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Discrete stochastic variational principles and quantum mechanics

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We consider stochastic variational principles for random processes taking values on a discrete configuration space. For a suitable time-reversal-invariant choice of the stochastic action, the resulting programming equations can be related to the Schrödinger equation for a discrete system. If the discrete system is considered as an approximation of a continuous system, then the limit reproduces Nelson's stochastic mechanics and allows one to derive the assumptions on the random noise acting on the system. In fact, the variational principle also gives information about the osmotic behavior of the process. Finally, we show that there are also critical processes whose behavior can be interpreted as a model for quantum measurement, because they relax in time to mixtures of processes. Therefore, stochastic variational principles can provide a very simple conceptual model simulating quantum behavior, both from the point of view of unperturbed time evolution and of the measurement phenomena leading to wave-function collapse.

I. INTRODUCTION

In previous work[1] it was shown that stochastic variational principles for controlled diffusion processes can provide a very simple basis for quantization of classical dynamical systems, in the frame of stochastic mechanics, originally introduced by Nelson.[2]

This general scheme, extended to the quantum field case (see, for example, Ref. 3), can be interpreted as a physical motivation for the great success[4] of probabilistic methods in the study of relativistic quantum field theory.

The main objective of this paper is to extend the methods of stochastic quantization to general quantum systems. Therefore, we consider Markov random processes taking values on a discrete (finite or denumerable) configuration space and introduce a suitable form of time-reversal-invariant stochastic action. Compared with the case of a continuous configuration space we cannot introduce *a priori* hypotheses on the properties of the random noise acting on the system. In particular, we cannot write stochastic differential equations for the controlled process, isolating a regular-drift part and a Brownian-disturbance part. On the other hand, in the discrete case the variational principle also gives information on the osmotic part of the process, so that no such *a priori* assumption is necessary.

For a suitable choice of the stochastic action the programming equations, coming from the variational principle, can be easily connected to the quantum Schrödinger equation.

Moreover, in the case when the discrete system can be considered as an approximation of a continuous system, of the type considered in Refs. 1 and 2, it can be easily shown that the processes corresponding to the discrete system have, as a limit, the processes corresponding to the continuous system, constructed according to the methods of Refs. 1 and 2. From this point of view, the basic background-field hypothesis of stochastic quantization can also be considered as a result coming from the choice of the stochastic action for the discrete approximating system.

Beyond the solutions corresponding to the Schrödinger evolution, the stochastic variational principle also provides other solutions corresponding to random processes, whose behavior resembles, in this general stochastic framework, the behavior of quantum systems subject to measurement. In particular, the asymptotic time behavior of these processes produces mixtures equivalent to those resulting from wave-function collapse in the conventional formulation of quantum mechanics. Here the collapse is not instantaneous but corresponds to a time-asymptotic relaxation behavior.

The organization of this paper is as follows. In Sec. II we briefly recall the discrete form of the Madelung fluid coming from the Schrödinger equation. This form is associated to a particular choice of canonical variables for the Hamiltonian system provided by the wave function and the evolution equation. These canonical variables are very natural to consider in a frame where discrete random processes and stochastic variational principles play a major role.

In Sec. III we collect all properties of discrete Markov processes necessary for our subsequent treatment. In particular, we recall the time-reversal properties and write in explicit form the variations of transition probabilities and densities corresponding to variations of the controlling parameters.

Section IV is devoted to a general discussion of stochastic variational principles coming from a choice of time-reversal-invariant action. We show that criticality for some action implies a double set of conditions related to the current and osmotic parts of the controlling parameters. The first set gives programming equations of the Hamilton-Jacobi type (adapted to a discrete setting). The second set determines the osmotic part of the transition probabilities per unit time.

In Sec. V we introduce our proposed form of stochastic action. This form is strongly reminiscent of analogous forms of local Lagrangians for canonical systems for which a "configuration" space is chosen, such that the phase space does not correspond to its cotangent bundle. We explicitly recall the elementary case of a plane pendulum where angular momentum is taken as the "configuration" of the system.

Section VI is dedicated to the derivations of properties of critical processes associated to the action introduced in Sec. V. We show the existence of solutions connected to Schrödinger evolution and the existence of nonstandard solutions whose properties are investigated in Sec. VIII and connected to measurement processes in quantum mechanics.

Section VII is dedicated to the study of the limit of discrete systems, considered as approximations of continuous systems. In this way we recover all of the structure of Nelson's stochastic mechanics, without any assumption on the Brownian disturbance, because the discrete stochastic variational principle gives all necessary information on the osmotic part of the process, as explained in Secs. IV and VI.

Finally, Sec. IX deals with possible further developments and applications of the general scheme outlined here.

II. THE DISCRETE FORM OF THE MADELUNG FLUID

Consider a quantum system on a separable (or even finite-dimensional) Hilbert space. If φ_i, $i=1,2,\dots$, is an orthonormal basis, let us introduce, for a generic wave function ψ, the components $\psi_i=\langle\varphi_i,\psi\rangle$ and consider the Schrödinger equation in the abstract form

$$i\hbar\partial_t\psi=H\psi \tag{1}$$

or, for the components,

$$i\hbar\dot{\psi}_i=\sum_j h_{ij}\exp(i\alpha_{ij}/\hbar)\psi_j\ . \tag{2}$$

In the right-hand side (RHS) of (2) the matrix elements of the Hamiltonian H have been written so that $h_{ij}\geq 0$. Self-adjointness of H implies

$$h_{ij}=h_{ji},\quad \alpha_{ij}=-\alpha_{ji},\quad \alpha_{ii}=0\ . \tag{3}$$

It is convenient to introduce real variables (ρ_i,S_i), $\rho_i\geq 0$, such that

$$\psi_i=\rho_i{}^{1/2}\exp(iS_i/\hbar)\ . \tag{4}$$

Then the complex equation (2) splits into two real ones:

$$\dot{\rho}_i=\sum_j(2h_{ij}/\hbar)(\rho_i\rho_j)^{1/2}\sin\beta_{ij}\ , \tag{5}$$

$$\dot{S}_i=-\sum_j h_{ij}(\rho_j/\rho_i)^{1/2}\cos\beta_{ij}\ , \tag{6}$$

where

$$\beta_{ij}=\frac{\alpha_{ij}+S_j-S_i}{\hbar},\quad \beta_{ij}=-\beta_{ji},\quad \beta_{ii}=0\ . \tag{7}$$

