its dynamics and behavior in a general way.

We define the relativistic isospin currents in the U(N) model as

$$j^{A,\mu}(x)=\left[g^{0\mu}+\frac{\epsilon^{0\mu}}{1+g/\pi}\right]\times\frac{1}{4}\lim_{\epsilon\to 0}\left[\bar{\psi}\left[x+\frac{\epsilon}{2}\right],\gamma^\mu\frac{\lambda^A}{2}\psi\left[x-\frac{\epsilon}{2}\right]\right]. \quad (4.4)$$

Due to the isospin structure, the equal-time commutation rules between the currents and the Fermi fields becomes

$$[j^{A,\mu}(x),\psi^a(y)]=\left[g^{0\mu}+\frac{\epsilon^{0\mu}}{1+g/\pi}\gamma^5\right]\left[\frac{\lambda^A}{2}\psi\right]^a\delta(x-y), \quad (4.5)$$

$$[j^{A,\mu}(x),j^{B,\nu}(y)]=-\frac{i}{\pi}\left[\frac{\epsilon^{\mu\nu}}{1+g/\pi}\right]\frac{\partial}{\partial x}\delta(x-y)\delta^{AB}+if^{ABC}\left[\left[g^{0\mu}g^{0\nu}+\frac{g^{1\mu}g^{1\nu}}{(1+g/\pi)^2}\right]j_0^C(x)-(g^{0\mu}g^{1\nu}+g^{1\mu}g^{0\nu})j_1^C(x)\right]\delta(x-y), \quad (4.6)$$

whereas the energy-momentum tensor is

$$T^{\mu\nu}=\left[1+\frac{g}{\pi}\right]\frac{\pi}{2}\sum_{A=0}^{N^2-1}(j^{A,\mu}j^{A,\nu}+j^{A,\nu}j^{A,\mu}-g^{\mu\nu}j^A_\lambda j^{A,\lambda}). \quad (4.7)$$

The equation of motion is obtained by combining the time and space derivatives

$$\partial_0\psi^a=-i\pi\left[1+\frac{g}{\pi}\right]\sum_{A=0}^{N^2-1}\left[j^{A,0}+\frac{\gamma^5}{1+g/\pi}j^{A,1}\right]\times\left[\frac{\lambda^A}{2}\psi\right]^a, \quad (4.8)$$

$$\partial_1\psi^a=i\pi\left[1+\frac{g}{\pi}\right]\sum_{A=0}^{N^2-1}\left[j^{A,1}+\frac{\gamma^5}{1+g/\pi}j^{A,0}\right]\times\left[\frac{\lambda^A}{2}\psi\right]^a, \quad (4.9)$$

into the Dirac equation

$$i\gamma^\mu\partial_\mu\psi^a=\sum_{A=0}^{N^2-1}g\gamma^\mu j_\mu^A\left[\frac{\lambda^A}{2}\psi\right]^a. \quad (4.10)$$

The isospin currents $j^{A,\mu}$ can be divided into two groups: one consisting of all the currents formed from the diagonal U(N) matrices λ^d; the other, of those from the nondiagonal matrices λ^n. The diagonal currents $j^{d,\mu}$ are conserved, i.e.,

$$\partial_\mu j^{d,\mu}(x)=0\ ,\quad d=1,\ldots,N\ . \quad (4.11)$$

They can therefore be expressed in terms of N scalar fields $\tilde{\phi}^d(x)$ in the form

$$j^{d,\mu}(x)=\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\tilde{\phi}^d(x)\ . \quad (4.12)$$

The fields $\tilde{\phi}^a(x)$ and their conjugates $\tilde{\pi}^a(x)$ are related to the canonical scalar fields $\phi^a(x)$ and their conjugates $\pi^a(x)$ by

$$\tilde{\phi}^a(x)=\frac{\phi^a(x)}{(1+g/\pi)^{1/2}}\ ,\qquad \tilde{\pi}^a(x)=(1+g/\pi)^{1/2}\pi^a(x)\ , \quad (4.13)$$

where

$$[\phi^a(x),\pi^b(y)]=i\delta(x-y)\delta^{ab}\ . \quad (4.14)$$

It follows from Eqs. (4.9) and (4.11) that the Fermi fields can be represented according to

