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Renormalization-scheme ambiguity and perturbation theory near a fixed point

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We consider the perturbative calculation of critical exponents in massless, renormalizable theories having a nontrivial fixed point. In conventional perturbation theory, all results depend on the arbitrary renormalization scheme used. We show how to resolve this problem, following the "principle of minimal sensitivity" approach. At least three orders of perturbation theory are required for quantitative results. We give scheme-independent criteria for determining the presence or absence of a fixed point in nth order, and discuss the conditions under which perturbative results might be reliable. As illustrations we discuss QED with many flavors, and $(\phi^4)_4$ theory. In neither case do we find a fixed point, in contrast to naive perturbative expectations.

I. INTRODUCTION

Consider a renormalizable theory, asymptotically free in the ultraviolet region, which has an infrared fixed point $a=a^*$. (Or, alternatively, consider an infrared-free theory with an ultraviolet fixed point.) If a^* is small, it is tempting to think that perturbation theory will apply even up to the fixed point, so that the physics in both ultraviolet and infrared regions is perturbatively calculable.

There are two ways in which this idea might fail. (i) There might be nonperturbative terms [e.g., $\exp(-a^*/a)$], invisible in perturbation theory, which become large near the fixed point, no matter how small a^* is; or (ii) perturbation theory might become internally inconsistent near the fixed point. The first question cannot be addressed by a perturbative analysis, and we shall have nothing more to say about it: We assume here that such terms do not arise. The second question, however, can be studied by analyzing the structure of perturbation theory at or near a fixed point. It is this issue that we propose to address here. Our conclusion will be cautiously positive: Under certain circumstances, it is perfectly possible that perturbation theory yields believable results at the fixed point.

Central to this issue is the problem of the renormalization-scheme ambiguity of perturbation theory. While physical quantities are, in principle, independent[1] of the arbitrary choice of renormalization scheme (RS), this invariance is inevitably spoiled by truncating the perturbation series. Consequently, at any finite order, the results of perturbation theory are ambiguous. For example, the fixed point a^* is supposedly the zero of the renormalization-group β function, but the coefficients of this function are RS dependent, except for the first two: thus, at finite order, the position of the fixed point—even its existence— can be altered by a change of RS. How, then, can we tell whether a "fixed point," found perturbatively, is likely to be real or spurious?

We stress that we are interested here in renormalizable, and not superrenormalizable, theories. That is, we consider the theory in the critical number of dimensions for which the coupling constant is dimensionless. In many critical-phenomena applications one is interested in a superrenormalizable theory, and one employs the ϵ expansion,[2] where $\epsilon=d_{\rm crit}-d$. There is no scheme-dependence problem in the ϵ expansion: the results *at each order in* ϵ are independent of the RS choice. The point is that ϵ is a RS-invariant expansion parameter. However, there are some critical-phenomena applications (e.g., Refs. 3 and 4), as well as many high-energy physics applications, in which one is interested in the renormalizable theory itself, and would like to employ perturbation theory. The problem is then that one's expansion parameter, the renormalized couplant a, is RS dependent.

A philosophy for dealing with the RS-dependence problem has been given by one of the authors in Ref. 5 (hereafter referred to as I). The idea is that, since the exact result is exactly RS independent, we want our approximate result to share this property, at least in an approximate sense. That is, we should examine our approximate result as a function of RS and ask where it becomes insensitive to small variations in scheme. Only in this region does the result have a chance of being believable. We mention that this line of argument is supported by examples in I and elsewhere.[6–8]

The purpose of this paper is to develop the "optimization" formalism of I to apply to the problem of calculating critical exponents, and other physical quantities, at a nontrivial fixed point. This provides us with a framework within which we can meaningfully discuss the questions raised earlier. We find that the optimization conditions at the fixed point simplify to algebraic, rather than transcendental, equations. A less-welcome result is that, in order to obtain quantitative results at the fixed point, the formalism requires at least three orders of perturbation theory. Because of this fact, we are unable to offer many examples. We discuss two cases: (i) QED with many flavors, where the conventional RS would indicate a fixed point at third order. In our method this fixed point is seen to be spurious, in agreement with the large-n_f analysis of Ref. 9. (ii) $(\phi^4)_4$ theory, where we find that, even though the first two terms of the β function have opposite sign, there is no indication of a fixed point in third order. This is in accord with various nonperturbative approaches.[2,10]

II. FORMALISM

A. Conventional approach

Consider the critical exponent γ^* of some particular Green's function (correlation function) $\Gamma=\Gamma(p_i,\mu,a(\mu))$. Here p_i denote the physical external momenta, μ is the unphysical arbitrary mass scale introduced in the renormalization procedure, and $a(\mu)$ is the renormalized coupling constant (couplant). Γ has an anomalous dimension defined by

$$\gamma(a)\equiv\frac{\mu}{\Gamma}\frac{d\Gamma}{d\mu}=\frac{\mu}{Z_\Gamma}\frac{dZ_\Gamma}{d\mu}\ , \tag{1}$$

where

$$\Gamma=Z_\Gamma(a(\mu),\mu,\text{cutoff})\Gamma_{\text{bare}}(p_i,\text{cutoff})\ . \tag{2}$$

We shall loosely call Z_Γ the "wave-function renormalization constant," though strictly it is a combination of wave-function renormalization constants appropriate to the particular Γ in question.

