On various joint measurements of position and momentum observables in quantum theory

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(Received 27 September 1983)

In this paper the problem of simultaneous measurability of complementary physical quantities, and the relevance of the uncertainty relations to this question, is analyzed. Formalizing carefully the crucial notations involved in the problem it is shown that the well-known apparently contradictory viewpoints on the subject matter are all true in pointing to a different aspect of one and the same fundamental fact expressed here as complementarity.

I. INTRODUCTION

The problem of joint measurement (simultaneous measurability) of complementary physical quantities, and the relevance of the uncertainty principle to this question, is one of the central issues on the foundations of quantum theory. This problem has a long history which begins with the discovery of the fundamental "exchange relation" $QP-PQ=(ih/2\pi)I$ and the uncertainty relations $\Delta Q\Delta P\geq h/4\pi$. In the relevant literature one may distinguish between three basically different viewpoints on that problem. They might be called *the standard view, the first extreme view,* and *the second extreme view.* According to the standard view *complementary physical quantities* (like position Q and momentum P) *can simultaneously be measured only to the accuracy allowed by the* (relevant) *uncertainty relations* (like $\Delta Q\Delta P\geq h/4\pi$). This rather common view belongs to the conceptual core of the so-called Copenhagen interpretation of quantum theory, and it can be traced back to Heisenberg.[1] The first extreme view says that *complementary physical quantities cannot simultaneously be measured at all.* Such a view, which is compatible with the intuitive notion of complementary physical quantities as developed by Pauli[2] and Bohr,[3] has, in particular, been supported by the nonexistence theorems for the relevant joint probability distributions. For example, Suppes[4] (see p. 385) argued that "the conclusion that momentum and position are not simultaneously measurable at all does not follow from the Heisenberg relation but from the more fundamental results about the absence of a genuine joint distribution." (See also Jauch.[5]) The second extreme view claims also that the uncertainty principle is irrelevant to the problem of simultaneous measurability of complementary physical quantities, but that *such quantities can simultaneously be measured within any accuracy.* A prominent advocate of this view is Margenau, with his co-workers, who paid much attention to provide "an empirical method for the simultaneous measurement of Q and P" through the so-called "time-of-flight" method (Park and Margenau,[6] p. 239ff; see also Ballentine[7]).

We do not aim to review the rather extensive and divergent discussion on the problem of joint measurability of complementary physical quantities, and the importance of the uncertainty principle to this problem. Rather, we wish to analyze the problem with respect to some recent theoretical developments. We carry out our analyses within the so-called convexity approach which is flexible enough to allow one to formulate the relevant (intuitive) notions involved in the above three different viewpoints. With respect to the proposed formalizations of the crucial notions the apparent contradiction between the standard view and the two extreme views will be resolved. Moreover, each view will be seen to be true in pointing to a different aspect of one and the same fundamental fact—*complementarity,* in the sense defined below.

II. ON THE NOTION OF MEASUREMENT

In this preliminary section we shall formalize the notion of measurement needed in analyzing the simultaneous measurability of complementary physical quantities. In the approach followed here the description of a physical system is based on its set of states. Within such an approach a measurement on the system is most properly described through transformation of states of the system caused by the measurement performed on it. Peculiar features of measurements can then be analyzed and characterized through different properties of the associated transformations of states. Although the problem on simultaneous measurability of complementary physical quantities will be analyzed within the Hilbert state space we find it advisable to introduce the basic notions and terminology within a more general frame, the convexity frame. Moreover, we believe that the problem may be formulated independently of the Hilbert-space frame of quantum theory.

A. The convexity frame

In the convexity approach the description of a physical system is exclusively based on its set of *states,* and on the distinction between the *pure* and the *mixed* states. The basic assumption of this approach is conveniently summarized as follows: The set of states of a physical system is represented by a norm closed generating cone $\underline{V}^+$ for a complete base norm space $(\underline{V},\underline{B})$ (see Mielnik,[8] Davies and Lewis,[9] Edwards,[10] and Davies[11]). Note that $\underline{V}=\underline{V}^+-\underline{V}^+=\bigcup(\lambda\underline{B}:\ \lambda\geq 0)-\bigcup(\lambda\underline{B}:\ \lambda\geq 0)$, and $\underline{B}=\{\alpha\in\underline{V}^+:\ e(\alpha)=1\}$, where $e:\underline{V}\rightarrow\underline{R}$ is the strictly positive linear functional associated with the base $\underline{B}$ of $\underline{V}$. $\underline{V}$ is partially ordered by the cone $\underline{V}^+$: for any α,β in $\underline{V}$, $\alpha\leq\beta$ if and only if (iff) $\beta-\alpha\in\underline{V}^+$. The functional $e:\underline{V}\rightarrow\underline{R}$ is occasionally called the strength functional, and the set $\underline{B}$ can be recognized as the set of *normalized states.* Owing to the convex structure of $\underline{B}$ the distinction between the pure states, the extreme elements of $\underline{B}$, and the mixed states, the nonextreme elements of $\underline{B}$, can be made. $\mathrm{Ex}(\underline{B})$ denotes the set of pure states in $\underline{B}$.

The basic idea of the approach is that any change on the system like those caused by measurements on it or those associated with its evolution, can be described through transformations of states of the system. To this end the important notion of *operation* (state transformation) is introduced. It is assumed that an operation, when performed on the system, will change a given initial state into a well-defined final state, it does not increase the strength of any state, and it acts additively and homogeneously on states. Formally, an operation is thus defined as a positive, norm-nonincreasing, linear mapping $\phi:\underline{V}\rightarrow\underline{V}$, $\alpha\mapsto\phi\alpha$, and the set $\underline{O}(\underline{V})$ of all (formally possible) operations on the space $\underline{V}$ is represented as the set of all positive elements in the unit ball of $L(\underline{V})$, the space of bounded linear operations on $\underline{V}$, equipped with the strong operator topology.

The set $\underline{O}(\underline{V})$ is a semigroup with zero 0 and unit I, allowing one thus to perform sequences of operations on the system. Further, it is naturally ordered as follows: for any ϕ_1 and ϕ_2 in $\underline{O}(\underline{V})$, $\phi_1\leq\phi_2$ iff $(\phi_2-\phi_1)(\alpha)\in\underline{V}^+$ for any $\alpha\in\underline{V}^+$. Note, in particular, that any operation ϕ in $\underline{O}(\underline{V})$ with the property $e(\phi\alpha)=e(\alpha)$ for any α in $\underline{B}$ is maximal with respect to that order.

Any operation ϕ leads to a detectable *effect* when combined with detecting the strength of a state after it has undergone the operation ϕ. Thus, for any ϕ in $\underline{O}(\underline{V})$ there is associated its detectable effect, denoted as $e\circ\phi$, which is a positive linear functional on $\underline{V}$ with $0\leq e\circ\phi\leq e$. On the other hand, for any positive linear functional a on $\underline{V}$ with $0\leq a\leq e$ there is associated an operation ϕ_a in $\underline{O}(\underline{V})$ whose associated effect $e\circ\phi_a$ equals a: fix any $\beta\in\underline{B}$ and define $\phi_a:\alpha\rightarrow\phi_a\alpha=a(\alpha)\beta$. Thus the set of all (formally possible) effects of the physical system is represented by the set $\underline{E}(\underline{V})$ of elements a in the dual space $(\underline{V}^*,e)$ of $(\underline{V},\underline{B})$ satisfying $0\leq a\leq e$, where the ordering is defined by the dual cone $\underline{V}^{+*}$ of $\underline{V}^+$.