$$\psi_p^a(x)=\begin{bmatrix}\psi_+^a(x)\\ \psi_-^a(x)\end{bmatrix}=\frac{1}{(2\pi\alpha)^{1/2}}\begin{bmatrix}\exp[+i\sqrt{4\pi}\,\tilde{\phi}_+^a(x)]\\ \exp[-i\sqrt{4\pi}\,\tilde{\phi}_-^a(x)]\end{bmatrix}, \quad (4.15)$$

where the chiral components of different fields

$$\widetilde{\phi}^a_{\pm}(x)=\frac{1}{2}\left[\frac{\phi^a(x)}{\gamma}\mp\gamma\int_{-\infty}^{x}\pi^a(x')dx'-\frac{\gamma}{2}\epsilon^{ab}Q^b\right] \tag{4.16}$$

are mixed by the Klein factors Q^a, which are themselves defined by the various axial charges q^a by

$$Q^a=\int_{-\infty}^{\infty}\pi^a(x')dx'=-\sqrt{\pi}\int_{-\infty}^{\infty}j^{a,1}(x')dx'=-\sqrt{\pi}\,q^a . \tag{4.17}$$

The term ϵ^{ab} in Eq. (4.16) is antisymmetric, i.e., $\epsilon^{ab}=-\epsilon^{ba}$; $\epsilon^{ab}=1$ whenever a,b are cyclic in order. The addition of these Klein factors enable the different chiral components to satisfy the following commutation rules:

$$[\widetilde{\psi}^a_{\pm}(x),\widetilde{\psi}^a_{\pm}(y)]=\pm\frac{i}{4}\epsilon(x-y) , \tag{4.18}$$

$$[\widetilde{\psi}^a_{\pm}(x),\widetilde{\psi}^b_{\pm}(y)]=\frac{i}{4} , \tag{4.19}$$

for any combination of $+,-$. These in turn guarantee the anticommutativity property among the N types of fermion, i.e.,

$$\{\psi^a_\rho(x),\psi^{b\dagger}_\sigma(y)\}=Z\delta(x-y)\delta^{ab}\delta_{\rho\sigma} . \tag{4.20}$$

From the structure of the isospin currents, the Hamiltonian obtained from Eq. (4.7) can again be separated into the part containing the N diagonal currents $j^{d,\mu}$ and the part containing the $N(N-1)$ nondiagonal currents $j^{n,\mu}$. Such a decomposition is particularly useful in the boson formulation since according to Eq. (4.12), the contribution from the diagonal currents is simply the kinetic part of the Hamiltonian, i.e.,

$$\left[1+\frac{g}{\pi}\right]\frac{\pi}{2}[(j^{d,0})^2+(j^{d,1})^2]=\gamma^2\sum_{a=1}^{N}[\tfrac{1}{2}(\partial_0\widetilde{\phi}^a)^2+\tfrac{1}{2}(\partial_1\widetilde{\phi}^a)^2] , \tag{4.21}$$

whereas the contribution from the nondiagonal currents gives the interacting part. Due to the special structure of the SU(N) matrices, these nondiagonal currents can be grouped into exactly $N(N-1)/2$ pairs. The result is

$$\left[1+\frac{g}{\pi}\right]\frac{\pi}{2}\sum_n[(j^{n,0})^2+(j^{n,1})^2]=-\sum_{a\neq b=1}^{N}\frac{g}{4\pi^2\alpha^2}\cos\sqrt{4\pi}(\widetilde{\phi}^a-\widetilde{\phi}^b) . \tag{4.22}$$

The entire U(N) Hamiltonian in Bose form is then given by

$$T^{00}=\gamma^2\sum_{a=1}^{N}[\tfrac{1}{2}(\partial_0\widetilde{\phi}^a)^2+\tfrac{1}{2}(\partial_1\widetilde{\phi}^a)^2]-\sum_{a\neq b=1}^{N}\frac{g}{4\pi^2\alpha^2}\cos\sqrt{4\pi}(\widetilde{\phi}^a-\widetilde{\phi}^b) , \tag{4.23}$$

while the corresponding Lagrangian is

$$\mathscr{L}=\gamma^2\sum_{a=1}^{N}\tfrac{1}{2}\partial_\mu\widetilde{\phi}^a\partial^\mu\widetilde{\phi}^a+\sum_{a\neq b=1}^{N}\frac{g}{4\pi^2\alpha^2}\cos\sqrt{4\pi}(\widetilde{\phi}^a-\widetilde{\phi}^b) . \tag{4.24}$$

Thus the U(N) chiral-invariant model is equivalent to a system of N-coupled sine-Gordon theories in the boson representation. In subsequent developments, we shall encounter several interesting consequences when the theory is examined in its boson version, according to the Lagrangian (4.24).