The critical exponent γ^* is the value of $\gamma(a)$ at the fixed point $a=a^*$, where a^* is a positive zero of the β function, $\beta(a)$ being defined as

$$\beta(a)\equiv\mu\frac{da}{d\mu}=\left[\frac{\mu}{Z_a}\frac{dZ_a}{d\mu}\right]a\ , \tag{3}$$

where

$$a(\mu)=Z_a(a(\mu),\mu,\text{cutoff})a_{\text{bare}}(\text{cutoff})\ . \tag{4}$$

(If β has more than one nontrivial zero we shall be interested only in the one closest to the origin, since we presuppose our theory to lie in the perturbative domain $0<a<a^*_{\min}$.)

Since γ^* is a physically measurable quantity, it must be renormalization-group invariant.[1] This means that, if one could somehow obtain $\gamma(a)$ and $\beta(a)$ exactly, the value of $\gamma(a)$ at a positive zero of $\beta(a)$ would always be the same, irrespective of the particular scheme used to renormalize the results.[11] However, the various intermediate quantities $\Gamma,\gamma(a),\beta(a),a^*$ are not invariant. [For example, with a different coupling-constant renormalization $a'=Z'_a a_{\text{bare}}=(Z'_a/Z_a)a=f(a)$, the position of the fixed point is moved to $a^{*\prime}=f(a^*)$.]

In the nth order of perturbation theory, the naive procedure for calculating γ^* is to first calculate

$$\gamma^{(n)}(a)=\gamma_1 a+\gamma_2 a^2+\cdots+\gamma_n a^n \tag{5}$$

and

$$\beta^{(n)}(a)=-ba^2(1+ca+c_2a^2+\cdots+c_{n-1}a^{n-1}) \tag{6}$$

from Feynman diagrams, then to find a^* as the smallest positive root of $\beta^{(n)}(a^*)=0$, and finally to evaluate $\gamma^{(n)}(a)$ at the fixed point $a=a^*$. In the calculation one can employ any renormalization scheme (RS). The difference between two RS's corresponds to a different choice for the finite parts of the coefficients $z_\Gamma^{(i)}$ and $z_a^{(i)}$ in

$$\begin{aligned}Z_\Gamma&=1+z_\Gamma^{(1)}a+z_\Gamma^{(2)}a^2+\cdots+z_\Gamma^{(n)}a^n\ ,\\ Z_a&=1+z_a^{(1)}a+z_a^{(2)}a^2+\cdots+z_a^{(n)}a^n\ .\end{aligned} \tag{7}$$

While, in principle, the value of γ^* should be independent of the RS choice, this invariance property is spoiled by series truncations in (5) and (6).

There are two effects here. First, away from the fixed point, $\gamma(a)$ is not a RS-invariant quantity: it depends intrinsically upon the wave-function renormalization. This dependence turns out to be proportional to $\beta(a)$, and so, in principle, it will go away at $a=a^*$.[11] However, the required factorization property is lost when the series is truncated. Second, the absence of coupling-constant RS dependence in γ^* is due to a subtle cancellation between the RS dependence of a^* and the RS dependence of the coefficients of γ and β. [Except for γ_1, b, and c, all the coefficients in (5) and (6) are RS dependent.] Again, this cancellation mechanism is disrupted by the series truncations.

The resulting RS ambiguity makes perturbative results difficult to interpret. For example, in one RS one might find $c_{n-1}<0$, whereas, in another scheme, $c_{n-1}>0$. So, in nth order, one may find a β-function zero in one scheme which is simply absent in another scheme. This illustrates the seriousness of the RS-dependence problem.

To cope with the RS dependence we proceed in two stages. First, we eliminate all dependence on the wave-function renormalization Z_Γ by shifting our attention from the anomalous dimension $\gamma(a)$ to a closely related quantity, $\mathscr{R}_\Gamma$, which is RS invariant even away from the fixed point. We can then apply the "optimization" procedure of I to deal with the coupling-constant RS dependence present in perturbative approximations to $\mathscr{R}_\Gamma$.

B. The RS-invariant anomalous dimension

The quantity we propose to consider is

$$\mathscr{R}_\Gamma\equiv-\frac{\lambda}{\Gamma}\frac{d}{d\lambda}\Gamma(\lambda p_i,\mu,a(\mu))\Big|_{\lambda=1} \tag{8}$$

which describes the behavior of Γ under a homogeneous, infinitesimal scaling of its momentum arguments. As we shall show, $\mathscr{R}_\Gamma$ is RS invariant, even away from the fixed point. Potentially a physically measurable quantity, $\mathscr{R}_\Gamma$ is really of more direct interest than $\gamma(a)$ itself. What we are doing, in fact, is simply to go back one step in the usual RG procedure.