The set $\underline{E}(\underline{V})$ of effects is naturally ordered as follows: for any a, b in $\underline{E}(\underline{V})$, $a\leq b$ iff $(b-a)(\alpha)\geq 0$ for any $\alpha\in\underline{V}^+$. $(\underline{E}(\underline{V}),\leq)$ is a bounded poset with bounds 0 and e. Moreover, $\underline{E}(\underline{V})$ is closed under the mapping $a\rightarrow a^{\perp}\equiv e-a$, which has the properties $(a^{\perp})^{\perp}=a$; if $a\leq b$ then $b^{\perp}\leq a^{\perp}$. In general, however, $a\rightarrow a^{\perp}$ is not an orthocomplementation as $a\wedge a^{\perp}=0$ does not necessarily hold in $\underline{E}(\underline{V})$.

Two more basic notions of the approach are *instruments* and *observables.* An instrument corresponds to an experimental arrangement, defining thus a regular family of operations which can be performed on the system with the arrangement. Thus an instrument $\underline{I}$ is defined as a map from the Borel sets $\underline{B}(\underline{R})$ of the real line $\underline{R}$ into the set of operations which satisfies (i) $e(\underline{I}(\underline{R})v)=e(v)$ for any $v\in\underline{V}$; (ii) for any countable family (X_i) of pairwise disjoint sets in $\underline{B}(\underline{R})$, $\underline{I}(\bigcup X_i)=\sum\underline{I}(X_i)$ where the sum converges in the strong operator topology. To each instrument there is associated an observable corresponding to the family of the detectable effects of the operations performable with the instrument. Thus an observable A is defined as an effect-valued measure on the real Borel space $(\underline{R},\underline{B}(\underline{R}))$ with the properties (i) $A(\underline{R})=e$; (ii) for any countable family (X_i) of pairwise disjoint sets in $\underline{B}(\underline{R})$ $A(\bigcup X_i)=\sum A(X_i)$ where the sum converges in the weak $*$-topology of $\underline{V}^*$.

The above ingredients constitute the convexity or $(\underline{V},\underline{B})$ scheme for describing a physical system. The Hilbert realization of this scheme is provided by the usual Hilbert state space $(T_s(\underline{H}),T_s(\underline{H})_1^+)$, where $T_s(\underline{H})$ denotes the set of all self-adjoint trace class operators on the underlying Hilbert space $\underline{H}$. The strength functional associated with the base $T_s(\underline{H})_1^+$ is the usual trace functional. Recall, in particular, that the set $Ex(T_s(\underline{H})_1^+)$ of pure states of the description consists exactly of the one-dimensional projections $|\varphi\rangle\langle\varphi|$ on $\underline{H}$, which thus can be identified, modulo complex multipliers of absolute value one, with unit vectors (vector states) φ of $\underline{H}$. $\underline{H}_1$ denotes the set of all unit vectors in $\underline{H}$. The set $\underline{O}(\underline{H})$ of all operations on the Hilbert state space $(T_s(\underline{H}),T_s(\underline{H})_1^+)$ is the set $L(T_s(\underline{H}))_{\leq 1}^+$. Hence the set $\underline{E}(\underline{H})$ of all effects is the set $\{\mathrm{tr}[\phi\cdot]:\ \phi\in\underline{O}(H)\}$, which, due to the duality $T_s(\underline{H})^*\simeq L_s(\underline{H})$, can now be identified with the set $\{E\in L_s(\underline{H}):\ 0\leq E\leq 1\}$. Thus, in particular, the set $Ex(\underline{E}(\underline{H}))$ of extreme effects shall be identified with the set $\underline{P}(\underline{H})=\{P\in L_s(\underline{H}):\ P=P^+=P^2\}$ of projections on $\underline{H}$. The effect-valued measures, i.e., the observables of the description, are exactly the semispectral measures.

Standard Hilbert-space quantum mechanics operates also with the state space $(T_s(\underline{H}),T_s(\underline{H})_1^+)$, but considers only such observables which are represented as spectral measures (projection-valued measures). *A fortiori*, the standard frame considers only such operations which give rise to projections as their effects, i.e., $\underline{O}_s(\underline{H})=\{\phi\in\underline{O}(\underline{H}):\ \mathrm{tr}[\phi\cdot]\in\underline{P}(H)\}$. Thus $\underline{E}_s(\underline{H})=Ex(\underline{E}(\underline{H}))=\underline{P}(\underline{H})$. In particular, the von Neumann-Lüders operations $\phi_P:T_s(\underline{H})\rightarrow T_s(\underline{H})$, $\alpha\mapsto\phi_P\alpha\equiv P\alpha P$, $P\in\underline{P}(\underline{H})$, are such. They correspond to the so-called pure, ideal, first-kind measurements. In Sec. V we shall meet an important argument which leads to an extension of the standard Hilbert-space quantum mechanics in the lines described above.

B. On various kinds of measurements

According to the basic idea of the convexity approach, a measurement on the system is described through a transformation of the states of the system $\alpha_i\rightarrow\alpha_f$, where the postmeasurement (final) state α_f, depends, in general, on the premeasurement (initial) state α_i, on the measuring result, and on the measuring instrument employed. Thus, within the present approach, we shall consider only such measurements on the system which can be described, to their essence, as operations on the state space $(\underline{V},\underline{B})$ of the system. This agrees with the general ideas on the quantum theory of measurement (see, e.g., Beltrametti and Cassinelli,[12]). Moreover, the results of Kraus[13] show that each Hilbert operation admits a measurement theoretical interpretation.

We shall now specify some measurement theoretical notions which will be needed subsequently.

Let $(\underline{V},\underline{B})$ be the state space of a physical system under consideration. We recall first that there is a many-to-one correspondence between the sets $\underline{O}(\underline{V})$ of operations and $\underline{E}(\underline{V})$ of effects, $\Psi:\underline{O}(\underline{V})\to\underline{E}(\underline{V})$, $\phi\mapsto\Psi(\phi)$, $\Psi(\phi)(\alpha)=e(\phi\alpha)$, $\alpha\in\underline{B}$. The set $\Psi^{-1}(a)\equiv\{\phi\in\underline{O}(\underline{V})\colon e\circ\phi=a\}$ contains exactly those operations ϕ in $\underline{O}(\underline{V})$ which uniquely define the effect a. For any effect $a\in\underline{E}(\underline{V})$, the *certainly-yes-domain* $a^1\equiv\{\alpha\in\underline{B}\colon a(\alpha)=1\}$ may or may not be empty. If a^1 is not empty the effect a can be *actualized,* or realized, with preparing the system in any of the states in a^1. The notion of *state preparation* can be described as a constant operation $P_\alpha:x\mapsto P_\alpha x=\alpha$. State preparations are trivially idempotent. For any state $\alpha\in\underline{B}$, the certainly-yes-domain is $\alpha^1\equiv\{a\in\underline{E}(\underline{V})\colon a(\alpha)=1\}$.

A *measurement of an effect a* is any operation ϕ in $\underline{O}(\underline{V})$ which provides (probabilistic) information on a. Thus any ϕ in $\bigcup(\Psi^{-1}(c)\colon c\leq a)$ is a measurement of a. A measurement thus always is a measurement of some effect. For any $c\leq a$, $c^1\subset a^1$, so that $a^1=\varnothing$ implies $c^1=\varnothing$ for any $c\leq a$. Any ϕ in $\Psi^{-1}(a)$ is, in particular, a measurement of the effect a. They uniquely define a. Moreover, such measurements of the effect a are distinguished by their minimal informative content, i.e., they are measurements of a which do not provide any (probabilistic) information beyond a. We call them *minimal* measurements of the effect a. In the standard Hilbert space frame of the quantum theory the minimal measurements are just the Lüders-type measurements, whereas the von Neumann-type measurements have been called maximal measurements (cf. Süssmann[14]).