V. DYNAMICAL MASS GENERATION WITH ASYMPTOTIC FREEDOM AND CHIRAL SYMMETRY

An unexpected and important consequence of boson representation is that it can provide a mechanism for the generation of mass in an interacting theory. Under the continuous chiral transformation

$$\psi^a_\rho\to[\exp(i\beta\gamma^5)]_{\rho\sigma}\psi^a_\sigma , \tag{5.1}$$

β being an arbitrary parameter, a quantity such as the mass term $m\bar{\psi}\psi$ will not, in general, be invariant. In the U(N) theory, it can be inferred from the connection formula in Eq. (4.15) as well as the Hamiltonian in Eq. (4.23) that $1/\alpha$ has the dimension of a mass. The original massless fermion theory has become a theory of massive interacting bosons. We shall understand this generation of mass in the following consideration.

In the boson Hamiltonian, all interaction terms have the form

$$\frac{g}{4\pi^2\alpha^2}\cos\sqrt{4\pi}(\widetilde{\phi}^a-\widetilde{\phi}^b) , \tag{5.2}$$

which are linear combinations of $\exp(\pm i\sqrt{4\pi}\,\widetilde{\phi}^{a,b})$. We may normal order such exponential fields by bringing the creation operator to the right of the annihilation operator, for example,

$$e^{i\beta\phi(x)}=e^{i\beta\phi^{(-)}(x)}e^{i\beta\phi^{(+)}(x)}e^{-(\beta^2/2)[\phi^{(+)}(x),\phi^{(-)}(x)]} , \tag{5.3}$$

where the positive- and negative-frequency parts are

$$\phi^{(\pm)}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2k^0}}c^{(\pm)}(k)e^{\pm ik\cdot x}e^{-\alpha|k|/2} . \tag{5.4}$$

The commutator in Eq. (5.3) is easily evaluated; the result for small value of α is given as

$$[\phi^{(+)}(x),\phi^{(-)}(x)]\sim-\frac{1}{2\pi}\ln\frac{c\alpha\mu}{2} . \tag{5.5}$$

The number $c=0.577$ is Euler's constant and μ denotes the physical mass of the boson. It follows from Eqs. (5.3) and (5.5) that the exponential field, and hence the sine and cosine fields, is related to its normal-ordered form by

$$e^{i\beta\phi(x)}=\frac{c^2\alpha^2}{4}\left[\frac{c^2\kappa^2\mu^2}{4}\right]^{\beta^2/8\pi-1}\mu^2{:}e^{i\beta\phi(x)}{:} . \tag{5.6}$$

Accordingly, a cosine interaction term will acquire the proper mass parameter μ^2. Introducing the sine-Gordon coupling a_0 via $g=a_0/4\pi\beta^2$, as well as the renormalization factor

$$Z=\left[\frac{c^2\mu^2\alpha^2}{4}\right]^{1-\beta^2/8\pi}, \tag{5.7}$$

we define the renormalized sine-Gordon coupling

$$a_r=Z^{-1}a_0 . \tag{5.8}$$

The interaction in (5.2) then has the form

$$\frac{a_0}{\alpha^2\beta^2}\cos\beta\phi=\frac{a_0}{\beta^2 Z}\left[\frac{c^2\mu^2}{4}\right]:\cos\beta\phi:$$
$$=\left[\frac{a_r}{\beta^2}\right]\kappa^2:\cos\beta\phi: , \tag{5.9}$$

where $\kappa^2=c^2\mu^2/4$.

An important property of the U(N) fermion theory is the dynamical generation of mass with asymptotic-freedom behavior. Typically, the bare coupling constant g in such a theory vanishes as the ultraviolet cutoff Λ is taken to infinity according to

$$g^{(\Lambda)}\sim\frac{1}{\ln(\Lambda/m)} \quad\text{or}\quad m=\Lambda e^{-\text{const}/g} \tag{5.10}$$

with m being the dynamically generated mass. To see such behavior in the boson representation, we observe from Eqs. (5.7) and (5.8) that the renormalized sine-Gordon coupling can be written as

$$a_r=4\pi\beta^2 g(\kappa^2\alpha^2)^{\beta^2/8\pi-1} , \quad \beta^2=\frac{8\pi}{1+g/\pi} . \tag{5.11}$$