The relation beween $\mathscr{R}_\Gamma$ and γ follows from dimensional analysis, the only subtlety being the μ dependence of a. (For an elementary, pedagogical discussion of these issues, see Ref. 12.) Ignoring a, Γ depends only on the ratios p_i/μ, and so we have

$$\mathscr{R}_\Gamma = \frac{\mu}{\Gamma}\frac{\partial}{\partial\mu}\Gamma(p_i,\mu,a)\bigg|_{a=\text{fixed}} . \tag{9}$$

Allowing for the μ dependence of a we can convert this into a total derivative, as in Eq. (1), and so obtain

$$\mathscr{R}_\Gamma = \gamma - \frac{\beta(a)}{\Gamma}\frac{d\Gamma}{da} . \tag{10}$$

We see that, at the fixed point, $\mathscr{R}_\Gamma$ and γ coincide.

The RS invariance of $\mathscr{R}_\Gamma$ is easily seen using (2):

$$\mathscr{R}_\Gamma = \frac{-\lambda}{Z_\Gamma\Gamma_{\text{bare}}}\frac{d}{d\lambda}(Z_\Gamma\Gamma_{\text{bare}}) = \frac{-\lambda}{\Gamma_{\text{bare}}}\frac{d\Gamma_{\text{bare}}}{\partial\lambda} . \tag{11}$$

The point is that the renormalization constant Z_Γ, being momentum independent, simply cancels out in $\mathscr{R}_\Gamma$. Thus, even though Γ itself is an intrinsically RS-dependent object, its scaling behavior, as measured by $\mathscr{R}_\Gamma$, is RS independent and hence meaningful. To stress this simple point we express it as a theorem.

Theorem. The normalized derivative $(Q/\Gamma)(d\Gamma/dQ) = d(\ln\Gamma)/d(\ln Q)$ of any multiplicatively renormalized function Γ with respect to its external momentum arguments (or any combination thereof) is RS invariant.

The proof follows from Eq. (11). [In gauge theories, however, one must check that $\mathscr{R}_\Gamma$ is gauge independent; i.e., independent of the bare gauge parameter, a possibility not excluded by the proof in (11).[13]]

C. Optimization

We have now eliminated the problem of wave-function renormalization dependence; it cancels out, order by order, in $\mathscr{R}_\Gamma$. The second stage of our program is to deal with the coupling-constant RS dependence of perturbative approximations to $\mathscr{R}_\Gamma$. As explained in the Introduction, our strategy is to study the RS dependence of the approximant and seek the point at which it becomes least sensitive to RS variations; i.e., we are looking for

$$\frac{d\mathscr{R}_\Gamma^{(n)}}{d(RS)}\bigg|_{\text{RS=opt RS}} = 0 , \tag{12}$$

which we shall call the "optimization condition." Our analysis thus follows the spirit of I. However, since we are primarily interested in γ^*—the limiting value of $\mathscr{R}_\Gamma$—we can take advantage of certain simplifications which occur at the fixed point.

The symbolic equation (12) can be given a concrete meaning. As shown in I, the RS may be parametrized by the set of variables $\{\tau, c_2, c_3, \ldots, c_{n-1}\}$, where

$$\tau = b\ln\frac{\mu}{\tilde{\Lambda}} . \tag{13}$$

b and c_i are the β-function coefficients appearing in (6), and $\tilde{\Lambda}$ the μ-independent mass scale which enters through the boundary condition on the β function. A given set $\{\tau, c_i\}$ corresponds to a particular choice of the finite parts of the coefficients $z_a^{(i)}$ in (7), and so defines a (coupling-constant) RS.

In the nth order of perturbation theory $\mathscr{R}_\Gamma$ has the form

$$\mathscr{R}_\Gamma^{(n)}(a^{(n)}) = \gamma_1 a(1 + r_1 a + r_2 a^2 + \cdots + r_{n-1}a^{n-1}) , \tag{14}$$

where r_i are obtained by Feynman-diagram calculations in a given RS. The critical exponent γ^* is given by

$$\gamma^{*(n)} = \mathscr{R}_\Gamma^{(n)}(a^{*(n)}) \tag{15}$$

in this approximation, where $a^{*(n)}$ is the smallest positive root of

$$\beta^{(n)}(a^{*(n)}) = -ba^{*2}(1 + ca^* + c_2 a^{*2} + \cdots + c_{n-1}a^{*n-1}) = 0 . \tag{16}$$

The superscript (n) is used to denote "the nth-order approximation to . . . ." For brevity we shall often drop this qualifier on a^*, important though it is.