A *joint measurement of two effects a and b* is any measurement which provides (probabilistic) information on both of the two effects. Thus, any operation ϕ in $\bigcup(\Psi^{-1}(c)\colon c\leq a, c\leq b)$ is a joint measurement of a and b. For any $c\leq a, c\leq b$ it is $c^1\subset a^1\cap b^1$, so that $a^1\bigcap b^1=\varnothing$ implies $c^1=\varnothing$. For any $a,b\in\underline{E}(\underline{V})$, if their meet (the greatest lower bound) $a\wedge b$ exists in $\underline{E}(\underline{V})$, then the measurements in $\Psi^{-1}(a\wedge b)$ are minimal joint measurements of a and b, i.e., they provide no information beyond the two effects a and b.

The main purpose of measurements in physics is to provide information about the physical system under consideration, or, more specifically, to determine the state of the system. To be precise, we have to distinguish two possibilities: measuring results may have significance either to the system *before* or *after* the measuring act. Whereas measurements are always determinative (see below) they may, or may not, be preparatory. One cannot decide *a priori* whether or not the measuring act has or has not any influence on the observed system—quantum theory tells that in general it does have; only if the premeasurement state α belongs to the certainly-yes-domain of the measured effect a it is possible to register the result "yes" without "disturbing" the system. We shall now formulate some distinctive features of measurements relevant to the subsequent discussion.

By definition, any measurement ϕ of an effect a is *determinative,* i.e., with performing the measurement ϕ in a state α we get probabilistic information on the effect a in the premeasurement state α. The statistics of the measurement results of a sufficiently large class of effects (of a complete set of observables) allows one to determine the prepared, initial, state.

A measurement ϕ is *predictable,* if there exists a state α in $\underline{B}$ such that $e(\phi\alpha)=1$. The resulting effect $e\circ\phi$ is actualizable as $(e\circ\phi)^1\neq\varnothing$. A state preparation P_α is a predictable measurement for those effects which belong to the certainly-yes-domain of the prepared state α.

A measurement ϕ is *preparatory* (or weakly repeatable) iff $e(\phi^2\alpha)=e(\phi\alpha)$ for any state α. If an effect a admits a preparatory measurement then a is actualizable. A measurement ϕ is of the *first kind* (or strongly repeatable) iff it is preparatory and $e(\phi\alpha)=e(\alpha)$ implies $\phi\alpha=\alpha$ for any state α.

We recall that in the Hilbert case, an effect E is actualizable iff E has the eigenvalue 1 iff E admits a preparatory measurement iff E admits a first-kind measurement.

A measurement ϕ is *probabilistically preparatory* iff $e(\phi^2\alpha)\cdot e(\phi\alpha)^{-1}\geq e(\phi\alpha)$ for all states α, that is, the measured effect $e\circ\phi$ is more probable in the (normalized) postmeasurement state $\phi\alpha/e(\phi\alpha)$ as compared with the premeasurement state α. This seems to be the weakest possible kind of "preparatory" influence through measurements on the system. In the Hilbert framework it can be realized for any effect E in several ways, e.g., by $\phi\alpha\equiv E^{1/2}\alpha E^{1/2}$. Clearly any preparatory measurement is also probabilistically preparatory.

III. COMPLEMENTARY PHYSICAL QUANTITIES

Physical quantities are complementary, if the experimental arrangements which permit their unambiguous definitions are mutually exclusive (see, e.g., Bohr,[3] or Pauli[2]). We shall now formulate this intuitive idea within the Hilbert frame of quantum theory.

In the standard Hilbert-space formulation of quantum theory observables, physical quantities, are represented (and identified) as self-adjoint operators $A:\mathrm{dom}(A)\to\underline{H}$, $\varphi\to A\varphi$, or, equivalently, as spectral measures $A:\underline{B}(\underline{R})\to\underline{P}(\underline{H})$, $X\mapsto A(X)$ on the relevant Hilbert space $\underline{H}$. For each observable $A:\underline{B}(\underline{R})\to\underline{P}(\underline{H})$, $X\mapsto A(X)$ there is associated at least one instrument (operation-valued measure) $\underline{I}^A:\underline{B}(\underline{R})\to\underline{O}(\underline{H})$, $X\mapsto\underline{I}^A(X)$ which uniquely defines this observable, through the relation $\mathrm{tr}[\underline{I}^A(X)\alpha]=\mathrm{tr}[\alpha A(X)]$ for all $X\in\underline{B}(\underline{R})$, $\alpha\in T_s(\underline{H})_1^+$. Through its operations (state transformations) such an instrument characterizes an experimental arrangement which can be used to measure all the possible values of the observable, or which serves to define it unambiguously. Actually, for each observable A there is associated a (unique) family of instruments $\underline{I}_i^A$, $i \in I$ a suitable index set, which contains all the possible A measurements, i.e., A measurements which are describable within the theory as state transformations. We shall recall that in the family $\underline{I}_i^A$, $i \in I$, there is a special instrument which consists of the von Neumann-Lüders operations $T_s(\underline{H}) \to T_s(\underline{H})$, $\alpha \mapsto A(X)\alpha A(X)$, $X \in \underline{B}(\underline{R})$, associated with the ideal first-kind measurements of the (standard) observables $\underline{A}:\underline{B}(\underline{R}) \to \underline{P}(\underline{H})$.

Let $\underline{I}_i^A$, $i \in I(A)$, and $\underline{I}_j^B$, $j \in I(B)$, be any two instruments associated with the observables A and B, respectively. The operations $\underline{I}_i^A(X)$ and $\underline{I}_j^B(Y)$, $X, Y \in \underline{B}(\underline{R})$, describe some particular measurements of the two properties $A(X)$ and $B(Y)$ of the system. If there exists an operation $\phi \in \underline{O}(\underline{H})$, $\phi \neq 0$, which is contained both in $\underline{I}_i^A(X)$ and $\underline{I}_j^B(Y)$, this operation, when performed on the system, provides information on both of the two properties. It is a joint measurement of $A(X)$ and $B(Y)$. Obviously, in such a case the instruments $\underline{I}_i^A$ and $\underline{I}_j^B$ were not mutually exclusive. We come thus to the following definition of the notion of mutual exclusiveness of instruments, experimental arrangements (cf. Lahti,[15–17]).

Definition 3.1. Instruments $\underline{I}_1:\underline{B}(\underline{R}) \to \underline{O}(\underline{H})$ and $\underline{I}_2:\underline{B}(\underline{R}) \to \underline{O}(\underline{H})$ are *mutually exclusive* if and only if *lower bound* $\{I_1(X), \underline{I}_2(Y)\} = \{0\}$ for any two bounded X and Y in $\underline{B}(\underline{R})$ for which neither $I_1(X)$ nor $I_2(Y)$ is maximal.

As noted above the set $\underline{O}(\underline{H})$ of operations on the state space $T_s(\underline{H})$ is naturally ordered by the base $T_s(\underline{H})_1^+$ of the positive cone $T_s(\underline{H})^+$ of the state space: for any two operations ϕ_1 and ϕ_2 in $\underline{O}(\underline{H})$, $\phi_1 \leq \phi_2$ iff $\phi_1\alpha \leq \phi_2\alpha$ for all α in $T_s(\underline{H})_1^+$. The zero operation $0:\alpha \mapsto 0\alpha = 0$ is the least element of the poset $(\underline{O}(\underline{H}), \leq)$ whereas each $\phi \in \underline{O}(\underline{H})$ with the property $\mathrm{tr}[\phi\alpha] = \mathrm{tr}[\alpha]$ $\forall$ $\alpha \in T_s(\underline{H})_1^+$ is maximal. The identity operation $I:\alpha \mapsto I\alpha = \alpha$ is maximal, though not the only one. The restriction to nonmaximal operations allows the possibility that also bounded instruments, i.e., instruments with bounded value sets, might be mutually exclusive. The restriction to bounded Borel sets may be motivated by considering the possibilities for an operational definition of an observable. Actually, such considerations would propose that closed intervals should already suffice here. Finally, *lower bound* $\{\phi_1, \phi_2\} \equiv \{\phi \in \underline{O}(\underline{H}): \phi \leq \phi_1, \phi \leq \phi_2\}$.