For small values of g, β^2 is approximately 8π, and

$$a_r\simeq 32\pi^2 g(\kappa^2\alpha^2)^{-g/\pi} . \tag{5.12}$$

By keeping the renormalized sine-Gordon coupling a_r and the dynamically generated mass κ fixed, we obtain the relation

$$\kappa^2=\Lambda^2 e^{-(\pi/g)(\ln a_r/32\pi^2-\ln g)} , \quad \Lambda=1/\alpha \tag{5.13}$$

which shows the asymptotic behavior of g as a function of the cutoff, analogous to Eq. (5.10).

It would appear that the emergence of an explicit mass parameter in the boson version would have destroyed the chiral symmetry of the massless theory. This is, however, not the case for the U(N) theory. The effect of a continuous chiral transformation on the bosons is a translation $\tilde{\phi}^a\to\tilde{\phi}^a+\beta/\sqrt{\pi}$. Since all interactions in the massive theory are of the form $\cos\sqrt{4\pi}(\tilde{\phi}^a-\tilde{\phi}^b)$, the boson Hamiltonian is actually invariant under all continuous chiral transformations. Dynamical mass generation in the model is thus possible without the breaking of its chiral symmetry and is further independent of the number of fermions. This conclusion can be compared with the result from a $1/N$ expansion study of the model in which chiral symmetry is intact only when N goes to infinity.[17]

We may always perform an orthogonal transformation Λ^{ij} on the N Bose fields such that

$$\begin{aligned}
\tilde{B}^1&=(\tilde{\phi}^1+\tilde{\phi}^2+\cdots+\tilde{\phi}^N)/\sqrt{N} ,\\
\tilde{B}^2&=(\tilde{\phi}^1-\tilde{\phi}^2)/\sqrt{2} ,\\
&\cdots\\
\tilde{B}^N&=[\tilde{\phi}^1+\cdots+\tilde{\phi}^{N-1}-(N-1)\tilde{\phi}^N]/[N(N-1)]^{1/2} .
\end{aligned} \tag{5.14}$$

The above transformation, when applied to Lagrangian (4.24), allows a massless boson to decouple from the remaining theory of $N-1$ interacting massive bosons. The new Lagrangian is

$$\mathscr{L}=\gamma^2\tfrac{1}{2}\partial_\mu\tilde{B}^1\partial^\mu\tilde{B}^1+\left[\gamma^2\sum_{k=2}^{N}\tfrac{1}{2}\partial_\mu\tilde{B}^k\partial^\mu\tilde{B}^k+\sum_{i\neq j=2}^{N}\frac{g}{4\pi^2\alpha^2}\cos\sqrt{4\pi}(\Lambda^{li}\tilde{B}^l-\Lambda^{mj}\tilde{B}^m)\right] . \tag{5.15}$$

Each $\tilde{B}^k$, $k=2,\ldots,N$, is a linear combination of the Bose fields $\tilde{\phi}^a$ such that the interaction part of Lagrangian (5.15) remains automatically invariant under continuous chiral transformations on the original Fermi fields ψ^a. An interaction term involving the field $\tilde{B}^1$ would not possess such an invariance. This requirement therefore demands that $\tilde{B}^1$ can only appear as a kinetic term and hence a massless boson.

We have shown that dynamical mass generation without breaking the continuous chiral symmetry of the U(N) theory, in addition to asymptotic-freedom behavior, is tenable; contrary to naive expectation. The result is therefore consistent with Coleman's theorem[18] that there is no spontaneous symmetry breakdown in two dimensions. We shall find in the next section that the boson formulation has an unexpected utility in uncovering this chiral symmetry.

VI. SOME O(2N) MODELS AS U(N) THEORIES

We now illustrate with several examples some unusual metamorphoses of the U(N) models via their boson formulations. We shall, first of all, consider the U(2) model involving two massless fermions. The transformation (5.14) in this case results in the following boson Lagrangian:

$$\mathscr{L}=\gamma^2\tfrac{1}{2}\partial_\mu\tilde{B}^1\partial^\mu\tilde{B}^1$$
$$+\left[\gamma^2\tfrac{1}{2}\partial_\mu\tilde{B}^2\partial^\mu\tilde{B}^2+\frac{g}{4\pi^2\alpha^2}\cos\sqrt{8\pi}\tilde{B}^2\right] . \tag{6.1}$$

Here $\tilde{B}^1$ is the massless decoupled field and $\tilde{B}^2$ the sine-Gordon scalar field.