Due to the truncation, $\gamma^{*(n)}$ is RS dependent; by performing the above calculations in different RS's one will end up with different values of $\gamma^{*(n)}$. To deal with this RS dependence, we must identify the RS dependence in r_i and in $a^{*(n)}$ separately. The former follows from the self-consistency condition that the $\mathscr{R}_\Gamma^{(n)}$'s in different RS's can differ only by terms which are formally higher order in a, i.e.,

$$\frac{d\mathscr{R}_\Gamma^{(n)}(a)}{d(\text{RS})} = O(a^{n+1}) . \tag{17}$$

Here (RS) stands for the set of parameters $\{\tau, c_i\}$. Thus (17) leads to the following equations (see I) [for later use we quote the results for the general case in which $\gamma(a)$ starts at order a^P, though for the present we are considering only $P=1$]:

$$\begin{aligned}
&\frac{\partial r_1}{\partial\tau} = P ,\\
&\frac{\partial r_2}{\partial\tau} = (P+1)r_1 + Pc, \quad \frac{\partial r_2}{\partial c_2} = -P ,\\
&\frac{\partial r_3}{\partial\tau} = (P+2)r_2 + (P+1)r_1 c + Pc_2 ,\\
&\frac{\partial r_3}{\partial c_2} = -(P+1)r_1, \quad \frac{\partial r_3}{\partial c_3} = -\frac{P}{2} ,
\end{aligned} \tag{18}$$

etc. They may readily be solved, and the results can be summarized as saying that the combinations

$$\rho_1=\tau-r_1/P\ ,$$
$$\rho_2=r_2+Pc_2-\frac{P+1}{2P}\left[r_1+\frac{P}{P+1}c\right]^2\ , \tag{19}$$
$$\rho_3=r_3+\frac{P}{2}c_3$$
$$-r_1\left[c_2+\frac{P+2}{P}r_2-\frac{(P+1)(P+2)}{3P^2}r_1{}^2-\frac{cr_1}{2P}\right]\ ,$$

etc., are RS invariant; the ρ_i arise as constants of integration and hence are independent of $\{\tau,c_i\}$.

To find the RS dependence of a^* we simply need to ask how the root of (16) changes if we vary one of the coefficients c_j. Differentiating, we find

$$\frac{\partial a^*}{\partial c_j}=-a^{*j+1}/[ca^*+2c_2a^{*2}+\cdots+(n-1)c_{n-1}a^{*n-1}]$$
$$=ba^{*j+2}/B^{(n)}\ , \tag{20}$$

where

$$B^{(n)}\equiv\left.\frac{d\beta^{(n)}}{da}\right|_{a=a^*}$$
$$=-ba^*(1+2ca^*+\cdots+nc_{n-1}a^{*n-1})\ . \tag{21}$$

Note that it is not possible to assign an order in a^* to any quantity; the condition $\beta^{(n)}(a^*)=0$ involves equating different powers of a^*, and by using it we can alter the apparent order in a^* at will. The two parts of Eq. (20) illustrate this phenomenon.

It can easily be verified that the above result also follows from the general formula

$$\frac{\partial a}{\partial c_j}\equiv\beta_j(a)=-b\beta(a)\int_0^a dx\frac{x^{j+2}}{[\beta(x)]^2}\ ,$$

given in I, in the limit $a\rightarrow a^*$ in which $\beta^{(n)}\rightarrow(a-a^*)B^{(n)}$.

Armed with these results we can now analyze the RS dependence of the approximant $\gamma^{*(n)}$ in (15) and (16). At second order we have

$$\gamma^{*(2)}=\gamma_1a^*(1+r_1a^*) \tag{22}$$

with

$$1+ca^*=0\ . \tag{23}$$

We can immediately see the difficulty with second order. The only scheme dependence resides in r_1, since both γ_1 and c are RS invariant. The scheme-dependence cancellation mechanism has not yet got going, and the approximant is a monotonic function of scheme: there is no stationary point where it is locally insensitive to scheme changes. This is analogous to the situation in *first* order in the general case away from the fixed point, where the only scheme dependence is in $a^{(1)}(\tau)$. In second order one normally has a (partial) cancellation between the τ dependences of a and r_1. The trouble is that as one approaches the fixed point, the τ dependence of the couplant a disappears, since it tends to the constant value a^*. One has to proceed to at least third order to see the beginnings of the cancellation mechanism and to extract a meaningful result.

In third order we have

$$\gamma^{*(3)}=\gamma_1a^*(1+r_1a^*+r_2a^{*2}) \tag{24}$$

with $a^*\equiv a^{*(3)}$ being the root of

$$1+ca^*+c_2a^{*2}=0\ . \tag{25}$$

The approximant $\gamma^{*(3)}$ depends on the two RS parameters τ and c_2. Explicitly, using (18) and (20), we have

$$\frac{\partial\gamma^{*(3)}}{\partial\tau}=\gamma_1a^{*2}[1+(2r_1+c)a^*]\ , \tag{26}$$
$$\frac{\partial\gamma^{*(3)}}{\partial c_2}=-\gamma_1a^{*3}+\gamma_1(1+2r_1a^*+3r_2a^{*2})\frac{ba^{*4}}{B^{(3)}}\ . \tag{27}$$