We are led now to the following formal definition of the notion of complementary physical quantities.

Definition 3.2. Observables $A:\underline{B}(\underline{R}) \to \underline{E}(\underline{H})$ and $B:\underline{B}(\underline{R}) \to \underline{E}(\underline{H})$ are *complementary* iff any two instruments $\underline{I}_i^A$, $i \in I(A)$, and $\underline{I}_j^B$, $j \in I(B)$, uniquely defining these observables are mutually exclusive.

For our subsequent needs, we have formulated this definition, as also the preceding one, not in the standard frame, but in the extended frame which results from considering the whole set $\underline{O}(\underline{H})$ of operations. As a consequence, the standard set of propositions $\underline{P}(\underline{H})$ extends to the set of effects $\underline{E}(\underline{H}) = \{E \in \underline{L}_s(\underline{H}): 0 \leq E \leq I\}$, and the standard notion of observable as a $\underline{P}(\underline{H})$-valued measure extends to a generalized observable as an $\underline{E}(\underline{H})$-valued measure. Note that these definitions apply, in particular, to the standard observables and to their instruments.

An immediate consequence of the above two definitions is that observables A and B are complementary iff $A(X) \wedge B(Y) = 0$ for any two bounded X and Y in $\underline{B}(\underline{R})$ for which neither $A(X)$ nor $B(Y)$ is the unit element I of the poset $\underline{E}(\underline{H})$. This then shows that the notion of complementarity expresses the strongest case of the so-called incommensurability, or incompatibility which usually is defined through the commutation relation $[A,B] \neq 0$. Incompatibility of A and B means that there is no complete set of eigenvectors but it does not exclude the possible existence of some common eigenvectors. Complementary observables do have no common eigenvectors.

Let A and B be complementary observables so that, by definition, any two instruments $\underline{I}_i^A$ and I_j^B which uniquely define these observables are mutually exclusive, i.e., *lower bound* $\{\underline{I}_i^A(X), \underline{I}_j^B(Y)\} = \{0\}$ for all bounded value sets X and Y and for all $i \in I(A)$ and $j \in I(B)$. Thus, corresponding to the intuitive idea that mutually exclusive experimental arrangements cannot be applied at the same time, there are no joint measurements (describable as state transformations and associated with bounded value sets) of A and B. Complementary observables cannot be measured, or defined, at the same time. We formulate this important, though obvious, result as a corollary.

Corollary 3.1. Complementary observables do not admit any joint measurements.

There is a relaxation of the notion of complementary observables, which opens the possibility for their joint measurements. We shall now work it out.

Theorem 3.1. Let A and B be any two self-adjoint operators with $0 \leq A \leq I$, $0 \leq B \leq I$, and *lower bound* $\{A,B\} = \{0\}$, i.e., $A \wedge B = 0$. If $(\varphi, A\varphi) = 1$ for some $\varphi \in \underline{H}_1$, then $(\varphi, B\varphi) < 1$, and conversely.

Proof. Assume that there is a $\varphi \in \underline{H}_1$ such that $(\varphi, A\varphi) = 1$ and $(\varphi, B\varphi) = 1$. As $0 \leq A \leq I$ and $0 \leq B \leq I$, this implies that $A\varphi = \varphi$ and $B\varphi = \varphi$ which means that 1 is an eigenvalue of both A and B, and φ is their common eigenvector. In particular, this would mean that the meet $A(\{1\}) \wedge B(\{1\})$ of the corresponding eigenprojections $A(\{1\}) \leq A$ and $B(\{1\}) \leq B$ is nonzero, which contradicts with the assumption *lower bound* $\{A,B\} = \{0\}$.

The above result shows that if the observables A and B are complementary in the sense of definition 3.2, then they also share the property:

if $(\varphi, A(X)\varphi) = 1$ for some vector state $\varphi \in \underline{H}_1$, then

$(\varphi, B(Y)\varphi) < 1$, and conversely, for all bounded X and

Y in $\underline{B}(\underline{R})$ for which $A(X) \neq I \neq B(Y)$. (3.1)

As it will turn out, there are (generalized) observables which share the property (3.1) but which are not complementary in the sense of definition (3.2). This suggests the following definition.

Definition 3.3. Observables $A:\underline{B}(\underline{R}) \to \underline{E}(\underline{H})$ and $\underline{B}:(\underline{R}) \to \underline{E}(\underline{H})$ are *probabilistically complementary* iff they share the property (3.1).

We then have:

complementarity (in the sense of mutual exclusiveness) implies probabilistic complementarity, but, in general, not conversely . (3.2)

Note that it is only in the case of standard observables where the two notions of complementary observables and probabilistically complementary observables are equivalent. The term probabilistic complementarity is motivated by the fact that if the observables A and B are of that kind, then the certainly-yes predictions concerning their possible values are mutually exclusive. This feature of complementarity is emphasized by Pauli,[2] and further elaborated, e.g., by Beltrametti and Cassinelli,[12] and Lahti.[16,17]

Let A and B be two probabilistically complementary observable, which need not be complementary in the sense of mutual exclusiveness. Assume further that for some bounded value sets X and Y there is a predictable joint measurement $\phi \in \underline{O}(\underline{H})$ of $A(X)$ and $B(Y)$. But as $\mathrm{tr}[\phi\alpha]=1$ for some $\alpha=|\varphi\rangle\langle\varphi|\in T_s(\underline{H})_1^+$, we then would have $(\varphi,A(X)\varphi)=1$ and $(\varphi,B(Y)\varphi)=1$ which is excluded by the probabilistic complementarity of A and B. Thus we have the following corollary.

Corollary 3.2. Probabilistically complementary observables do not admit any predictable joint measurements.

We recall that preparatory measurements are also predictable measurements. Finally, we note that the canonically conjugate position and momentum observables Q and P are complementary. This follows from the Paley-Wiener theorem as Q and P are Fourier-Plancherel equivalent physical quantities, i.e., $P=(h/2\pi)F^{-1}Q$ with F denoting the Fourier-Plancherel operator of the underlying Hilbert space $\underline{H}$. (For a full discussion of this topic, with the relevant literature, see Busch,[18] Busch and Lahti,[19] Jammer,[20] and Lahti.[15-17] The components of the spin observable $s=(s_1,s_2,s_3)$, $s=0,\frac{1}{2},1,\ldots$ are pairwise complementary as well (Beltrametti and Cassinelli[12], p. 31).

IV. JOINT MEASUREMENTS OF Q AND P WITHIN THE STANDARD FRAME

We shall now begin to study the problem of joint measurability of complementary physical quantities with respect to the uncertainty relations. We start with analyzing the problem first within the standard Hilbert space formulation of quantum theory.