Note that (5) is a continuity equation and implies the conservation of the total probability so that, at all times,

$$\sum_i\rho_i(t)=1\ . \tag{8}$$

On the other hand (6) is of Hamilton-Jacobi type; it can also be written in the form

$$\dot{S}_i+H_i=0 \tag{9}$$

with $-H_i$ defined by the RHS of (6).

It is convenient to introduce the canonical Hamiltonian

$$\mathscr{H}=\langle\psi,H\psi\rangle=\sum_{i,j}h_{ij}(\rho_i\rho_j)^{1/2}\cos\beta_{ij}=\sum_i H_i\rho_i\ . \tag{10}$$

Then we can define Poisson brackets for functions F, G of (ρ_i,S_i),

$$\{F,G\}=\sum_i\left[\frac{\partial F}{\partial\rho_i}\frac{\partial F}{\partial S_i}-\cdots\right], \tag{11}$$

where $\cdots$ contains terms with F,G exchanged. In particular,

$$\{\rho_i,S_j\}=\delta_{ij},\quad \{\rho_i,\rho_j\}=0,\quad \{S_i,S_j\}=0\ . \tag{12}$$

This symplectic structure is the natural one associated to a Euclidean space. In fact we can consider Cartesian variables (x_i,y_i) such that

$$\psi_i=(x_i+iy_i)/\sqrt{2} \tag{13}$$

and note that for the basic one-forms

$$\omega_1\equiv\sum_i y_i dx_i=-\sum_i\rho_i dS_i/\hbar\ . \tag{14}$$

With respect to this natural symplectic structure, Eqs. (5) and (6) are the canonical Hamilton equations for the Hamiltonian (10). In fact

$$\dot{\rho}_i=\frac{\partial\mathscr{H}}{\partial S_i}=\{\rho_i,\mathscr{H}\},\quad \dot{S}_i=\frac{-\partial\mathscr{H}}{\partial\rho_i}=\{S_i,\mathscr{H}\}\ . \tag{15}$$

Note that (5) and (6) are invariant under local gauge transformations

$$S_i\to S_i+\chi_i,\quad \alpha_{ij}\to\alpha_{ij}+\chi_i-\chi_j\ , \tag{16}$$

corresponding to an independent rephasing of all wave functions of the basis.

The global invariance $S_i\to S_i+\chi$, $\mathscr{H}(\rho,S)=\mathscr{H}(\rho,S+\chi)$ generates the constant of motion $\sum_i\rho_i$ which makes it possible to keep (8) at all times. This shows that only the relative phases in (4) are physically relevant, as it is well known in quantum mechanics. This general frame about the canonical structure of the Schrödinger equation has also been considered in Ref. 5.

Our basic strategy, to give a stochastic background to this dynamical system, will be to consider ρ_i as the probability that a controlled Markov process is at site i, so that the continuity equation (5) is satisfied. On the other hand we will try to interpret (6) as the programming equation for a stochastic variational principle on the process. It is therefore necessary to recall some basic facts about discrete Markov processes.

III. DISCRETE MARKOV PROCESSES

Consider a Markov process $q(t)$, taking values on $\{1,2,\dots\}$. Let $\rho_i(t)$ be the occupation probability for the site i, so that (8) is satisfied and for the expectations we have $E(F(q(t),t))=\sum_i F_i(t)\rho_i(t)$. Introduce the transition probability $p(i,t;j,t')$, $t\geq t'$, so that

$$\begin{aligned} &p\geq 0,\quad \sum_i p(i,t;j,t')=1\ ,\\ &p(i,t;j,t')\to\delta_{ij}\quad \text{as } t\to t'^{+}\ ,\\ &p(i,t;j,t')=\sum_k p(i,t;k,t'')p(k,t'';j,t'),\quad t\geq t''\geq t'\ ,\\ &\rho_i(t)=\sum_j p(i,t;j,t')\rho_j(t')\ . \end{aligned} \tag{17}$$

Assume the existence of the limit

$$a_{ij}^{+}(t)=\lim_{\Delta t\to 0^{+}}[p(i,t+\Delta t;j,t)-\delta_{ij}]/\Delta t\ . \tag{18}$$

Then we have

$$\begin{aligned} &a_{ij}^{+}\geq 0\quad \text{for } i\neq j\ ,\\ &\textstyle\sum_i a_{ij}^{+}(t)=0\ ,\\ &a_{ii}^{+}\leq 0\ , \end{aligned} \tag{19}$$

and the forward diffusion equations

$$\partial_t p(i,t;j,t')=\sum_k a_{ik}^{+}(t)p(k,t;j,t')\ , \tag{20}$$

$$\dot\rho_i(t)=\sum_j a_{ij}^{+}(t)\rho_j(t)\ . \tag{21}$$

For a generic time-dependent site function $F(i,t)$ let us define forward and backward stochastic derivatives in analogy to those considered in Ref. 2:

$$(D_{(\pm)}F)(i,t)=\pm\lim_{\Delta t\to 0^{+}}(\Delta t)^{-1}E(F(q(t\pm\Delta t),t\pm\Delta t)-F(q(t),t)\,|\,q(t)=i)\ . \tag{22}$$