It is expedient to present the massive Thirring model again in the form

$$\mathscr{L}=i\bar{\chi}\gamma^{\mu}\partial_{\mu}\chi-\frac{g'}{2}k^{\mu}k_{\mu}-m'\bar{\chi}\chi\ , \qquad (6.2)$$

where k^{μ} are the currents. If we construct the Fermi field χ in terms of a canonical Bose field B according to

$$\chi_{\rho}(x)=\begin{bmatrix}\chi_{+}(x)\\ \chi_{-}(x)\end{bmatrix} =\frac{1}{(2\pi\alpha)^{1/2}}\begin{bmatrix}\exp[+i\sqrt{4\pi}\,\tilde{B}_{+}(x)]\\ \exp[-i\sqrt{4\pi}\,\tilde{B}_{-}(x)]\end{bmatrix}, \qquad (6.3)$$

where $\tilde{B}(x)=B(x)/(1+g'/\pi)^{1/2}$, and use the corresponding bilinear equivalences as in Eq. (2.57), then we see the Bose form of the Lagrangian (6.2) is straightforwardly

$$\mathscr{L}=\left[1+\frac{g'}{\pi}\right]\tfrac{1}{2}\partial_{\mu}\tilde{B}\partial^{\mu}\tilde{B}+\frac{m'}{\pi\alpha}\cos\sqrt{4\pi}\,\tilde{B}\ . \qquad (6.4)$$

This sine-Gordon Lagrangian will be identical to the sine-Gordon Lagrangian (6.1) from the U(2) model when

$$\tilde{B}=\sqrt{2}\,\tilde{B}^{2}\ ,\quad g'=\frac{g-\pi}{2}\ ,\quad m'=\frac{g}{4\pi\alpha}\ . \qquad (6.5)$$

Hence the SU(2) chiral-invariant theory is none other than the well-known massive Thirring model. The Fermi field χ is now a nonlinear composite of the original Fermi fields ψ^1,ψ^2 via their bosons.

It is of interest next to consider two independent U(2) theories represented by their separate Lagrangians $\mathscr{L}(\psi^1,\psi^2)$ and $\mathscr{L}(\chi^1,\chi^2)$. The Bose form for these systems, according to Eq. (4.24), is therefore two independent sine-Gordon theories:

$$\mathscr{L}=\gamma^2\left[\sum_{a=1}^{2}\tfrac{1}{2}\partial_{\mu}\tilde{\phi}^{a}\partial^{\mu}\tilde{\phi}^{a}+\sum_{b=1}^{2}\tfrac{1}{2}\partial_{\mu}\tilde{n}^{b}\partial^{\mu}\tilde{n}^{b}\right] +\frac{g}{4\pi^2\alpha^2}[\cos\sqrt{4\pi}(\tilde{\phi}^1-\tilde{\phi}^2) +\cos\sqrt{4\pi}(\tilde{n}^1-\tilde{n}^2)]\ . \qquad (6.6)$$

Here $\tilde{\phi}^a,\tilde{n}^b$ are the corresponding Bose fields of the Fermi fields ψ^a,χ^b. The Bose and the Fermi fields are related by their connections as that in Eq. (6.3). It is obvious that there is no coupling between the fields $\tilde{\phi}^a$ and $\tilde{n}^b$.

If we perform an orthogonal transformation L such that

$$\begin{aligned}\tilde{B}^1&=\tfrac{1}{2}(\tilde{\phi}^1+\tilde{\phi}^2+\tilde{n}^1+\tilde{n}^2)\ ,\\ \tilde{B}^2&=\tfrac{1}{2}(\tilde{\phi}^1+\tilde{\phi}^2-\tilde{n}^1-\tilde{n}^2)\ ,\\ \tilde{B}^3&=\tfrac{1}{2}(\tilde{\phi}^1-\tilde{\phi}^2+\tilde{n}^1-\tilde{n}^2)\ ,\\ \tilde{B}^4&=\tfrac{1}{2}(\tilde{\phi}^1-\tilde{\phi}^2-\tilde{n}^1+\tilde{n}^2)\ ,\end{aligned} \qquad (6.7)$$

then

$$\tilde{\phi}^1-\tilde{\phi}^2=\tilde{B}^3+\tilde{B}^4\ ,\quad \tilde{n}^1-\tilde{n}^2=\tilde{B}^3-\tilde{B}^4\ . \qquad (6.8)$$