The standard frame, though extremely useful, is highly idealized. This appears, in particular, in the fact that observables are represented as spectral measures $\underline{B}(\underline{R})\to\underline{P}(\underline{H})$, or, equivalently, that properties of the system are represented (and identified) as projections on the underlying Hilbert space $\underline{H}$. Thus, in particular, only such operations (and thus measurements) $\phi\in\underline{O}(\underline{H})$ can be considered within the theory which give rise to projections (i.e., which serve to define or measure properties) $\mathrm{tr}[\phi\cdot]\in\underline{P}(\underline{H})$. [Here we use the identification $T_s(\underline{H})^*\cong L_s(\underline{H})$.] Further, each property $P\in\underline{P}(\underline{H})$ admits a pure, ideal, first-kind measurement defined by the operation $\phi_P:\alpha\mapsto\phi_P\alpha\equiv P\alpha P$. Closely related to that is the fact that within the standard frame the notions of complementarity (in the sense of mutual exclusiveness) and probabilistic complementarity are equivalent. Also an operational definition of an observable $A:\underline{B}(\underline{R})\to\underline{P}(\underline{H})$, $X\mapsto A(X)$ through its spectral projections $A([a,b])$, $a\leq b\in\underline{R}$, assumes that the value sets $[a,b]$ are sharply defined. No "defining ambiguities" or "measuring inaccuracies" are incorporated there. For each observable-state pair (A,α) the number $\Delta(A,\alpha)$ is the standard deviation of the probability distribution $\mu_\alpha^A:\underline{B}(\underline{R})\to[0,1]$, $X\mapsto\mu_\alpha^A(X)\equiv\mathrm{tr}[\alpha A(X)]$ of the possible values of the observable A in the state α. The interpretation of $\Delta(A,\alpha)$ is determined by, and thus consistent with, the interpretation of the distribution μ_α^A. This gives rise to probabilistic interpretation of the uncertainty relations. In the standard frame there is no formal reason to interpret the probabilistic number $\Delta(A,\alpha)$ as a kind of measuring (in-) accuracy. As a consequence, the standard view concerning the question of joint measurements of complementary physical quantities cannot be formulated in the usual Hilbert-space frame of quantum theory. The question remains which one of the two extreme views holds true therein. For explicitness, we shall refer to canonically conjugate position and momentum observables, though the considerations are equally valid for any pair of complementary observables satisfying the uncertainty principle.

Let $Q:\underline{B}(\underline{R})\to\underline{P}(\underline{H})$, $X\mapsto Q(X)$ and $P:\underline{B}(\underline{R})\to\underline{P}(\underline{H})$, $X\mapsto P(X)$ be canonically conjugate position and momentum observables. Owing to their Fourier-Plancherel equivalence Q and P are complementary,

$$\textit{lower bound}\{Q(X),P(Y)\}=\{0\} \tag{4.1}$$

for all bounded value sets X and Y in $\underline{B}(\underline{R})$. Consequently, any two instruments $\underline{I}_i^Q:\underline{B}(\underline{R})\to\underline{O}(\underline{H})$, $X\mapsto\underline{I}_i^Q(X)$, $i\in I(Q)$, and $\underline{I}_j^P:\underline{B}(\underline{R})\to\underline{O}(\underline{H})$, $X\mapsto\underline{I}_j^P(X)$, $j\in I(P)$, which uniquely define these observables are mutually exclusive, i.e.,

$$\textit{lower bound}\{\underline{I}_i^Q(X),\underline{I}_j^P(Y)\}=\{0\} \tag{4.2}$$

for all bounded value sets X and Y in $\underline{B}(\underline{R})$, and for all $i\in I(Q)$ and $j\in I(P)$. None of the instruments $\underline{I}_i^Q$, $i\in I(Q)$, and $\underline{I}_j^P$, $j\in I(P)$, can be applied simultaneously; there is no joint measurement of Q and P. The observables Q and P cannot be defined, or measured, at the same time. That is, in the standard frame the only acceptable standpoint seems to be the first exteme view.

The obstacle for the simultaneous definition, or measurement, of complementary physical quantities, like Q and P, lies in the fact that any two instruments associated with these observables are mutually exclusive. The problem is thus whether this obstacle can be removed or not. According to the standard view this should be possible by introducing some ambiguities, or inaccuracies, in the definitions, or measurements of the observables, to the extent expressed in the uncertainty relations

$$\Delta(Q,\varphi)\Delta(P,\varphi)\geq h/4\pi \quad \text{for all } \varphi\in\mathrm{dom}\,(Q)\cap\mathrm{dom}\,(P)\,. \tag{4.3}$$

Within standard quantum mechanics there is an intuitive idea how to account for the relevant ambiguities, or inaccuracies. This idea comes close to Heisenberg's original interpretation of his uncertainty relations. The standard deviation $\Delta(A,\alpha)$ of a physical quantity in a state α is here identified with the characteristic accuracy $\delta(A,\alpha)$ in determining the value of the quantity A in the state α, or, with the inaccuracy in an A measurement performed on the system in the state α. However, as already pointed out by Jammer,[21] see also Park and Margenau,[6] the identification of $\Delta(A,\alpha)$ with $\delta(A,\alpha)$ is not a logical must. Within the standard frame it is an *ad hoc* assumption, which, if accepted, would lead to difficulties with the fundamental probability postulate of the theory. The number $\Delta(A,\alpha)$ cannot be intepreted through some *ad hoc* assumptions, but its interpretation should be derived from the given interpretation of the measure μ_α^A as pointed out above. If such an *ad hoc* interpretation is, however, accepted, it remains totally unsuccessful within the standard frame, as it does not break the relation (4.2). In other words, within the standard frame, this interpretation does not lead to any new instruments associated with position and momentum which were not mutally exclusive. The obstacle for the simultaneous definitions, or measurements, of complementary physical quantities remains. This situation is closely related to the above discussed idealizations of the standard Hilbert space quantum theory. It contains neither the notion of ambiguous definition of a physical quantity nor the notion of inaccurate, or unsharp, measurement. The standard frame can, however, easily be extended to contain such notions. We shall continue our analyses of the problem within such an extended frame, which will be sketched in the next section.

We shall summarize the above discussion as the following result:

> Within the standard Hilbert-space quantum theory complementary physical quantities, like position and momentum, cannot be simultaneously measured at all. This does not follow from the uncertainty principle, which therein is to be interpreted probabilistically, but rather from the very idea of complementarity. (4.4)

V. UNSHARP OBSERVABLES—THE EXTENDED FRAME

In employing his γ-ray microscope (thought experiment) to determine the position of an electron Heisenberg argued that due to the finite resolution of the apparatus, the Q measurement is inaccurate (Heisenberg,[1] p. 198). A similar situation appears in a momentum measurement of a particle through the Doppler effect (see, e.g., Heisenberg[22]). Further, in the double-slit experiment, the measurement of the momentum transfer from the particle to the first diaphragm with one slit, say, requires that the diaphragm is not rigidly connected with the common support which serves to define the space frame of reference. Consequently, in giving up the exact controlling of the position of the slit, the slit (in the movable diaphragm) no more defines unambiguously the relevant value set, rather it becomes "unsharp" or "fuzzy" (with respect to the space frame of reference). *A fortiori*, a possibility for an unambiguous definition of the position of the particle is thereby lost (see Bohr[23]).

The notion of unsharp observables, introduced and discussed by several authors in recent years, provides a formal means to account for the intuitive ideas contained in the above examples on inaccurate measurements, or ambiguous definitions, of observables.

In order to appreciate the definition below of unsharp observable we precede it with some heuristics (cf. Ali and Emch[24]).

$\mu_\alpha^A(X)=\mathrm{tr}[\alpha A(X)]$ is the probability that a measurement of the observable $A:\underline{B}(\underline{R})\to\underline{P}(\underline{H})$, $X\mapsto A(X)$ on the system in the state $\alpha\in T_s(\underline{H})_1^+$ yields a result in the value set X. This formalism assumes that an experimental arrangement which serves to define the observable A, or a measuring apparatus which can be used to measure its values, is optimal in the sense that it admits a sharp (or unique or unambiguous or proper) definition of the value sets X. Assume, however, that the A-measuring apparatus, which is employed, is not such an ideal one, but it has a finite resolution δ, say. Thus it cannot distinguish between points lying within the distance δ of each other. Consequently, such an apparatus does not define (operationally) the value set

$$X=[x_1,x_2]=[x_0-\tfrac{1}{2}\,|x_2-x_1|\,,x_0+\tfrac{1}{2}\,|x_2-x_1|\,]\ ,$$

but rather it defines a "fuzzy" value set $\{x+X: x\in[-\delta/2,+\delta/2]\}$. Consistent with the standard frame, such an apparatus does not measure the probability $\mu_\alpha^A(X)$, but a weighted mean (or convex combination) of the probabilities $\mu_\alpha^A(x+X)$, $x\in[-\delta/2,+\delta/2]$ depending on the finite resolution δ and some other relevant properties of the system. In the approach followed here, it is assumed that such "relevant ambiguities" are properly accounted for through a probability density function.