Our optimization condition is that these variations should vanish in the "optimum scheme." Equating (26) and (27) to zero, together with the fixed-point condition (25) and the constraint that ρ_2 as given in (19) is a RS-invariant combination, we can obtain the "optimized" values of the scheme-dependent parameters r_1,r_2,c_2,a^*, which are

$$\bar{r}_1=-\frac{1}{2}\left[c+\frac{1}{\bar{a}^*}\right]\ ,$$
$$\bar{r}_2=-2\left[\rho_2+\frac{1}{4\bar{a}^{*2}}\right]\ , \tag{28}$$
$$\bar{c}_2=-\tfrac{3}{2}\bar{r}_2=3\left[\rho_2+\frac{1}{4\bar{a}^{*2}}\right]\ ,$$

where $\bar{a}^*$ is given by the quadratic equation

$$\tfrac{7}{4}+c\bar{a}^*+3\rho_2\bar{a}^{*2}=0\ . \tag{29}$$

Putting all this together we can write the optimized result for $\gamma^{*(3)}$, provided that $\bar{a}^*$ exists, as

$$\gamma^{*(3)}_{\rm opt}=\gamma_1\bar{a}^*(\tfrac{7}{6}+\tfrac{1}{6}c\bar{a}^*)\ . \tag{30}$$

The practical procedure can be summarized thus: first one needs to compute $\mathscr{R}_\Gamma$ and β to third order (this is the hard part). This can be done in any RS that one finds computationally convenient, though, of course, it must be the same RS for both calculations. The coefficients γ_1 and c are already RS invariant, and from the others one can compute the RS-invariant combination ρ_2 of (19). It is then trivial to solve (29) for $\bar{a}^*$, and hence to evaluate $\gamma^{*(3)}_{\rm opt}$ from (30).

As we noted before, in a random RS we may or may not find a fixed point, depending entirely on the scheme label c_2. Now, however, we have in (29) a RS-independent criterion for whether third-order perturbation theory indicates a fixed point, or not. Equation (29) has a root if $\rho_2\le c^2/21$. A positive root $\bar{a}^*$ exists in the following two cases:

$$\text{case (i): } \rho_2<0 \text{ and } c \text{ arbitrary },$$
$$\text{case (ii): } 0\le\rho_2<c^2/21 \text{ and } c<0 . \quad (31)$$

If $\bar{a}^*$ is too large, the perturbation approach becomes unreliable. If we require that $\bar{a}^*<1$, then we must have, for case (i) $\rho_2<-(7+4c)/12$; and for case (ii) $c<-\frac{7}{4}$ and $\rho_2<(4|c|-7)/12$.

One may compare this with a naive guess which uses only the RS-independent terms in (5) and (6):

$$\gamma^*=\gamma_1 a^* ,$$
$$\beta(a^*)=-ba^{*2}(1+ca^*)=0 .$$

In this case, $c<0$ indicates a nontrivial fixed point. In the optimization approach, however, c negative is neither a necessary nor a sufficient condition for a fixed point. On the other hand, the naive guess $\gamma^*\simeq\gamma_1/|c|$ would be justified if $\rho_2\simeq-c^2/4$.

D. Fourth order and generalizations

To illustrate the method further we briefly consider fourth order, where we have

$$\gamma^{*(4)}=\gamma_1 a^*(1+r_1a^*+r_2a^{*2}+r_3a^{*3}) \quad (32)$$

with $a^*\equiv a^{*(4)}$ being the root of

$$1+ca^*+c_2a^{*2}+c_3a^{*3}=0 . \quad (33)$$

We now have to consider three RS labels; τ, c_2, and c_3. Differentiating, and using (18) and (20) gives

$$\frac{\partial\gamma^{*(4)}}{\partial\tau}=\gamma_1a^{*2}[1+(2r_1+c)a^*+(3r_2+2r_1c+c_2)a^{*2}] ,$$
$$\frac{\partial\gamma^{*(4)}}{\partial c_2}=-\gamma_1a^{*3}(1+2r_1a^*)+\gamma_1(1+2r_1a^*+3r_2a^{*2}+4r_3a^{*3})\frac{ba^{*4}}{B^{(4)}} , \quad (34)$$
$$\frac{\partial\gamma^{*(4)}}{\partial c_3}=\gamma_1a^{*4}(-\tfrac{1}{2})+\gamma_1(1+2r_1a^*+3r_2a^{*2}+4r_3a^{*3})\frac{ba^{*5}}{B^{(4)}} .$$

Requiring these expressions to vanish, together with the fixed-point condition (33), and the expressions (19) for the invariants ρ_2 and ρ_3 we find, after some algebra, that

$$\tfrac{83}{64}+\tfrac{13}{16}c\bar{a}^*+\tfrac{3}{4}(\rho_2+\tfrac{1}{4}c^2)\bar{a}^{*2}+4\bar{a}^{*3}\rho_3=0 \quad (35)$$

and

$$\gamma^{*(4)}_{\text{opt}}=\gamma_1\bar{a}^*[\tfrac{249}{256}+\tfrac{13}{64}c\bar{a}^*+\tfrac{1}{16}(\rho_2+\tfrac{1}{4}c^2)\bar{a}^{*2}] . \quad (36)$$

These two equations are the analogs of Eqs. (29) and (30). They determine the optimized fourth-order result for γ^* in terms of RS-invariant, calculable quantities.

