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Five-dimensional projective unified theory and the principle of equivalence

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We investigate the physical consequences of a new five-dimensional projective theory unifying gravitation and electromagnetism. Solving the field equations in the linear approximation and in the static limit, we find that a celestial body would act as a source of a long-range scalar field, and that macroscopic test bodies with different internal structure would accelerate differently in the solar gravitational field; this seems to be in disagreement with the equivalence principle. To avoid this contradiction, we suggest a possible modification of the geometrical structure of the five-dimensional projective space.

I. INTRODUCTION

A new interesting unified theory for the electromagnetic and gravitational fields has recently been proposed by Schmutzer.[1–3] The starting point of his theory is the minimal formulation of the laws of the gravitational interaction in a five-dimensional projective space, endowed with a nonsymmetric affine connection. After dimensional reduction, i.e., after the projection of the gravitational equations on the four-dimensional physical space-time, the field content of the theory is the following: a symmetric second-rank tensor, representing the metric of the four-dimensional Riemann space, and related as usual to the gravitational field; an antisymmetric second-rank tensor, interpreted as the electromagnetic field; and finally a massless scalar field, acting as the source of the vector part of the torsion tensor associated with the five-dimensional affinity, and interpreted as a new fundamental field of nature.

In this theory the electromagnetic and gravitational interactions are unified, like in the old Kaluza-Klein theory[4,5(a)] (see also the work of Kalinowski which investigates the Kaluza-Klein theory with a nonvanishing torsion[5(b)]), in the sense that the Einstein and Maxwell equations are both contained in a compact, five-dimensional geometrical formulation. The additional scalar field interacts universally with the electromagnetic field, and is coupled also to the matter substrate; in the presence of a nonzero scalar field the Maxwell equations are modified; however, no variation of the gravitational coupling constant is predicted.

In this paper we show that the direct coupling between the electromagnetic and scalar fields implies, in particular, that test bodies with different electromagnetic structure would accelerate differently in a given external scalar field. Therefore, since each macroscopic and astronomical body is the source of a long-range scalar field produced by its own content of electromagnetic energy, disagreement with the high-precision experimental tests of the equivalence principle performed by Dicke[6] and Braginsky[7] appears to arise.

In Sec. II of this paper, after a short review, we present the basic Lagrangian and the field equations of the five-dimensional projective theory. Then, in Sec. III, we calculate the acceleration of a neutral, macroscopic test body, under the action of an external scalar field and, in Sec. IV, we evaluate the scalar field produced by a celestial body like the Sun.

Finally, in Sec. V, comparing the predictions of the projective theory with experimental data, we show that they disagree; in order to overcome this difficulty, without dropping the hypothesis of a unification in the framework of the projective formalism, we suggest then a possible modification in the geometrical structure of the five-dimensional projective space.

II. FIELD EQUATIONS OF THE PROJECTIVE THEORY

We start by giving a short review of Schmutzer theory.[3] Our notations are as follows: capital latin indices run from 1 to 5 in the five-dimensional projective space, lower-case latin letters run from 1 to 4 and are world indices of the physical four-dimensional space-time. Moreover x^i and g_{ik} are, respectively, the coordinates and the (symmetric) metric tensor of four-dimensional space-time, while X^A and g_{AB} are coordinates and (symmetric) metric of the projective space.

The four-dimensional projection of a vector V^A is given by

$$V^i = g^i{}_A V^A \tag{2.1}$$

and the mixed metric (projection cosines) g_{iA} is fixed by Schmutzer according to the following postulate:

$$g^i{}_A = x^i{}_{,A} \exp(-\sigma/2) \,, \qquad \exp(\sigma) = S/S_0 \,, \tag{2.2}$$

where S_0 is a free constant with the dimensions of length and

$$S^2 = g_{AB} X^A X^B \,. \tag{2.3}$$

The physical four-dimensional space is endowed with a Riemannian geometric structure, and then the covariant derivative is defined as usual by means of the Christoffel symbols, i.e.,

$$V^i{}_{;k} = V^i{}_{,k} + \left\{ {}^{\,i}_{lk} \right\} V^l \,. \tag{2.4}$$

The five-dimensional projective space, however, is endowed with a nonsymmetric connection $\Gamma_{AB}{}^C$, and the general covariant derivative is

$$V^A{}_{||B} = V^A{}_{,B} + \Gamma_{CB}{}^A V^C = V^A{}_{;B} - K_{CB}{}^A V^C \,, \tag{2.5}$$