Let $A:\underline{B}(\underline{R})\to\underline{P}(\underline{H})$, $X\mapsto A(X)$ be a projection-valued measure (i.e., a standard observable), and $f:\underline{R}\to\underline{R}$, $x\mapsto f(x)$ a probability density function ($f\geq 0$, $\int_{\underline{R}} f=1$). Any such couple (A,f) defines, in the weak sense, an effect-valued measure (known also as a generalized or fuzzy or unsharp or approximate or improper observable) $A_f:\underline{B}(\underline{R})\to\underline{E}(\underline{H})$, $X\mapsto A_f(X)$ through the relation

$$\mathrm{tr}[\alpha A_f(X)]=\int_{\underline{R}} f(x)\,\mathrm{tr}[\alpha A(x+X)]dx \tag{5.1}$$

for all states α in $T_s(H)_1^+$. The function f is taken to describe the unsharpness involved in the measurement, or the ambiguity involved in the definition of the observable. That is, we may alternatively interpret this function as attached *either* to the measuring apparatus employed for a *measurement* the values of the observables (characterizing thereby, e.g., the finite resolution of the measuring apparatus), *or* to the experimental arrangement used to *define* the observables (characterizing thus the ambiguities involved in the defining procedure). For a more detailed exposition of this notion we refer to Ali and Emch,[24] Davies,[11] Prugovecki,[25] and Ali and Prugovecki.[26] Its measurement theoretical interpretation has been discussed by Busch.[27]

The notion of unsharp observable is consistent with the standard notion of observable emphasizing, however, the ideality of the latter. Really, if f tends to the Dirac δ function, which corresponds to the idea that the measuring accuracy increases, or the defining ambiguities decrease, then the unsharp A_f tends to the sharp, i.e., standard, A (in the weak sense). Further, the set function $\mu_\alpha^{A_f}:\underline{B}(\underline{R})\rightarrow[0,1]$, $X\rightarrow\mu_\alpha^{A_f}(X)\equiv\mathrm{tr}[\alpha A_f(X)]$ is a probability measure for any state $\alpha\in T_s(\underline{H})_1^+$ so that, in particular, the number $\mu_\alpha^{A_f}(X)$ admits a probabilistic interpretation. Consistent with the probability postulate of the standard theory, the number $\mu_\alpha^{A_f}(X)$ may be interpreted as the probability that a measurement of the observable A in the state α yields a result in the value set X within the measuring accuracy characterized by the function f [or that the value of the unsharp observable A_f lies in the set X in the state α, or that the system possesses the effect (unsharp property) $A_f(X)$ in the state α]. We note also that for each effect $A_f(X)$ there exists at least one operation $\phi\in\underline{O}(\underline{H})$ which uniquely defines the effect, through the relation $\mathrm{tr}[\phi\alpha]=\mathrm{tr}[\alpha A_f(X)]$ for all $\alpha\in T_s(\underline{H})_1^+$. Similarly, to each generalized observable $A_f:\underline{B}(\underline{R})\rightarrow\underline{E}(\underline{H})$ there corresponds at least one instrument (operation-values measure) $\underline{I}^{A_f}:\underline{B}(\underline{R})\rightarrow\underline{O}(\underline{H})$ which uniquely defines the observable.

The theory has now been extended through a generalization of the notion of observable. This generalization resulted from introducing the notion of unsharp measurement, or ambiguous definition, of an observable in the form of a couple (A,f) defining the unsharp observable A_f. In particular, what has been gained is that the standard deviation Δf of a probability density function $f:\underline{R}\rightarrow\underline{R}$ may consistently be taken to describe, e.g., the characteristic accuracy in determining the value of the observable A in a given state α. The conceptual difficulties involved in the above-discussed identification of $\delta(A,\alpha)$ with $\Delta(A,\alpha)$ can thus be overcome, e.g., through an identification of $\delta(A,\alpha)$ with Δf. Furthermore, this extension allows a double role for the basic probability measures μ_α^A of the theory: On one hand μ_α^A is the probability distribution of the possible values *of the observable A in the state of the system, on the other hand* μ_α^A defines a probability density function $f^{\mu_\alpha^A}$, which, in such, may be taken to describe some ambiguities, or inaccuracies, involved in a given A measurement (giving thus rise to the unsharp observable $A_{f^{\mu_\alpha^A}}$). It is just this feature of the extended frame which makes it possible that the uncertainty relations may have something to do with the imprecisions of measurements as stated in the standard view. It is important to note that this distinction between the two interpretations of the probability measures μ_α^A appears already in Heisenberg[1] though he did not explicitly work out the formal consequences of the second interpretation. It is certainly true that within the standard frame of quantum theory "the Heisenberg uncertainty relations have *nothing* to do with the imprecisions of measurement . . . . On the contrary the Heisenberg relation $\Delta q\ \Delta p\geq h/4\ldots$ is deduced in the framework of the theory under the assumption that the observables position and momentum are measured with absolute precision" (Ludwig,[28] p. 17). However, as soon as one may consistently speak of "the imprecisions of measurements" such a strong conclusion is hardly tenable, as an extended frame is needed. In the present approach, it would simply overlook the double role played by the probability measures μ_α^A of the theory. The extended frame thus opens the possibility for an alternative of the probabilistic interpretation of the uncertainty relations, and it admits a quantitative formulation of the standard view. It is to be noted that in general the probability density function f of an unsharp observable A_f need not be given by the relevant measure μ_α^A though this appears to be the case in the standard view. Finally, we point out that the introduction of the unsharp observable A_f, A a standard observable, f a probability density function, does not exhaust the sets $\underline{O}(\underline{H})$ and $\underline{E}(\underline{H})$ of all operations and effects on the Hilbert state space $T_s(H)$.

VI. JOINT MEASUREMENTS OF Q AND P WITHIN THE EXTENDED FRAME

We shall now analyze the validity of the three views concerning the simultaneous measurability of complementary physical quantities within the extended Hilbert-space formulation of the quantum theory. For explicitness, we shall refer again to canonically conjugate position and momentum observables Q and P, with $L_2(\underline{R})$ as the underlying Hilbert space.

We have

$$\textit{lower bound}\{Q(X),P(Y)\}=\{0\}\quad[\text{for all bound } X \text{ and } Y \text{ in } \underline{B}(\underline{R})]\ , \tag{6.1}$$

$$\Delta(Q,\varphi)\Delta(P,\varphi)\geq h/4\pi\quad[\text{for all } \varphi \text{ in dom } (Q)\cap\text{dom } (P)]\ , \tag{6.2}$$

and we ask whether the measuring inaccuracies, or defining ambiguities, which agree with the (probabilistic) uncertainty relations (6.2) are needed, or are enough, to break the complementarity of Q and P, expressed in (6.1) and thus to admit their joint measurements.