It is straightforward to continue to still higher orders, though we are unable to give explicit formulas for the general, nth-order case. General expressions for $\partial r_l/\partial\tau$ and $\partial r_l/\partial c_j$ are given in I, but no explicit formula for the nth-order invariant ρ_n is known.

Let us suppose that we do find a solution to (29), (35), etc., and so obtain a result for $\gamma^{*(n)}_{\text{opt}}$, in some particular application. Under what circumstances can we believe the result? As usual with any approximation method, one can examine how much the results change from one order to the next. Only if the results appear to be settling down can one have confidence that one is getting reliable results. We emphasize that this is an essentially *numerical* matter; comparing the numerical result of $\gamma^{*(n)}$ with that of $\gamma^{*(n+1)}$, so as to obtain a numerical estimate of the error. It would be dangerous to use formal considerations based on "orders in a^*," since, as we mentioned earlier, that concept is meaningless.

We conclude this section by quoting the formulas for the general case in which $\mathscr{R}_\Gamma(a)$ has the form

$$\mathscr{R}_\Gamma=\gamma_1a^P(1+r_1a+r_2a^2+\cdots) \quad (37)$$

for general values of P. [We assumed $P=1$ until now, except when specifying the invariants ρ_i in (19).] The third-order result is

$$\frac{(2P^2+3P+2)}{2(P+1)}+Pca^*+\frac{(P+2)}{P}\rho_2a^{*2}=0 \quad (38)$$

and

$$\gamma^{*(3)}_{\text{opt}}=\frac{\gamma_1a^{*P}}{(P+1)(P+2)}[(2P^2+3P+2)+P^2ca^*] , \quad (39)$$

which generalize Eqs. (29) and (30), respectively. The fourth-order result is

$$\frac{P(24P^3+77P^2+100P+48)}{24(P+1)^2}+\frac{P(4P^2+6P+3)}{4(P+1)}ca^*+\frac{P(P+2)}{(P+1)}\tilde{\rho}_2a^{*2}+2(P+3)\rho_3a^{*3}=0 \quad (40)$$

and

$$\gamma^{*(4)}_{\text{opt}}=\frac{\gamma_1a^{*P}}{48(P+1)^2(P+3)}[(72P^3+231P^2+300P+144)+12P(4P^2+6P+3)ca^*+24P(P+1)\tilde{\rho}_2a^{*2}] , \quad (41)$$

where

$$\tilde{\rho}_2 \equiv \rho_2 + \frac{P}{2(P+1)} c^2 . \quad (42)$$

We should emphasize that our procedure is no more and no less than the infrared (ultraviolet) limit, in a fixed-point theory, of the optimization procedure of I. That is, we can imagine calculating the quantity $\mathscr{R}_\Gamma$ at a finite energy scale Q and optimizing the approximation by the method described in I. In the limit $Q \to 0$ ($Q \to \infty$), given that we have a fixed-point theory, one would obtain a limiting result which coincides with the one obtained here. The important point is that the optimization conditions greatly simplify at the fixed point, becoming algebraic, rather than transcendental, equations.

III. ILLUSTRATIVE EXAMPLES

A. QED with many flavors

As a first example, we consider QED with n_f integer-charge fermions, in third-order perturbation theory. The coefficients of the β function are

$$b = -\tfrac{2}{3} n_f , \qquad c = \tfrac{3}{4} , \quad (43)$$

which are RS independent, and

$$c_2^{\text{on}} = -\tfrac{1}{96}(112 n_f + 9) , \qquad c_2^{\text{MS}} = -\tfrac{1}{96}(22 n_f + 9) , \quad (44)$$

in the on-shell[14] and minimal-subtraction (MS)[15] schemes, respectively. Naively, one would look for a fixed point as a positive root of the equation

$$1 + c a^* + c_2 a^{*2} = 0 . \quad (45)$$

In either RS one would find such a root, since c_2 is negative. Moreover, for large n_f, the root is given by

$$a^* \simeq (\tfrac{7}{6} n_f)^{-1/2} \ \text{(on-shell)} , \qquad a^* \simeq (\tfrac{11}{48} n_f)^{-1/2} \ \text{(MS)} , \quad (46)$$

and so a^* becomes small for large n_f. Thus, one would be tempted to conclude that there is a nontrivial ultraviolet fixed point in the perturbative regime, when n_f is sufficiently large.