where

$$K_{AB}{}^C = \left\{ {}^{\,C}_{AB} \right\} - \Gamma_{AB}{}^C = -Q_{AB}{}^C - Q^C{}_{Ab} + Q_B{}^C{}_A \tag{2.6}$$

and $Q_{AB}{}^{CC} = \Gamma_{[AB]}{}^C$ is the torsion tensor. Using the metric postulate $g_{AB||C} = 0$, one can obtain the explicit form of the torsion tensor of projective space. In Schmutzer theory, the result is[3]

$$Q_{ABC} = \tfrac{1}{2} [S_{BA} S_C + \tfrac{1}{2} (g_{BC} - S_B S_C) \sigma_{,A} - \tfrac{1}{2} (g_{AC} - S_A S_C) \sigma_{,B}] \,, \tag{2.7}$$

where

$$S_A = X_A / S \,, \qquad S_{AB} = S_{B,A} - S_{A,B} = (x_{B,A} - X_{A,B}) S^{-1} + S_A \sigma_{,B} - S_B \sigma_{,A} \,. \tag{2.8}$$

Notice that, in this case, the scalar field σ is the source of the vector part of the torsion, i.e., $Q_A \equiv Q_{AB}{}^B = \frac{1}{4}\sigma_{,A}$. The basic Lagrangian chosen by Schmutzer in the projective space is

$$\mathscr{L}^5 = \sqrt{-g^5} \frac{R^5}{2\chi_0 S_0} + \mathscr{L}^5_M \,, \tag{2.9}$$

where $g^5 = \det g_{AB}$, R^5 is the five-dimensional scalar curvature, χ_0 is a coupling constant, and $\mathscr{L}^5_M$ is the Lagrangian density of the "substrate" (i.e., nongeometrized matter fields). Notice that in this paper we neglect, for simplicity, the possibility of adding to Eq. (2.9) a cosmological term.

By variation of the corresponding action, one gets the five-dimensional Einstein-type field equations

$$G^{AB} = \chi_0 \theta^{AB} \,, \tag{2.10}$$

where G_{AB} is the Einstein tensor and

$$\theta_{AB} = -\frac{2S_0}{\sqrt{-g^5}} \frac{\delta \mathscr{L}^5_M}{\delta g^{AB}} \,. \tag{2.11}$$

The projection of these equations on the physical space-time leads to the four-dimensional field equations (see Ref. 3 for a detailed computation), which can be deduced from the following four-dimensional Lagrangian density:

$$\mathscr{L}^4 = \frac{\sqrt{-g^4}}{2\chi_0} (R^4 - \tfrac{3}{2} \sigma_{,k} \sigma^{,k}) - \frac{\sqrt{-g^4}}{16\pi} B_{ik} H^{ik} + \mathscr{L}^4_M \,. \tag{2.12}$$

Here $g^4 = \det g_{ik}$, R^4 is the scalar curvature obtained from g_{ik}, $\mathscr{L}^4_M$ is the matter Lagrangian density, and the two antisymmetric tensors B_{ik} and H_{ik} are defined by

$$B_{ik} = e_0 (S_0/S^2) S_{ik} = e_0 (S_0/S^2) g_i{}^A g_k{}^B S_{AB} \,, \tag{2.13}$$

$$H_{ik} = b_0 S S_{ik} = (b_0 S^3 / e_0 S_0) B_{ik} \,, \tag{2.14}$$

where e_0 and b_0 are free constant parameters.

As B_{ik} satisfies the cyclic relation $B_{[ik,j]} = 0$, one can put $B_{ik} = 2A_{[k,i]}$ and it is tempting to identify B_{ik} with the electromagnetic field-strength tensor F_{ik}; moreover, if H_{ik} is interpreted as an electromagnetic induction tensor, as in Ref. 3, then, from Eq. (2.14), the function

$$\epsilon = b_0 S^3 / e_0 S_0 \tag{2.15}$$

represent a sort of vacuum "dielectric constant," induced by the scalar field.

It is then rather natural, in the framework of this electromagnetic interpretation of the "projected" Lagrangian (2.12), to eliminate the free parameters e_0 and B_0 by imposing the boundary condition $\epsilon \to 1$ at infinity, and by demanding that the motion equations contain the Lorentz force term (see Ref. 3). It follows that

$$e_0 = S_0 (2\pi/\chi_0)^{1/2} \,, \qquad b_0 = S_0{}^{-1} (2\pi/\chi_0)^{1/2} \,, \qquad \epsilon = \exp(3\sigma) \,. \tag{2.16}$$

The last step is to identify, as usual, $\chi_0 = \chi = 8\pi G/c^4$, in order to obtain the Newtonian gravitational force in the weak-field limit.