We shall first consider such unsharp position and momentum observables Q_f and P_g whose "characteristic inaccuracies" Δf and Δg are *compatible* with the uncertainty relations, i.e., $\Delta f=\Delta(Q,\alpha)$ and $\Delta g=\Delta(P,\alpha)$ for some $\alpha\in T_s(L_2(\underline{R}))_1^+$. In other words, making use of the flexibility of the extended frame we now interpret the numbers $\Delta(Q,\alpha)$ and $\Delta(P,\alpha)$ as the measuring accuracies of the relevant Q and P measurements (performed on the system in a state α), so that the probability density functions f and g of the pair (Q_f,P_g) are the probability density functions of the probability measures μ_α^Q and μ_α^P, for some $\alpha\in T_s(L_2(\underline{R}))_1^+$, i.e.,

$$f=f^{\mu_\alpha^Q},\quad g=g^{\mu_\alpha^P},\quad \alpha\in T_s(L_2(\underline{R}))_1^+\ . \tag{6.3}$$

As Q and P are Fourier-Plancherel equivalent physical quantities, the functions f and g above are Fourier-connected as well. More explicitly,

$$\begin{aligned} f(x)&=f^{\mu_\alpha^Q}(x)=\sum\lambda_i\,|\varphi_i(x)|^2\ ,\\ g(x)&=g^{\mu_\alpha^P}(x)=\frac{h}{2\pi}g^{\mu^Q_{F\alpha F^{-1}}}(x)\\ &=\frac{h}{2\pi}\sum\lambda_i\,|\tilde{\varphi}_i(x)|^2\ , \end{aligned} \tag{6.3a}$$

where $\sum \lambda_i |\varphi_i\rangle\langle\varphi_i|$ is any decomposition of α into pure states $|\varphi_i\rangle\langle\varphi_i|$ and $\tilde{\varphi}_i \equiv F\varphi_j$ is the Fourier transform of φ_i. With this choice we have

$$\Delta f \Delta g = \Delta(Q,\alpha)\Delta(P,\alpha) \geq h/4\pi , \qquad (6.3b)$$

which confirms that the characteristic inaccuracies involved in the pair (Q_f,P_g) agree with the uncertainty relations

We are now in position to apply the results of Davies[11] and Ali and Prugovecki[26] which show that in this case there exists an effect-valued measure $A_{f,g}:\underline{B}(\underline{R}^2) \to \underline{E}(L_2(\underline{R}))$, $Z \mapsto A_{f,g}(Z)$ which has unsharp position Q_f and momentum P_g as marginal observables, i.e., $A_{f,g}(X \times \underline{R}) = Q_f(X)$ and $A_{f,g}(\underline{R} \times Y) = P_g(Y)$ for any X and Y in $\underline{B}(\underline{R})$. Such an $A_{f,g}$ has the explicit form

$$(\varphi, A_{f,g}(Z)\varphi) = \frac{1}{2\pi}\int_Z (\varphi, \alpha_{qp}\varphi)dp\,dp, \quad \varphi \in L_2(\underline{R}) \qquad (6.3c)$$

where

$$\alpha_{qp} = \exp(ipQ)\exp(-iqP)\alpha\exp(iqP)\exp(-ipQ) \qquad (6.3d)$$

with $\alpha \in T_s(L_2(\underline{R}))_1^+$. Moreover, any effect-valued measure on $\underline{R} \times \underline{R}$ which has continuous positive definite density and unsharp position and momentum as marginals is of that type (Ali and Prugovecki[26]).

A pair (Q_f,P_g) of unsharp position and momentum observables with Fourier connected probability density functions (6.3a) shall be called a *Fourier couple.* It follows that only Fourier couples (Q_f,P_g) give rise to continuous joint observables $A_{f,g}:B(\underline{R}^2) \to \underline{E}(L_2(\underline{R}))$ with the corresponding measuring accuracies, or defining ambiguities, obeying the uncertainty relations. Since $A_{f,g}(X \times Y) \leq Q_f(X)$ and $A_{f,g}(X \times Y) \leq P_g(Y)$ we have

$$\textit{lower bound}\{Q_f(X),P_g(Y)\} \neq \{0\} \qquad (6.4)$$

for all nonempty sets X and Y in $\underline{B}(\underline{R})$. Unsharp observables Q_f and P_g, with Fourier connected probability density functions f ang g, are not complementary. Any operation ϕ in $\underline{O}(L_2(\underline{R}))$, whose effect lies in the set (6.4) is a joint measurement of $Q_f(X)$ and $P_g(Y)$. This then shows that the standard view is tenable, at least, with respect to all Fourier couples (Q_f,P_g).

Let (Q_f,P_g) be a Fourier couple. The existence of an $A_{f,g}$ implies the existence of a (continuous) instrument $\underline{I}^{Q_f,P_g}:\underline{B}(\underline{R}^2) \to \underline{O}(L_2(\underline{R}))$ which provides joint measurements $\underline{I}^{Q_f,P_g}(X \times Y)$ for all the possible values of the (continuous) position and momentum observables. It might turn out that some of the minimal measurements $\underline{I}^{Q_f,P_g}(X \times Y)$ of the effects $A_{f,g}(X \times Y)$ would also serve as minimal joint measurements of the pair $(Q_f(X),P_g(Y))$ but we leave this question open here. Anyway, it can be shown (Ali and Prugovecki[26]) that for any pair of nonnegative real numbers δq and δp the condition $\delta q \delta p \geq h/4\pi$ is necessary and sufficient for the existence of a Fourier couple (Q_f,P_g) with $\Delta f = \delta q$ and $\Delta g = \delta p$. It is in this, and only in this sense that the uncertainty relations express a necessary and sufficient condition to break the complementarity of position and momentum observables and thus to admit their joint measurability. It is to be emphasized, however, that these results refer to the continuous case only, i.e., to the existence of a continuous (joint-) instrument with minimal joint measurements. Although Fourier couples, and minimal joint measurements, obviously characterize an important class of joint measurements of position and momentum observables we still face the problem whether the uncertainty relations express, in general, a necessary and sufficient condition for their joint measurability. We shall now turn to this question which can be answered by the so-called generalized Jauch theorem (Busch[18]).

Let ν_X^f and ν_Y^g be the generalized characteristic functions defined by

$$\nu_X^f(q) = (\chi_X * f)(q) , \qquad (6.5)$$

where $\chi_X * f$ denotes the convolution of the probability density function f with the characteristic function χ_X of the set X. Let X and Y be bounded Borel sets of the real line $\underline{R}$. Then it holds (Busch[28])

$$\textit{lower bound}\{Q_f(X),P_g(Y)\} = \{0\}$$
$$\text{if both } \nu_X^f \text{ and } \nu_Y^g \text{ have bounded supports} \qquad (6.6a)$$

$$\textit{lower bound}\{Q_f(X),P_g(Y)\} \neq \{0\}$$
$$\text{if supp }(\nu_X^f) = \underline{R} \text{ or supp }(\nu_Y^g) = \underline{R} . \qquad (6.6b)$$

Let us call (Q_f,P_g) a *first-type Jauch couple* if both f and g are distribution functions with bounded supports. Then (6.6a) states the complementarity of all pairs of observables Q_f,P_g which are first-type Jauch couples, including the original pair (Q,P) for which f and g are the Dirac distributions. So we may say that the first extreme view refers (at least) to all first-type Jauch couples.

Let (Q_f,P_g) be a pair of unsharp position and momentum observables with f and g as the normalized characteristic functions of the symmetric intervals $[-n,n]$ and $[-m,m]$, $n,m \in \underline{N}$, say. Clearly, (Q_f,P_g) is a first-type Jauch couple and the characteristic inaccuracies involved in this pair satisfy the relation $\Delta f \Delta g = nm/3$. This then shows that the uncertainty relations do not express any sufficient amount of defining ambiguities, or measuring inaccuracies, to break the complementarity of position and momentum and thus to allow their joint measurements.