Then we can immediately show that

$$(D_{(\pm)}F)(i,t)=(\partial_t F)(i,t)+\sum_j F(j,t)a_{ji}^{\pm}(t)\ , \tag{23}$$

where a^{-} is easily found, as in Ref. 6, by exploiting the basic Nelson lemma

$$\frac{d}{dt}E(F(q(t),t)G(q(t),t))=E(FD_{(+)}G)+E(D_{(-)}FG)\ . \tag{24}$$

In fact, after a simple calculation we have

$$a_{ji}^{-}\rho_i+a_{ij}^{+}\rho_j=\delta_{ij}\sum_k a_{ik}^{+}\rho_k\ , \tag{25}$$

$$a_{ij}^{-}\leq 0\quad\text{for } i\neq j,\quad \sum_i a_{ij}^{-}(t)=0,\quad a_{ii}^{-}\geq 0\ , \tag{26}$$

$$\dot\rho_i(t)=\sum_j a_{ij}^{-}(t)\rho_j(t) \tag{27}$$

to be compared with (19) and (21).

Let us introduce

$$a_{ij}=\tfrac{1}{2}(a_{ij}^{+}+a_{ij}^{-}),\quad a_{ij}^{0}=\tfrac{1}{2}(a_{ij}^{+}-a_{ij}^{-}) \tag{28}$$

and note that for $i\neq j$

$$a_{ij}\rho_j=-a_{ji}\rho_i,\quad a_{ij}^{0}\rho_j=a_{ji}^{0}\rho_i,\quad a_{ij}^{0}\geq 0\ . \tag{29}$$

Moreover

$$\begin{aligned} &\dot\rho_i(t)=\sum_j a_{ij}(t)\rho_j(t)\ ,\\ &\sum_j a_{ij}^{0}(t)\rho_j(t)=0\ . \end{aligned} \tag{30}$$

In our strategy we consider a process $q(t)$ with initial distribution $\rho_i(t_0)$ and we let it evolve, according to (20), in the time interval $t_0\leq t\leq t_1$, assuming a^{+} as the basic controlling variable. Since (19) holds, we may consider only a_{ij}^{+} with $i\neq j$.

As an easy consequence of (20), (23), and (17) one immediately finds (see also Ref. 1)

$$D_{(+)}p(i_1,t_1;i,t)=0\ , \tag{31}$$

where $D_{(+)}$ acts on the (i,t) variables. Then, following the same line as in Ref. 1, it is very easy to prove the following formula giving the variation of p, as a consequence of (20), under variations $a^{+}\to a^{+}+\delta a^{+}$ of the controlling parameter:

$$\delta p(i_1,t_1;i_0,t_0)=\int_{t_0}^{t_1}\sum_{i,j}p(i_1,t_1;i,t)\delta a_{ij}^{+}(t)p(j,t;i_0,t_0)dt\ . \tag{32}$$

As a consequence of (32) and (17), the variation of ρ is given by

$$\delta\rho(i_1,t_1)=\int_{t_0}^{t_1}\sum_{i,j}{}'[p(i_1,t_1;i,t)-p(i_1,t_1;j,t)]\delta a_{ij}^{+}(t)\rho_j(t)dt+\sum_{i_0}p(i_1,t_1;i_0,t_0)\delta\rho_{i_0}(t_0)\ , \tag{33}$$

where $\sum'$ denotes a sum excluding terms with $i=j$ and also variations on the initial densities are allowed. It is also convenient to recall the transport equation for conditioned expectations

$$E(F(q(t_1),t_1)\,|\,q(t_0)=i_0)-F(i_0,t_0)$$

$$=\int_{t_0}^{t_1} E((D_{(+)}F)(q(t),t)\,|\,q(t_0)=i_0)dt\ , \quad (34)$$

an easy consequence of the definition (22).

For consideration of time-reversal-invariance properties it is also convenient to consider the time-inverted process q':

$$t\to t'=t,\ \ q(t)\to q'(t')\equiv q(t)\ . \quad (35)$$

Then one can easily show

$$a'^{+}_{ij}(t')=-a^{-}_{ij}(t)\ ,$$

$$a'_{ij}(t')=-a_{ij}(t)\ , \quad (36)$$

$$a'^{0}_{ij}(t')=a^{0}_{ij}(t)\ .$$

Therefore, the forward a^+, the backward a^-, the streaming a, and the osmotic a^0 behave exactly like their continuous counterparts v_+, v_-, v, and u in the continuous case (see Ref. 6).

Let us now relate the first equation in (30) with its analog (5). Note that in (5) there is no term in the sum with $j=i$, in fact, β is antisymmetric and $\beta_{ii}=0$. It is therefore convenient to write the first equation in (30) with a sum $\sum'$ excluding $j=i$. By exploiting $\sum_i a_{ij}=0$ [a consequence of (19), (26), and (28)] and (29) we have

$$a_{ii}\rho_i=-\sum_j{}' a_{ij}\rho_j\ , \quad (37)$$

$$\begin{aligned}\dot\rho_i&=\sum_j a_{ij}\rho_j\\ &=\sum_j{}' a_{ij}\rho_j+a_{ii}\rho_i\\ &=2\sum_j{}' a_{ij}\rho_j\ .\end{aligned} \quad (38)$$

Therefore, we can recognize

$$a_{ij}=(h_{ij}/\hbar)(\rho_i/\rho_j)^{1/2}\sin\beta_{ij},\ \ i\neq j\ . \quad (39)$$

On the other hand, the first equation in (19) tells us that

$$a^{+}_{ij}=a_{ij}+a^{0}_{ij}\geq 0,\ \ i\neq j\ . \quad (40)$$

In the continuous case the osmotic velocity is a function only of the density and not of the phase. This remark, together with (40) and (39), leads us to the natural consideration of processes for which the osmotic part a^0 has the form

$$a^{0}_{ij}=(h_{ij}/\hbar)(\rho_i/\rho_j)^{1/2},\ \ i\neq j \quad (41)$$

(compare also with Ref. 7).