In conclusion, the five-dimensional projective theory of Schmutzer reduces, in four dimensions, to a field theory for electromagnetism interacting with a scalar field (plus eventually other matter fields) in a curved Riemann space, and it is described by the following "effective" Lagrangian (henceforth we drop the superscript "4" as we consider only four-dimensional quantities):

$$\mathscr{L} = \frac{\sqrt{-g}}{2\chi} (R - \tfrac{3}{2} \sigma_{,k} \sigma^{,k}) - \frac{\sqrt{-g}}{16\pi} F_{ik} F^{ik} e^{3\sigma} + \mathscr{L}_M \,. \tag{2.17}$$

Here $F_{ik} = 2A_{[k,i]}$ and $\mathscr{L}_M$ is the Lagrangian for the matter fields interacting with gravitation and electromagnetism in the usual minimal way, and in general also coupled to the scalar field σ, since $\mathscr{L}_M$ is obtained from the corresponding five-dimensional Lagrangian according to the projective formalism. From independent variation of g_{ik}, A_k, and σ in the total action corresponding to the Lagrangian (2.17), we obtain the field equations, respectively, for the gravitational field

$$G_{ik} = \chi(\theta_{ik} + E_{ik} + \Sigma_{ik}) \;, \tag{2.18}$$

for the electromagnetic field

$$H^{ik}{}_{;k} = 4\pi J^i, \quad H^{ik} = \exp(3\sigma) F^{ik} \;, \tag{2.19}$$

and for the scalar field

$$\sigma^{,k}{}_{;k} = \chi \left[\tfrac{2}{3}\vartheta + \frac{1}{8\pi} F_{ik} F^{ik} e^{3\sigma} \right] , \tag{2.20}$$

where the gravitational sources are defined by

$$\theta_{ik} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathscr{L}_M}{\delta j^{ik}} \;, \tag{2.21}$$

$$\Sigma_{ik} = -\frac{3}{2\chi} (\sigma_{,i}\sigma_{,k} - \tfrac{1}{2} g_{ik} \sigma_{,a} \sigma^{,a}) \;, \tag{2.22}$$

$$E_{ik} = \frac{e^{3\sigma}}{4\pi} (F_{ia} F^a{}_k + \tfrac{1}{4} g_{ik} F_{ab} F^{ab}) \;. \tag{2.23}$$

The electromagnetic current of matter is given by

$$J^i = \frac{1}{\sqrt{-g}} \frac{\delta \mathscr{L}_M}{\delta A^i} \tag{2.24}$$

and finally the matter contribution to the scalar field is

$$\vartheta = -\frac{1}{\sqrt{-g}} \frac{\delta \mathscr{L}_M}{\delta \sigma} \;. \tag{2.25}$$

It must be noticed, from Eq. (2.20), that the scalar field can be produced by the electromagnetic field even if matter is decoupled from σ (i.e., $\vartheta = 0$).

Moreover, the Maxwell equations (2.19) are modified by the scalar field, which induces an electric current density, but no magnetic current.

In the following sections we will investigate some physical implications of the basic Lagrangian (2.17) and of the field equations (2.18)—(2.20).

III. TEST-BODY ACCELERATIONS

In order to calculate the acceleration of a neutral and macroscopic test body in a given external scalar field according to Schmutzer theory, we will apply the method developed in Refs. 8 and 9, where the same computation was performed, in the framework of a different theory.

The four-momentum vector of a test body is

$$P_k = \int_v \sqrt{-g}\; T_k{}^0 d^3x \;, \tag{3.1}$$

where T_{ik} is its total stress-energy tensor including also the electromagnetic contributions, and the integration is performed over the volume v of the body. Neglecting second-order derivatives of σ, as in Refs. 8 and 9, we start with the interaction Lagrangian of the projective theory

$$\mathscr{L}' = -\frac{\sqrt{-g}}{16\pi} F_{ik} F^{ik} e^{3\sigma} + \mathscr{L}_P \;, \tag{3.2}$$

where $\mathscr{L}_P$ is the test-body Lagrangian density. Using the matter-response equations

$$(\sqrt{-g}\; T_i{}^k)_{,k} = \mathscr{L}'_{,i} \tag{3.3}$$

we get, from (3.1),

$$\dot{P}_k = -\frac{1}{8\pi} (\sqrt{-g}\; e^{3\sigma})_{,k} \int_v (\,|\vec{\mathrm{B}}\,|^2 - |\,\vec{\mathrm{E}}\,|^2) d^3x + \tfrac{1}{2} (\sqrt{-g}\; g_{ab})_{,k} \int_v T_P^{ab} d^3x \;, \tag{3.4}$$

where T_P is the stress energy of the test body, including all interactions but the electromagnetic one, and $\vec{\mathrm{E}}$ and $\vec{\mathrm{B}}$ are the electric and magnetic fields.