We now show that, according to our method, this fixed point is a spurious one. For this purpose we consider the inverse photon propagator

$$(-q^2 g_{\mu\nu} + q_\mu q_\nu)[1 + \Pi(-q^2/m^2, a)] + \xi q_\mu q_\nu , \quad (47)$$

where $a \equiv e^2/4\pi^2 = \alpha/\pi$, and ξ is the gauge parameter. The self-energy function Π is ξ independent. We define the corresponding RS-invariant anomalous dimension as

$$\mathscr{R}_\Pi = -2 \frac{d}{d \ln x} \{ \ln[1 + \Pi(x,a)] \} , \quad (48)$$

where $x \equiv -q^2/m^2$. We shall make use of the calculations of de Rafael and Rosner,[14] performed in the on-shell scheme. Following their work [cf. their Eq. (2.14)] we may write

$$\Pi(x,a) = (A_1 + B_1 \ln x) a + (A_2 + B_2 \ln x) a^2 + (A_3 + B_3 \ln x + C_3 \ln^2 x) a^3 + \cdots , \quad (49)$$

if we neglect terms that vanish in the ultraviolet limit $-q^2/m^2 \to \infty$. Reference 14 gives numerical values for all these coefficients (except for A_3, which we do not need), and derives the relationship to the β-function coefficients:

$$b = 2B_1, \quad c = B_2/B_1, \quad c_2^{\text{on}} = B_3/B_1 + A_2 , \quad (50)$$

and also the constraint

$$2C_3 + B_1 B_2 = 0 . \quad (51)$$

Evaluating

$$\mathscr{R}_\Pi = \gamma_1 a (1 + r_1 a + r_2 a^2 + \cdots) , \quad (52)$$

one finds from (48) and (49) that

$$\begin{aligned} \gamma_1 &= -2B_1 = \tfrac{2}{3} n_f , \\ r_1^{\text{on}} &= B_2/B_1 - A_1 - B_1 \ln x , \\ r_2^{\text{on}} &= (A_1^2 - A_2 - A_1 B_2/B_1 + B_3/B_1) \\ &\quad + 2(A_1 B_1 - B_2 + C_3/B_1) \ln x + B_1{}^2 \ln^2 x . \end{aligned} \quad (53)$$

From these results it is straightforward to construct ρ_2, as defined in Eq. (19) (with $P=1$). The $\ln x$ terms all cancel, leaving

$$\rho_2 = 2cA_1 + 2c_2^{\text{on}} - 2A_2 - \tfrac{9}{4} c^2 . \quad (54)$$

Using c, c_2^{on}, quoted above, together with[14]

$$A_1 = \tfrac{5}{9} n_f , \qquad A_2 = [\tfrac{5}{24} - \zeta(3)] n_f ,$$

where $\zeta(3) = \sum_1^\infty n^{-3} = 1.202 \ldots$, one obtains finally

$$\rho_2 = -\tfrac{93}{64} + [2\zeta(3) - \tfrac{23}{12}] n_f \simeq -1.453 + 0.487 n_f . \quad (55)$$

As a check on our results we have also constructed ρ_2 from the minimal-subtraction (MS) scheme calculations of Ref. 15. These give

$$\begin{aligned} r_1^{\text{MS}} &= -\tfrac{1}{3} n_f L - \tfrac{5}{9} n_f + \tfrac{3}{4} , \\ r_2^{\text{MS}} &= \tfrac{1}{9} n_f{}^2 L^2 + (\tfrac{10}{27} n_f{}^2 - \tfrac{3}{4} n_f) L + \tfrac{25}{81} n_f{}^2 \\ &\quad + [2\zeta(3) - \tfrac{47}{16}] n_f - \tfrac{3}{32} , \end{aligned} \quad (56)$$

where $L = \ln(\mu^2/Q^2) + \ln 4\pi - \gamma_E$, where γ_E is the Euler constant, and μ is the arbitrary unit of mass. Constructing ρ_2 [Eq. (19)] from the MS quantities, one observes that the L terms cancel, leaving exactly the same result as quoted in Eq. (55). This explicitly verifies the RS invariance of ρ_2.

Since $c=\frac{3}{4}>0$, a fixed point exists only if $\rho_2<0$, according to the discussion near Eq. (31) of Sec. II C. Therefore, for $n_f \geq 3$ we find no fixed point. (For $n_f=1$ and 2 we find $a^*=0.917$ and 1.395, respectively. Since these values are rather large, our perturbative approach is not reliable, and the reality of these fixed points is doubtful.) For large n_f, since ρ_2 becomes large and positive, our fixed-point criterion, Eq. (29), becomes increasingly emphatic that there is no fixed point. This is in sharp contrast to the naive expectation of Eq. (46). The spurious fixed point (46) is merely an artifact of an inappropriate RS choice.

Our view is confirmed by a recent analysis of the large-n_f limit in QED,[9] which shows that there is indeed no fixed point at large n_f.