The main purpose of this paper is to show that there are choices of stochastic action, such that (39), (41), and (6) are all consequences of the variational principle.

IV. DISCRETE STOCHASTIC VARIATIONAL PRINCIPLE

Let us consider a forward Lagrangian density $\mathscr{L}^{(+)}_i$, as a function only of a^{+}_{ij}, $i\neq j$, and its time-inverted one $\mathscr{L}^{(-)}_i$, such that for the expectations we have

$$L=E(\mathscr{L}^{(+)})=E(\mathscr{L}^{(-)})=E(\mathscr{L}) \quad (42)$$

for some time-reversal-invariant $\mathscr{L}$ function of a and a^0.

Let us assume a given initial density $\rho_i(t_0)$ and consider $a^{+}_{ij}(t)$, $i\neq j$ as controlling parameters in the time interval $t_0\leq t\leq t_1$. Introduce the stochastic action

$$A=A(t_0,t_1;\rho_0,a^+)=\int_{t_0}^{t_1} E(\mathscr{L}^{(+)})dt \quad (43)$$

and define

$$I_i(t)=-\int_{t_0}^{t_1}\sum_j \mathscr{L}^{(+)}_j(t')p(j,t';i,t)dt'\ , \quad (44)$$

so that

$$(D_{(+)}I)_i(t)=\mathscr{L}^{(+)}_i(t),\ \ I_i(t_1)=0\ , \quad (45)$$

$$A=-\sum_{i_0} I_{i_0}(t_0)\rho_{i_0}(t_0) \quad (46)$$

with obvious shorthand notations. Consider variations $\delta\rho(t_0)$, δa^+. Introduce an arbitrary function S^1 and define

$$S_i(t)=I_i(t)+\sum_{i_1} S^{1}_{i_1}p(i_1,t_1;i,t) \quad (47)$$

so that

$$(D_{(+)}S)_i(t)=\mathscr{L}^{(+)}_i(t),\ \ S_i(t_1)=S^1_i\ . \quad (48)$$

Note that in our shorthand notation we have suppressed the dependence of S on a^+ and S^1, which must be tacitly understood. A straightforward application of (34) and (33) allows us to find immediately the following expression for the variation of the action A defined in (43):

$$A=\int_{t_0}^{t_1}\sum_i\left[\delta\mathscr{L}^{(+)}_i(t)-\sum_j{}'[S_j(t)-S_i(t)]\delta a^{+}_{ji}(t)\right]\rho_i(t)dt+\sum_{i_1}S_{i_1}(t_1)\delta\rho_{i_1}(t_1)-\sum_{i_0}S_{i_0}(t_0)\delta\rho_{i_0}(t_0)\ . \quad (49)$$

The proof is standard and follows the same line as in the continuous case explained in Ref. 1. Note that (49) is completely insensible to the choice of S^1, but each single one of the three terms in the RHS of (49) does depend on S^1 (only the sum does not).

Let us now explore the consequences of a stationary action principle requiring $\delta A=0$ under the constraint that the variations $\delta\rho(t_0)$ and δa^+ are such that the two terms containing $\sum_{i_1}$ and $\sum_{i_0}$ in (49) balance each other. Since $\mathscr{L}^{(+)}$ is a function of a^+ let us introduce the generalized momentum p_+ through

$$\delta \mathscr{L}_k^{(+)}(t) = \sum_{j,i}{}' p_+(k,j,i;t)\delta a_{ji}^+(t) \ , \tag{50}$$

in analogy with $p_i = \partial\mathscr{L}/\partial q_i$ in classical mechanics. Then (49) gives the following condition for a critical process (i.e., in the case where the variational principle is satisfied):

$$\sum_k \frac{\rho_k(t)}{\rho_i(t)} p_+(k,j,i;t) = S_j(t) - S_i(t) \ . \tag{51}$$

By splitting symmetric and antisymmetric terms for $i \leftrightarrow j$ we get the osmotic formula

$$\sum_k \rho_k(t)[\rho_i{}^{-1}(t)p_+(k,j,i;t) + \rho_j{}^{-1}(t)p_+(k,i,j;t)] = 0 \tag{52}$$

and the current formula

$$\tfrac{1}{2}\sum_k \rho_k(t)[\rho_i{}^{-1}(t)p_+(k,j,i;t) - \rho_j{}^{-1}(t)p_+(k,i,j;t)] = S_j(t) - S_i(t) \tag{53}$$

[we assume $\rho_i(t) > 0$].

Formulas (52) and (53) are the discrete analogs of $u = \nu\nabla\rho/\rho$ and $p = \nabla S$, respectively, of the continuous case (compare with Ref. 1). It is important to remark that the osmotic formula (52) here is a consequence of the stochastic variational principle and not an independent assumption as in the continuous case.

The resulting programming equation is (48) where now a^+ is restricted by the conditions (52) and (53). Our objective is to find a particular form of $\mathscr{L}^{(+)}$ as a function of a^+, such that (52) and (53) are equivalent to (41) and (39), respectively, while (48) reduces to (6).