We arrive immediately at the following equivalent Lagrangian for (6.6), i.e.,

$$\mathscr{L}=\gamma^2\sum_{i=1}^{2}\tfrac{1}{2}\partial_{\mu}\tilde{B}^{i}\partial^{\mu}\tilde{B}^{i} +\left[\gamma^2\sum_{i=3}^{4}\tfrac{1}{2}\partial_{\mu}\tilde{B}^{i}\partial^{\mu}\tilde{B}^{i} +\frac{g}{2\pi^2\alpha^2}\cos\sqrt{4\pi}\,\tilde{B}^3\cos\sqrt{4\pi}\,\tilde{B}^4\right]. \qquad (6.9)$$

In this case, we obtain two decoupled massless bosons $\tilde{B}^1$ and $\tilde{B}^2$, and a pair of interacting bosons $\tilde{B}^3$ and $\tilde{B}^4$.

We shall at this point introduce the O(4) Gross-Neveu model[14] of four interacting Majorana fermions ξ^{κ}_{M}, with M denoting their Majorana character. The Lagrangian is of the form

$$\mathscr{L}=\sum_{\kappa=1}^{4}i\bar{\xi}^{\kappa}_{M}\gamma^{\mu}\partial_{\mu}\xi^{\kappa}_{M}+\frac{g}{2}\left[\sum_{\kappa=1}^{4}\bar{\xi}^{\kappa}_{M}\xi^{\kappa}_{M}\right]^2 . \qquad (6.10)$$

By definition, a Majorana fermion has the self-conjugating property

$$\bar{\xi}_M=\xi^{T}_{M}C\ ,\quad C=\gamma^0\ , \qquad (6.11)$$

and hence it is a real two-component spinor. If we construct two complex fermions from the above four Majorana ones according to the combinations

$$\begin{aligned}\xi^3&=\frac{1}{\sqrt{2}}(\xi^1_M+i\xi^2_M)\ ,\\ \xi^4&=\frac{1}{\sqrt{2}}(\xi^3_M+i\xi^4_M)\ ,\end{aligned} \qquad (6.12)$$

we find the O(4) Lagrangian can be written in its own identical form with only two fermions, i.e.,

$$\mathscr{L}=\sum_{i=3}^{4}i\bar{\xi}^{i}\gamma^{\mu}\partial_{\mu}\xi^{i}+\frac{g}{2}\left[\sum_{i=3}^{4}\bar{\xi}^{i}\xi^{i}\right]^2 . \qquad (6.13)$$

By using further the connection

$$\begin{bmatrix}\xi^{i}_{+}(x)\\ \xi^{i}_{-}(x)\end{bmatrix}=\frac{1}{(2\pi\alpha)^{1/2}}\begin{bmatrix}\exp[+i\sqrt{4\pi}\,\tilde{B}^{i}_{+}(x)]\\ \exp[-i\sqrt{4\pi}\,\tilde{B}^{i}_{-}(x)]\end{bmatrix},\quad i=3,4\ , \qquad (6.14)$$

in the Lagrangian (6.13), we obtain the Bose form of the O(4) model,[19] which is precisely the interacting part in the Lagrangian (6.9). The U(2) chiral-invariant theory is thus connected with the O(4) Gross-Neveu model by the local isomorphism SU(2)×SU(2)≃O(4).

Another relevant case to consider is the U(4) chiral-invariant model of four massless fermions ψ^a, $a=1,2,3,4$. The Bose form for this model is

$$\mathscr{L}=\gamma^2\sum_{a=1}^{4}\tfrac{1}{2}\partial_{\mu}\tilde{\phi}^{a}\partial^{\mu}\tilde{\phi}^{a} +\sum_{a\neq b=1}^{4}\frac{g}{4\pi^2\alpha^2}\cos\sqrt{4\pi}(\tilde{\phi}^a-\tilde{\phi}^b)\ . \qquad (6.15)$$