Choosing a local inertial frame, such that the Christoffel symbols vanish at the location of the body, to first order in $\sigma_{,k}$ Eq. (3.4) reduces to

$$\dot{P}_k = 3\sigma_{,k} (\mathscr{E}_e - \mathscr{E}_m) \;, \tag{3.5}$$

where $\mathscr{E}_e$ and $\mathscr{E}_m$ are the total electric and magnetic energies of the test body.

Finally, defining the coordinates of the center of mass as[8,9]

$$\xi^k = P_0{}^{-1} \int_v \sqrt{-g}\; T_0{}^0 x^k d^3x \;, \tag{3.6}$$

we get, in a local inertial frame, in first approximation, $P_0 \ddot{\xi}_k = m \ddot{\xi}_k = \dot{P}_k$, where m is the mass of the test body. The acceleration is then

$$\ddot{\xi}_k = \frac{3}{m} \sigma_{,k} (\mathscr{E}_e - \mathscr{E}_m) \;. \tag{3.7}$$

Therefore, according to the unified theory of Schmutzer, an external scalar field induces different accelerations on test bodies with different electromagnetic content.

IV. THE SCALAR FIELD OF AN ASTRONOMICAL BODY

In this section we will compare the scalar field σ produced by a celestial body like the Sun with its Newtonian gravitational potential φ.

We have to solve Eqs. (2.19) and (2.20). We suppose, for simplicity, that matter is not coupled to σ directly, but only indirectly through its own electromagnetic field, that is, we put $\vartheta = 0$.

We get, in this way, an estimate of the scalar field which may be regarded as a sort of lower limit for σ, since an additional matter contribution would produce a stronger scalar field. To first order in σ, Eqs. (2.19) and (2.20) decouple, and become ($c = 1$)

$$F^{ik}{}_{;k} = 4\pi J^i \;, \tag{4.1}$$

$$\sigma^{,k}{}_{;k} = G F_{ik} F^{ik} \;. \tag{4.2}$$

In the weak-field approximation, we suppose that g_{ik} and σ are uncoupled; then the equation for the scalar field, in the static limit, reduces to

$$\nabla^2 \sigma = 2G(\,|\vec{\mathrm{B}}\,|^2 - |\,\vec{\mathrm{E}}\,|^2) \;, \tag{4.3}$$

where the electric and magnetic fields are solutions of the ordinary Maxwell equations.

The solution of Eq. (4.3) is given by

$$\sigma(x) = -\frac{G}{2\pi} \int_v \frac{d^3x'}{|\,\vec{\mathrm{x}} - \vec{\mathrm{x}}\,'\,|} (\,|\vec{\mathrm{B}}\,|^2 - |\,\vec{\mathrm{E}}\,|^2) \;, \tag{4.4}$$

where V is the volume of the source. As we are concerned with the field far from the body, expanding the solution (4.4) and considering only the first term of the expansion (to a good accuracy, in the case of a spherical source), we obtain

$$\sigma(x)=\frac{4G}{r}(\mathscr{E}_e-\mathscr{E}_m)\ , \tag{4.5}$$

where $r=|\vec{x}|$.

Therefore, according to the theory of Schmutzer, the scalar field produced by a macroscopic body is proportional to its own content of electromagnetic energy.

In the case of the Sun, for example, the energy due to its macroscopic magnetic field is negligible compared to its microscopic internal energy, and the dominant contribution to Eq. (4.5) comes from the nuclear electric energy.[9] Equation (4.5) then becomes

$$\sigma(x)\simeq 4(\mathscr{E}_{\rm ne}/M)\varphi(x)\ , \tag{4.6}$$

where M is the mass of the Sun, $\mathscr{E}_{ne}$ is the total nuclear electric energy, and $\varphi=GM/r$ is the gravitational potential. For the Sun one can estimate[9] $\mathscr{E}_{\rm ne}/M\simeq 10^{-4}$, and then the scalar field of the Sun is

$$\sigma(x)\simeq 4\times 10^{-4}\varphi(x)\ . \tag{4.7}$$

It must be stressed that we have considered only the electromagnetic contribution to σ. To evaluate the matter contributions, we may consider, for example, the case in which the source of the fields is a Dirac particle. In this case the four-dimensional matter Lagrangian of the projective theory is[3] ($\hbar=c=1$)

$$\mathscr{L}_M=\frac{\sqrt{-g}}{2}e^{-3\sigma/2}[\bar{\psi}\gamma^k(\psi_{;k}-i\alpha A_k\psi)-(\bar{\psi}_{;k}+i\alpha A_k\bar{\psi})\gamma^k\psi+2me^{-\sigma/2}\bar{\psi}\psi] \tag{4.8}$$