V. THE STOCHASTIC ACTION

Let us now consider the problem of finding a suitable form of $\mathscr{L}^{(+)}$ such that the resulting programming equation is (6). Our strategy will be the following. In a first stage we assume (6) and derive $\mathscr{L}^{(+)}$ as a function of a^+ by exploiting (39), (48), and (41). This will be done in this section. We will find that it is very convenient to introduce auxiliary parameters β_{ij} related to the parameters a^+. Then the Lagrangian also involves β_{ij} (which in fact should be considered as a function of a^+). The situation is very similar to the case when we try to introduce a Lagrangian theory for a system for which we have assumed a "configuration" space, such that the real canonical phase space is not its cotangent bundle. We give explicitly the case of the plane pendulum in order to show that in these cases the Lagrangian can be defined only locally on the tangent bundle and some additional auxiliary parameters are naturally involved.

In this way we find a candidate for the Lagrangian which surely agrees with the true Lagrangian on the physical orbits. Then in Sec. VI, we verify the highly nontrivial result that this candidate, assumed as the basic starting point, really does give rise to the correct equations through the variational principle. As a matter of fact, some nonstandard solutions will also be found.

Therefore, let us start from (6) and (48). Then we have

$$\begin{aligned}\mathscr{L}_i^{(+)}(t) &= (D_{(+)}S)_i(t) \\ &= \sum_j{}'(\hbar\beta_{ij} + \alpha_{ji})a_{ji}^+ \\ &\quad - \sum_j{}' h_{ij}(\rho_j/\rho_i)^{1/2}\cos\beta_{ij} - h_{ii} \ .\end{aligned} \tag{54}$$

Clearly h_{ij} and α_{ij} can be naturally considered as fixed parameters in the theory, playing a role analogous to the external fields. Therefore, we must find $(\rho_j/\rho_i)^{1/2}$ and β_{ij} as functions of a^+, because $\mathscr{L}^{(+)}$, by basic assumption, must be a function of only a^+. Let us consider (39), (41), and (28); then we have

$$\begin{aligned}a_{ij}^+ a_{ji}^+ &= a_{ij}a_{ji} + a_{ij}^0 a_{ji}^0 \\ &= (h_{ij}/\hbar)^2\cos^2\beta_{ij}, \quad i \neq j \ .\end{aligned} \tag{55}$$

This is the basic formula relating β with a^+. Note that, even if we assume antisymmetric β_{ij}, formula (55) will give us in general four values of β_{ij}, on the circle, for given a^+. But this is not an obstacle because we can consider the antisymmetric β_{ij} as controlling parameters also, constrained to change so that (55) is always preserved. Note that (39), (41), and (28) also imply

$$a_{ij}^+ = (h_{ij}/\hbar)(\rho_i/\rho_j)^{1/2}(1 + \sin\beta_{ij}) \ . \tag{56}$$

Therefore, we have

$$h_{ij}(\rho_j/\rho_i)^{1/2} = \hbar a_{ji}^+/(1 - \sin\beta_{ij}) \ . \tag{57}$$

In conclusion (54) can be written in the form

$$\mathscr{L}_i^{(+)} = \sum_j{}'(\hbar f(\beta_{ij}) + \alpha_{ji})a_{ji}^+ - h_{ii} \ , \tag{58}$$

where

$$\begin{aligned}f(\beta) &= \beta - \cos\beta/(1 - \sin\beta) \\ &= \beta - (1 + \sin\beta)/\cos\beta \ ,\end{aligned} \tag{59}$$

and $\beta_{ij} = -\beta_{ji}$ is connected to a^+ through (55), which is an essential part in the definition (58).

Note that, as far as the consequences of the variational principle are concerned, the local Lagrangian (58) falls in the class of those considered in Sec. IV. In fact the presence of the additional parameters β_{ij} is irrelevant. For each couple $i \neq j$, we could solve (55) for β_{ij} as a function of a^+ (by taking one particular specification among the generic four) and substitute it in (58), so that $\mathscr{L}^{(+)}$ would be the only function of a^+. For small variations of a^+ and small time intervals the specification does not change so that (58) can be considered as giving rise to functions of a^+, to be exploited in the variational principle according to the general method developed in Sec. IV.

Note that (55) gives a bound on the possible values of a_{ij}^+, a_{ji}^+. In fact, in all situations we must have

$$a_{ij}^+ a_{ji}^+ \leq (h_{ij}/\hbar)^2, \quad i \neq j \ . \tag{60}$$

Recall that h_{ij} are fixed external parameters given once for all.

Bounds on the controlling parameters are not a surprise. Consider, for example, the following extremely simple case in classical mechanics. Let θ be an angle specifying the configuration of the plane pendulum with Lagrangian and Hamiltonian dynamics given by

$$\mathscr{L}(\theta,\dot{\theta})=\frac{I\dot{\theta}^2}{2}+k\cos\theta,\quad p=\frac{\partial\mathscr{L}}{\partial\dot{\theta}}=I\dot{\theta}\ , \tag{61}$$

$$H=\frac{p^2}{2I}-k\cos\theta,\quad \dot{p}=-k\sin\theta,\quad \dot{\theta}=p/I\ . \tag{62}$$

The phase space of the system is given by the cotangent bundle of U(1), i.e., $\Gamma=\mathrm{U}(1)\times\mathrm{R}$, in fact, $\theta\in\mathrm{U}(1)$, $p\in\mathrm{R}$. Suppose we try to build a Lagrangian theory assuming p as the "configuration." We are obliged in this case to work only locally. In fact, as a consequence of (62), $\dot{p}$ is constrained by $|\dot{p}|\le k$ and the corresponding Lagrangian is

$$\mathscr{L}(p,\dot{p})=k\cos\theta-\theta\dot{p}-\frac{p^2}{2I}\ , \tag{63}$$

where θ is introduced through the second formula in (62), which is considered here as connecting the control parameter $\dot{p}$ with some auxiliary parameter θ as a consequence of the bound $|\dot{p}|\le k$, which allows one to define θ uniquely as a (multivalued) function of $\dot{p}$ in the form $\dot{p}=-k\sin\theta$. We see a perfect analogy with (60), (55), and (58).