B. $(\phi^4)_4$ theory

Our second example is the massless ϕ^4 theory in four Euclidean dimensions. We define the couplant to be $a=\lambda/16\pi^2$, where λ is normalized such that the interaction Lagrangian in $4-\epsilon$ dimensions is $\mathscr{L}_{\rm int} = -(\lambda\mu^\epsilon/4!)\phi^4+$ counterterms. With this definition, the β-function coefficients are given by[16]

$$\begin{aligned} b&=-3\,,\\ c&=-\tfrac{17}{9}\,,\\ c_2^{\rm MS}&=\tfrac{145}{24}+4\zeta(3)\,, \end{aligned} \tag{57}$$

where the value of c_2 has been calculated in the MS scheme. The negative value of c might seem to indicate, at second order, a nontrivial fixed point at $a^*\simeq\frac{9}{17}=0.53$. However, we shall see that in third order there is no fixed point. To this end we consider the minimally subtracted four-point function $\Gamma^{(4)}$ at the symmetric point of its external momenta p_i $(i=1,2,3,4)$

$$p_i\cdot p_j = Q^2(\delta_{ij}-\tfrac{1}{4})\,. \tag{58}$$

The corresponding RS-invariant anomalous dimension (8) is given by

$$\mathscr{R}_\Gamma = -2Q^2\frac{\partial}{\partial Q^2}\ln\hat{\Gamma}^{(4)}(Q^2/\mu^2,a)\,, \tag{59}$$

where $\hat{\Gamma}^{(4)}\equiv(\lambda/4!)^{-1}\Gamma^{(4)}$. The coefficients of the ordinary anomalous dimension

$$\gamma^{(4)}=\mu\frac{d}{d\mu}\ln\hat{\Gamma}^{(4)}=\gamma_1^{(4)}a+\gamma_2^{(4)}a^2+\gamma_3^{(4)}a^3+\cdots \tag{60}$$

are known:[16]

$$\begin{aligned} \gamma_1^{(4)}&=-3\,,\\ \gamma_2^{(4)}&=6\,,\\ \gamma_3^{(4)}&=-[\tfrac{147}{8}+12\zeta(3)]\,. \end{aligned} \tag{61}$$

To obtain the coefficients r_1 and r_2 in the expansion of $\mathscr{R}_\Gamma$ [see Eq. (14)], one needs to compute the "correction term" $-\beta(a)d\ln\hat{\Gamma}/da$ [see Eq. (10)]. This requires the order-ϵ^1 (order-ϵ^0) terms in the first- (second-) order diagrams, which can be obtained from the explicit expressions for the one- and two-loop diagrams in Ref. 17. We obtain

$$\begin{aligned} r_1^{\rm MS}&=-(\tfrac{3}{2}L+5)\,,\\ r_2^{\rm MS}&=\tfrac{9}{4}L^2+\tfrac{107}{6}L+\tfrac{1291}{24}-\pi^2+6J_2+4\zeta(3)\,, \end{aligned} \tag{62}$$

where $L=\ln(\mu^2/Q^2)+\ln4\pi-\gamma_E$, and

$$J_2=-\int_0^1 dy\frac{y}{(1-y)}\int_0^1 dZ\ln\left[\frac{yZ(1-yZ)+y(1-y)(Z+3/4)}{Z(1-Z)}\right] \simeq -2.410\,. \tag{63}$$

Computing ρ_2, we observe the expected cancellation of the L terms, leaving

$$\rho_2=\tfrac{7937}{324}-\pi^2+6J_2+8\zeta(3)\simeq 9.79\,. \tag{64}$$

Since this exceeds $c^2/21\simeq0.17$ we conclude, from the analysis leading to Eq. (31), that there is no sign of a fixed point in the third order of perturbation theory. This is in agreement with various nonperturbative approaches to $(\phi^4)_4$ theory, which also find that the theory has no fixed point.[2,10]

IV. CONCLUSIONS

We have applied the principle-of-minimal-sensitivity criterion to deal with the RS ambiguity in perturbative calculations of anomalous dimensions at a nontrivial fixed point. Our analysis gives a criterion, involving only RS-invariant quantities, for whether finite-order perturbation theory indicates a fixed point or not. We see no sign that perturbation theory becomes internally inconsistent as one approaches the fixed point. (Of course, it could be that nonperturbative terms become large, but there seems no reason to suppose that this is inevitable.) Consequently, we believe that there may exist theories, free in the ultraviolet (infrared), for which perturbation theory remains a reasonable approximation even in the far infrared (ultraviolet). A potential candidate for such a theory is QCD with 16 massless flavors.[18] An example of an infrared-free theory with an ultraviolet fixed point is ϕ^6 in three dimensions, at least if the O(N) symmetry is sufficiently large.[19]

Our analysis shows that, while important simplifications occur at the fixed point, one unfortunately needs at least three orders of perturbation theory before meaningful results can be obtained. This fact prevents us from applying the method to the interesting cases mentioned above. Our examples have been of a somewhat negative character; showing that one does *not* expect a fixed point in large-n_f QED or in $(\phi^4)_4$ theory. However, these examples are of some interest in that more naive perturbative ideas might well have led to the opposite conclusion.

Undoubtedly, with improving techniques and increasing computer power, many more higher-order results will become available in the future. The method developed here will then be important for interpreting these results, and disentangling the physical content from the RS artifacts.

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