If we perform the same transformation L on the set of bosons $\tilde{\phi}^a$, i.e.,

$$\begin{aligned}\widetilde{B}^1&=\tfrac{1}{2}(\widetilde{\phi}^1+\widetilde{\phi}^2+\widetilde{\phi}^3+\widetilde{\phi}^4)\,,\\ \widetilde{B}^2&=\tfrac{1}{2}(\widetilde{\phi}^1+\widetilde{\phi}^2-\widetilde{\phi}^3-\widetilde{\phi}^4)\,,\\ \widetilde{B}^3&=\tfrac{1}{2}(\widetilde{\phi}^1-\widetilde{\phi}^2+\widetilde{\phi}^3-\widetilde{\phi}^4)\,,\\ \widetilde{B}^4&=\tfrac{1}{2}(\widetilde{\phi}^1-\widetilde{\phi}^2-\widetilde{\phi}^3+\widetilde{\phi}^4)\,,\end{aligned}\tag{6.16}$$

we find the following equivalent theory represented by

$$\mathscr{L}=\gamma^2\tfrac{1}{2}\partial_\mu\widetilde{B}^1\partial^\mu\widetilde{B}^1+\left[\gamma^2\sum_{i=2}^{4}\tfrac{1}{2}\partial_\mu\widetilde{B}^i\partial^\mu\widetilde{B}^i+\sum_{i\neq j=2}^{4}\frac{g}{2\pi^2\alpha^2}\cos\sqrt{4\pi}\,\widetilde{B}^i\cos\sqrt{4\pi}\,\widetilde{B}^j\right].\tag{6.17}$$

In this case, we have a single decoupled massless boson $\widetilde{B}^1$ and three massive interacting ones, $\widetilde{B}^2$, $\widetilde{B}^3$, and $\widetilde{B}^4$. The interacting theory here has the same type of structure as the Bose form of the O(4) model, except that it has one extra degree of freedom. It is therefore the boson theory of the O(6) Gross-Neveu model. This equivalence is clearly the realization of the isomorphism $SU(4)\simeq O(6)$.

We have realized that the specific form of the interaction $\cos\sqrt{4\pi}(\widetilde{\phi}^a-\widetilde{\phi}^b)$ involving the difference of any two Bose fields in the U(N) models is crucial for preserving exact continuous chiral symmetry. What happens to this symmetry when these models metamorphose into the O($2N$) cases? In generating the O(4) model, it can be seen from transformation (6.7) that the bosons $\widetilde{B}^3$ and $\widetilde{B}^4$ have the desired combinations involving the differences of two Bose fields, i.e.,

$$\widetilde{B}^{3,4}=\tfrac{1}{2}(\widetilde{\phi}^1-\widetilde{\phi}^2)\pm\tfrac{1}{2}(\widetilde{n}^1-\widetilde{n}^2)\,.\tag{6.18}$$

Thus the O(4) model is actually invariant under independent chiral transformations on the sets of U(2) fermions ψ^a and χ^b.

A similar situation exists in the generation of the O(6) model from the U(4) case. Each of the bosons $\widetilde{B}^2$, $\widetilde{B}^3$, and $\widetilde{B}^4$ in the O(6) model, according to transformation (6.16), is a linear combination of the four U(4) bosons $\widetilde{\phi}^a$, with equal numbers of plus and minus signs. A continuous chiral transformation performed on the U(4) fermions ψ^a again has no effect on the discrete chiral symmetry of the O(6) case.

We deduce from the above that the O($2N$) models which can be obtained from the U(N) cases actually possess an additional hidden continuous chiral symmetry since their fermions are nonlinear composites of the corresponding U(N) ones via their intermediate bosons. It is only obvious that a number of models of apparently different internal symmetries and interactions are related when they are formulated in this way. The existence of hidden duality symmetry in the U(N) models and triality symmetry in the O(8) Gross-Neveu model can be easily discovered in the same context.[7]

VII. CONCLUSION

The boson formulation of a multicomponent theory which we considered has led to some surprising developments. Along with the simplicity in understanding a given theory, we have been able to extend it as a possible guide to investigations in two dimensions. Some of the utilities of the boson formulation may be mentioned. They are used

(a) to find a direct boson equivalence of a fermion theory, thus relating various fermion and boson models which would otherwise exist independently;

(b) to relate apparently different fermion theories through the intermediate representations of bosons;

(c) to provide a mechanism for a dynamically generated mass; and

(d) to uncover unusual and hidden symmetries by performing transformations on the bosons.

Thus it is evident that a boson formulation can provide an alternate scheme for understanding fermion theories and that it has attained a wider role than the original scope of merely establishing equivalences between various models.

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