Of course (58) is, by now, only a candidate for a stochastic Lagrangian. Section VI explores the consequences of the stochastic variational principle applied to this Lagrangian.

VI. CONSEQUENCES OF THE STOCHASTIC VARIATIONAL PRINCIPLE

First of all let us summarize the general kinematical frame. We consider Markov processes controlled by a_{ij}^+, $i\neq j$. These control parameters are restricted, so that for each couple of sites $i\neq j$, there exist constants $\omega_{ij}=\omega_{ji}$ for which

$$a_{ij}^+a_{ji}^+\le\omega_{ij}{}^2\ . \tag{64}$$

Introduce antisymmetric $\beta_{ij}=-\beta_{ji}$, $i\neq j$ such that

$$a_{ij}^+a_{ji}^+=\omega_{ij}{}^2\cos^2\beta_{ij}\ . \tag{65}$$

For given $\alpha_{ij}=-\alpha_{ji}$, h_{ii} introduce the local Lagrangian (58) and consider stochastic variational principles of the type considered in Sec. IV. Note that $\hbar$ appears in (58) for dimensional reasons, while the constants in (64) are related to those in (55) through $\omega_{ij}=h_{ij}/\hbar$. In fact h_{ij} have the dimensions of energy, ω_{ij} are frequencies, and $\hbar$ is an action.

We can consider β_{ij} as additional parameters on which $\mathscr{L}^+$ depends, according to (58), but the variations of β_{ij} are restrained by (65) to the form

$$\frac{\delta a_{ji}^+}{a_{ji}^+}+\frac{\delta a_{ij}^+}{a_{ij}^+}=-2\tan\beta_{ij}\delta\beta_{ij}\ . \tag{66}$$

Through a long but straightforward calculation we can evaluate $E(\delta\mathscr{L}^{(+)})$, to be inserted in (49), in the form

$$E(\delta\mathscr{L}^{(+)})=\sum_{i,j}{}'\left[\hbar\beta_{ij}+\alpha_{ji}-\frac{\hbar(S_{ij}+A_{ij})}{\cos\beta_{ij}a_{ij}^+a_{ji}^+}\right]\delta a_{ji}^+\rho_i\ , \tag{67}$$

where the symmetric S_{ij} and the antisymmetric A_{ij} are given by

$$S_{ij}=a_{ji}(a_{ij}-a_{ij}^0\sin\beta_{ij}),\quad i\neq j\ , \tag{68}$$

$$A_{ij}=-a_{ji}^0(a_{ij}-a_{ij}^0\sin\beta_{ij}),\quad i\neq j\ . \tag{69}$$

In the derivation of (67), formulas (25), (28), and (29) play an essential role.

Let us now insert (67) in (49) and impose $\delta A=0$ under the usual constraints on the boundaries. As a consequence, formulas (52) and (53) assume the forms

$$\begin{aligned}&S_{ij}=0\ ,\\&\hbar\beta_{ij}+\alpha_{ji}-\frac{\hbar A_{ij}}{\cos\beta_{ij}a_{ij}^+a_{ji}^+}=S_j(t)-S_i(t)\ .\end{aligned} \tag{70}$$

Let us explore the consequences of $S_{ij}=0$. By taking into account (68) we must have either

$$a_{ij}=a_{ij}^0\sin\beta_{ij}\quad\text{(standard solution)} \tag{71}$$

or

$$a_{ij}=0\quad\text{(nonstandard solution)}\ . \tag{72}$$

Here we consider the standard case.

From (71) and (29) we derive

$$\begin{aligned}a_{ij}a_{ji}&=-a_{ij}^0a_{ji}^0\sin^2\beta_{ij}\\&=a_{ij}^0a_{ji}^0(\cos^2\beta_{ij}-1)\ .\end{aligned} \tag{73}$$

Therefore, taking into account the purely kinematical first equality in (55), we get

$$a_{ij}^0a_{ji}^0=(h_{ij}/\hbar)^2\ , \tag{74}$$

and finally, still exploiting (29), we find for a_{ij}^0 the expression (41).

Moreover (71) also implies $A_{ij}=0$, therefore (70) reproduces (7) and (71) is identical to (49). On the other hand it is also immediately seen that the transport equation (48) is equivalent to (6) in this case.

Therefore, we have one of the main results of this paper: *Stochastic variational principles, based on the Lagrangian (58), reproduce the osmotic expression (41) and the correct continuity equation (5) and Hamilton-Jacobi equation (6) of the Schrödinger theory for a discrete quantum system.* Nonstandard solutions will be investigated in Sec. VIII.

VII. DISCRETE APPROXIMATION OF CONTINUOUS SYSTEM

As an essential check of the consistency of our theory it is necessary to investigate the limiting behavior of discrete systems, considered as approximations for continuous systems. Since in the continuous case the procedure of stochastic quantization is well known,[2,1,6] it is important to verify that the theory outlined in this paper, for the discrete case, has the correct limit.