(α is the electromagnetic coupling constant). Performing the σ variation, and using the Dirac equations $\delta\mathscr{L}_M/\delta\psi=0=\delta\mathscr{L}_M/\delta\bar{\psi}$, one gets

$$\vartheta=-\frac{m}{2}e^{-2\sigma}\bar{\psi}\psi\ . \tag{4.9}$$

In the static limit Eq. (2.20) for a Dirac particle, neglecting the electromagnetic contributions, becomes, to first order in σ,

$$\nabla^2\sigma=-\frac{\chi}{3}m\psi^\dagger\psi \tag{4.10}$$

and the solution is

$$\sigma(x)=\frac{2}{3}G\frac{m}{r}\ . \tag{4.11}$$

Therefore each nucleon, for example, contained in a macroscopic body, gives a contribution to the scalar field proportional to its mass.

Since the matter contributions must be added to our previous estimate (4.7), it is justified then to regard Eq. (4.7) as a lower limit for the macroscopic scalar field produced by the Sun.

V. RESULTS AND CONCLUSION

Combining the results of Secs. III and IV, it follows, from the projective theory of Schmutzer, that test bodies of different electromagnetic content would accelerate differently in the solar gravitational field. Consider, for example, aluminum and platinum: the internal magnetic energy is negligible compared to the nuclear electric energy, and the $\mathscr{E}_e/m$ ratios are[9]

$$(\mathscr{E}_e/m)_{\rm Al}\simeq 1.7\times 10^{-3}\ ,$$
$$(\mathscr{E}_e/m)_{\rm Pt}\simeq 4.5\times 10^{-3}\ . \tag{5.1}$$

Therefore, combining Eqs. (3.7) and (4.7), it follows that an aluminum and a platinum test body would fall in the solar gravitational field with accelerations differing by

$$(\ddot{\vec{\xi}})_{\rm Pt}-(\ddot{\vec{\xi}})_{\rm Al}=3\left[\left[\frac{\mathscr{E}_e}{m}\right]_{\rm Pt}-\left[\frac{\mathscr{E}_e}{m}\right]_{\rm Al}\right]\vec{\nabla}\sigma \simeq 3.4\times 10^{-6}\vec{\nabla}\varphi\ , \tag{5.2}$$

where φ is the solar Newtonian potential. However, the experimental tests of the equivalence principle, performed by Dicke[6] and Braginsky,[7] show that the accelerations of aluminum and platinum, in the solar gravitational field, do not differ by more than 1 part in 10^{11} or 10^{12} of $\nabla\varphi$. Therefore the projective unified theory discussed in Sec. II disagrees with experimental data. This difficulty of course disappears if one rejects the identification of the antisymmetric tensor B_{ik}, in the projective Lagrangian, with the electromagnetic field. However, we think that one should not reject the interesting possibility of constructing a unified theory for gravitation and electromagnetism in the framework of the projective formalism, provided that Schmutzer theory is suitably modified. To this purpose, we notice that a similar situation arises in gauge theories when we try to couple torsion to the electromagnetic field without breaking gauge invariance. It is well known that the coupling is possible only with a vector or an axial-vector torsion tensor; however, in the vector case, an experimental disagreement occurs[9] like the one discussed in this paper, while in the axial case, a violation of the equivalence principle is predicted[10,11] only in the case of polarized test bodies, and then no disagreement is found with Dicke-Braginsky experiments.

Since the experimental disagreement, in the projective theory, is due to the interaction of the electromagnetic and scalar fields, and since the scalar field is the source of the vector part of the projective torsion, we suggest then to modify the geometric structure of the five-dimensional space, formulating the unified theory in a projective space endowed with a totally antisymmetric torsion tensor, so that the torsion vector is vanishing, and the electromagnetic field will be eventually coupled not to a scalar but to a pseudoscalar field, as done, for example, in Ref. 10 for the four-dimensional case.

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