Let us consider a quantum system on R^d, with the Schrödinger equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \ . \qquad (75)$$

Assume that R^d is discretized to ϵZ^d, the square lattice of spacing ϵ. Then $\psi(x,t)$ is replaced by $\psi_n(t)$, $n\in Z^d$, with

$$\sum_n |\psi_n|^2 = \sum_n \rho_n = 1 \ , \qquad (76)$$

and (75) can be approximated by exploiting a finite-difference expression for the Laplacian

$$ih\dot{\psi}_n = \sum_{n'} H_{nn'}\psi_{n'} \ , \qquad (77)$$

$$H_{nn'} = -\hbar^2/2m\epsilon^2 \ , \quad |n-n'|=1 \ , \quad H_{nn} = V_n + 2d\hbar^2/2m\epsilon^2 \ . \qquad (78)$$

We may relate to the general scheme exploited in this paper by assuming n as the discrete index labeling sites and putting

$$h_{nn'} = \hbar^2/2m\epsilon^2, \quad \alpha_{nn'} = \pi\hbar \ , \quad |n-n'|=1, \quad h_{nn} = H_{nn} \ . \qquad (79)$$

With these notations the discrete continuity equations are

$$\dot{\rho}_n = 2\sum_{n'}{}' a_{nn'}\rho_{n'} \ , \qquad (80)$$

$$a_{n'n} = \frac{\hbar}{2m\epsilon^2}\left[\frac{\rho_{n'}}{\rho_n}\right]^{1/2}\sin\frac{S_{n'}-S_n}{\hbar} \ , \qquad (81)$$

where $\sum'$ is restricted to $|n'-n|=1$.

Consider that $(S_{n'}-S_n)/\epsilon$ is the lattice approximation for the gradient ∇S. Therefore, we introduce the vector field $v(n',n)$ on the lattice, defined by

$$v(n',n) = 2\epsilon a_{n'n}(\rho_n/\rho_{n'})^{1/2} \ . \qquad (82)$$

Note that the RHS is antisymmetric as a consequence of (29). Clearly $v(n',n)$ acts as a lattice approximation for the current velocity field $v=\nabla S/m$, as (81) and (82) show. Then we can write (80) in the form

$$\begin{aligned}\dot{\rho}_n &= -\epsilon^{-1}\sum_{n'}{}'(\rho_n\rho_{n'})^{1/2}v(n',n)\\ &= -\rho_n{}^{1/2}\sum_{n'}{}'\epsilon^{-1}(\rho_{n'}{}^{1/2}-\rho_n{}^{1/2})v(n',n)\\ &\quad -\rho_n\sum_{n'}{}'\epsilon^{-1}v(n',n) \ . \end{aligned} \qquad (83)$$

Let us introduce the notation $\simeq$ to denote lattice approximations. Then we have

$$\epsilon^{-1}\sum_{n'}{}'v(n',n)\simeq\nabla\cdot v \ , \quad \epsilon^{-1}(\rho_{n'}{}^{1/2}-\rho_n{}^{1/2})\simeq\nabla\rho^{1/2} \ . \qquad (84)$$

Note also

$$v(n',n)\simeq v, \quad u(n',n)\simeq u \implies \sum_{n'}{}'v(n',n)u(n',n)\simeq 2v\cdot u \ . \qquad (85)$$

The factor 2 is necessary because there is a doubling $\sum'$ with respect to all directions. Collecting (85) and (84) we see that (83) reduces to

$$\dot{\rho}_n \simeq -2\rho^{1/2}v\cdot\nabla\rho^{1/2} - \rho\nabla\cdot v = -\nabla\cdot(\rho v) \ . \qquad (86)$$

Therefore, we see that the discrete continuity equations (80) and (81) reproduce correctly the continuous one

$$\partial_t\rho = -\nabla\cdot(\rho v), \quad v=\nabla S/m \ . \qquad (87)$$

In the same way one can show that (6) reduces in this case to the Hamilton-Jacobi-Madelung equation

$$\partial_t S + \frac{1}{2m}(\nabla S)^2 - \frac{(\hbar^2/2m)\Delta\rho^{1/2}}{\rho^{1/2}} + V = 0 \qquad (88)$$

by exploiting $\cos\alpha\simeq 1-\alpha^2/2$ for small α.

Let us now consider the limiting behavior of the discrete process associated to the quantum system according to the procedure outlined in the previous sections. We find it convenient to work with the forward derivative, which summarizes all random properties of the process. In this case, from (23) we have

$$(D_{(+)}F)_n(t) = \partial_t F_n + \sum_{n'}{}'(F_{n'}-F_n)a^+_{n'n} \ , \qquad (89)$$

where the second equation in (19) has been exploited in order to write $\sum'$ instead of $\sum$. Now we have

$$a^+_{n'n} = \frac{\hbar}{2m\epsilon^2}\left[\frac{\rho_{n'}}{\rho_n}\right]^{1/2}\left[1+\sin\frac{S_{n'}-S_n}{\hbar}\right] \ , \qquad (90)$$

where

$$\frac{S_{n'}-S_n}{\hbar}\simeq v(n',n) \ . \qquad (91)$$

If we write

$$(\rho_{n'}/\rho_n)^{1/2} = \rho_n{}^{1/2}(\rho_{n'}{}^{1/2}-\rho_n{}^{1/2})+1 \qquad (92)$$

we see immediately that (89) is the lattice approximation for

$$(D_{(+)}F)(x,t) = \partial_t F + v_+\cdot\nabla F + \frac{\hbar}{2m}\Delta F \ , \qquad (93)$$

$$v_+ = \nabla S/m + (\hbar/m)\nabla\rho^{1/2}/\rho^{1/2} \ . \qquad (94)$$

It is amusing to see through the splitting (92) that the osmotic term (41), in the continuous limit, gives both a contribution to v_+ and the reason for the appearance of the Laplace term in (93). Since (93) is the correct forward derivative for the continuous process (see Refs. 2, 1, and 6) we have given evidence that the correct limit is reproduced by the discrete stochastic process introduced in previous sections of this paper.

It is important to remark explicitly that, in the continuous case, the osmotic part, giving rise to the correction from v to v_+ and to the Laplacian in (93), must be postulated. On the other hand, in the discrete case the osmotic part is a consequence of the variational principle. Moreover it gives rise to the correct expression (93) in the limit.

Thus, we may conclude that the whole theory can be founded on the stochastic variational principle alone.

VIII. NONSTANDARD SOLUTIONS FOR STOCHASTIC VARIATIONAL PRINCIPLES

Let us now turn to the nonstandard case (72). Now we have $\dot{\rho}_i=0$ as a consequence of (38), i.e., the occupation probabilities stay constant in time on the average. On the other hand (55) gives

$$a_{ij}^0 a_{ji}^0=(h_{ij}/\hbar)^2\cos^2\beta_{ij} \ , \tag{95}$$

to be compared with (74). Therefore, from (29) we have

$$a_{ij}^0=(h_{ij}/\hbar)(\rho_i/\rho_j)^{1/2}\,|\cos\beta_{ij}\,| \tag{96}$$

instead of (41). In this case the second equation in (70) reduces to

$$\hbar(\beta_{ij}-\tan\beta_{ij})+\alpha_{ji}=S_j-S_i \ , \tag{97}$$

which is deeply different from (7). It is now easy to calculate $\dot{S}_i$ from (48). We find

$$\dot{S}_i=-\sum_j{}'\,\eta_{ij}(\rho_j/\rho_i)^{1/2}-h_{ii}=-E_i \ , \tag{98}$$

where η_{ij} is the sign of $\cos\beta_{ij}$ and ρ's are constant. Let us start from a situation at $t=t_0$ with a given sign for η_{ij}. Because of continuity it stays constant for t in some time interval around t_0 (a fact we will find for any t). In the generic case $E_i\neq E_j$, $i\neq j$. Thus, we see that $S_i-S_j\to\pm\infty$ as $t\to\infty$; therefore, $\beta_{ij}-\tan\beta_{ij}\to\pm\infty$, and β never crosses values where $\cos\beta$ changes sign, so that E_i stay constant. Necessarily we have $\cos\beta_{ij}\to 0$. As $\cos\beta_{ij}\to 0$, $a_{ij}^0\to 0$ and $a_{ij}^+\to 0$ also (because a_{ij} is already zero). Therefore, we see that in the nonstandard generic case as $t\to\infty$ the diffusion disappears completely in the limit, and the process reduces to a mixture of static distributions at each site.

Clearly we can choose the standard solution for all couples (i,j) belonging to a subset A of sites or to the complement A' and the nonstandard solution for all couples coupling A and A'. In this case, when $t\to\infty$, the process will not make jumps between sites of A and sites of A' and will reduce to a mixture of standard processes each jumping around in A and A', respectively. Thus, we see that nonstandard solutions tend to reproduce a behavior very similar to quantum measurement. It is the decay to zero of a_{ij}^+ which destroys quantum coherence between the sites i,j and forces the system to reproduce mixtures of states in the limit.

Therefore, we have the second main result of this paper: *Starting from a single stochastic variational principle we can also simulate the formation of mixtures in analogy with the quantum-measurement operations.*

Compared to the standard theory of quantum measurement (see, for example, Ref. 8, but compare also with Ref. 9), here the wave-packet collapse (mixture formation) is not instantaneous, but evolves according to (96)–(98). Note that the speed of mixture formation depends on the differences E_i-E_j. The larger they are, the faster is the progression to collapse. In order to see whether the phenomenon of nonstandard solutions found here may really have some physical implication for the quantum-measurement theory, it would be necessary to investigate the possibility of checking, at the experimental level, that the wave-packet collapse is not instantaneous. We plan to report on this problem in a future paper.

IX. CONCLUSION AND OUTLOOK

We have shown that it is possible to choose a stochastic Lagrangian for controlled diffusions on discrete configuration space, so that the whole structure of quantum-mechanical behavior is correctly simulated. In particular, we reproduce the right forms of the continuity equation and the discrete Hamilton-Jacobi–type equation. Moreover the stochastic variational principle also gives the expression for the osmotic (time-reversal-symmetric) part of the transition probabilities per unit time. This expression reduces to the standard one of stochastic mechanics in the continuous limit. A peculiar feature of this scheme is the existence of critical processes simulating the quantum-measurement wave-packet collapse and producing mixtures in the time-asymptotic region.

Any quantum system can be put in the form considered in this paper by choosing an appropriate complete set of compatible observables. In the stochastic framework these observables evolve according to a probabilistic scheme, whose foundation rests on variational principles, involving suitably averaged actions. Therefore, the scheme outlined here is completely general and can be applied to a large variety of cases, which were beyond the reach of the previous formulation of stochastic mechanics, as, for example, discrete spin systems or Fermi-Dirac fields.

On the other hand a physical justification for the assumed form of the stochastic Lagrangian seems to be very difficult and surely involves new ideas about the origin of the underlying Brownian motion in stochastic mechanics; it is clear that the Lagrangian is never unique. Therefore, as a preliminary step it would be interesting to find other forms, which reduces to that found here along the orbits of the stochastic dynamical system.

Also, a more detailed investigation on the bifurcation leading to the standard and nonstandard solutions seems to be worth considering. We have considered this problem for the simplest nontrivial case of random processes on the two-sites set Z_2 and plan to report this in a forthcoming note.[10]

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