A couple (Q_f,P_g) shall be called a *second-type Jauch couple* if $\text{supp}(f) = \underline{R}$ or $\text{supp}(g) = \underline{R}$. Then according to (6.6b) (at least) all second-type Jauch couples are jointly measurable. There is room both for the standard view [of course (almost all) Fourier couples are also second-type Jauch couples] as well as for the second extreme view. The result (6.6b), like (6.6a), holds true independently of the uncertainty relations. For example, any pair of independent Gaussian probability density functions f,g with arbitrarily small uncertainty product $\Delta f \Delta g$ gives rise to a second-type Jauch couple. Thus the uncertainty relations do not express any necessary condition for the joint measurability of complementary physical quantities, like position and momentum.

Up to this point we have found a formal characterization of the measuring situations which the three viewpoints refer to: the first extreme view claims that there exists a class of pairs of complementary observables Q_f, P_g; the standard view holds true with respect to Fourier couples, whereas the second extreme view refers to second-type Jauch couples which are also non-Fourier couples. All three classes of pairs have in common that they are probabilistically complementary.

A careful comparison of the determinative and preparatory features of the corresponding types of measurements will show the proper status of the uncertainty relation, that is, of the standard view. We shall turn to that in Sec. VIII. To end the present section we show that no joint measurement can be predictable which implies the probabilistic complementarity stated above.

Let $\phi \in \underline{Q}(L^2(\underline{R}))$ be a joint measurement of some given effects $Q_f(X)$ and $P_g(Y)$, associated with bounded value sets X and Y. Assume now that ϕ is *predictable,* i.e., there is a vector state $\varphi \in L^2(\underline{R})$ with $\mathrm{tr}[\phi \mid \varphi\rangle\langle\varphi \mid] = 1$. (Recall, in particular, that a first-kind measurement is also predictable.) In this case we would have $(\varphi, Q_f(X)\varphi) = 1$ and $(\varphi, P_g(Y)\varphi) = 1$ for the given φ. This would further imply that both f and g have bounded supports, thus (Q_f, P_g) must be a first-type Jauch couple which would contradict their supposed joint measurability. Thus we conclude that Q_f, P_g though not complementary are probabilistically complementary. Actually, although we can destroy the complementarity of Q and P through introducing some defining ambiguities or measuring inaccuracies in the form of unsharp Q_f and P_g we can never avoid their probabilistic complementarity, so far as we stick to the canonical relation $P = (h/2\pi)F^{-1}QF$. Finally as ϕ cannot be predictable it follows that it also cannot serve as a preparatory measurement, i.e., the postmeasurement or final state $\phi\alpha$ does not allow one to conclude that the system possess the measured "properties" $Q_f(X)$ and $P_g(Y)$ with certainty. We may conclude:

> Position and momentum observables Q and P can simultaneously be measured with measuring accuracies which may or may not obey the uncertainty relations. In particular, uncertainty relations do express neither necessary nor sufficient conditions for their joint measurability. None of the possible joint measurements of position and momentum observables is, however, predictable or preparatory.

VII. COMPARISON OF THE THREE VIEWPOINTS

In the following we shall elucidate the determinative and preparatory features of the various types of measurements which the three viewpoints refer to, respectively. It will turn out that there are considerable differences by which it becomes possible to resolve the apparent contradictions between the three views.

A. First extreme view: first-type Jauch couples

1. Determinative character

The relations $\mathrm{tr}[\underline{I}_f(X)\alpha] = \mathrm{tr}[\alpha Q_f(X)]$, $\mathrm{tr}[\underline{I}_g(Y)\alpha] = \mathrm{tr}[\alpha P_g(Y)]$ allow a determination of statistics of values, that is, of the respective probabilities in state α. If, for example, the state $\alpha = \mid\varphi\rangle\langle\varphi\mid$ is localized in some bounded interval X with measure $\mu(X) > \mu(\mathrm{supp} f)$, $\mathrm{supp}(\varphi(q)) \subseteq X$, then there exist bounded intervals $X' \supset X$ such that their registrations leads to the "yes" result with certainty. According to (6.6) the reason for the complementarity of a first-type Jauch couple (Q_f, P_g) can be said to lie in the fact that both Q_f and P_g allow predictable measurements of sharp values (i.e., sharply defined bounded Borel sets) whereas Corollary 3.2 tells us that there are no predictable joint measurements for Q_f and P_g.

2. Preparatory measurements

The predictability described in Sec. VII A 1 implies the existence of preparatory measurements in the following sense (slightly differing from the definition given in Sec. II): the final state of a $Q_f(X)$ measurement though only unsharply localized within X represents a particle which may be found with certainty in a somewhat larger set $X' = X \pm \Delta f$. Thus one may speak of the system as possessing an objective localization property.

B. Second extreme view: second-type Jauch couples

1. Determinative character

Let $0 \neq E \in$ *lower bound* $\{Q_f(X), P_g(Y)\}$, $\phi_E \in \underline{Q}(L^2(\underline{R}))$ defined through $\mathrm{tr}[\phi_E\alpha] = \mathrm{tr}[\alpha E]$. Then determination of the statistics for the effect E in state α represents an estimation of lower bounds of probabilities for $Q_f(X)$ and $P_g(Y)$. If (Q_f, P_g) is a non-Fourier couple ϕ_E cannot at all provide any information about sharp or unsharp position and momentum values simultaneously but only about probabilities. Obviously it is not trivial that this possibility exists.

2. Preparatory measurements

According to statement (6.7) there are no preparatory joint measurements. Let ϕ_E be a probabilistically preparatory measurement of effect E, that is $\mathrm{tr}[\phi_E{}^2\alpha]\mathrm{tr}[\phi_E\alpha]^{-1} \geq \mathrm{tr}[\phi_E\alpha]$, or with $\alpha' = \mathrm{tr}[\phi_E\alpha]^{-1}\phi_E\alpha$, $\mathrm{tr}(\phi_E\alpha') \geq \mathrm{tr}(\phi_E\alpha)$. Then there is only very little information about the (non-Fourier couple) $Q_f(X)$ and $P_g(Y)$ in the postmeasurement state α': after an individual "yes" registration for the effect E all one knows about $Q_f(X)$ and $P_g(Y)$ is that the lower bound for their probabilities has become larger than it was before the measurement. That is, one has improved one's chance of registrating each one of these effects, afterwards.

This type of measurement appears rather strange as it gives only probabilistic information about the system, namely, about probabilities for position and momentum values, but no information about the values themselves. Actually, no experimentalist who may primarily be interested in the determination of values would try to devise a measuring apparatus for the joint measurement of such non-Fourier second-type Jauch couple. Rather one may interpret, if possible, some real(istic) measurement of some other observable as such joint measurement. We shall give an example of this in the final section.

C. Standard view: Fourier couples

We have seen that there is a wide gap between the sharp measurements which are mutually exclusive and the joint measurements of unsharp second-type Jauch couples. In order to incorporate into the language of quantum physics the formal possibility of speaking about joint position and momentum measurements with arbitrary accuracy one has to give up all the strong determinative and preparatory features accessible in case of sharp measurements. In particular, one cannot speak of sharp path determinations. This is the point where the standard view comes into play. Only Fourier couples Q_f,P_g provide a means to define approximate classical trajectories for quantum-mechanical particles as has been worked out in detail by Busch.[27,29] The measurement theoretic analysis performed therein by means of models of joint measurements indicates that the following *conjectures* should be valid:

> The joint application of two devices suited for *minimal* measurements (see Sec. II) of $Q_{f_0}(X)$ and $P_{g_0}(Y)$ inevitably leads to a *minimal* measurement of some $A_{f,g}(X\times Y)$, that is, to a joint measurement of a Fourier couple $Q_f(X),P_g(Y)$. (7.1)

> If *one single* apparatus is suited to be applied by choice for a *minimal* measurement either of some $Q_f(X)$ or of some $P_g(Y)$ then (Q_f,P_g) necessarily represents a Fourier couple. (7.2)

In both cases one inescapably runs into the uncertainty relation. The proof of these conjectures would have to be formulated by means of a general description of the respective measuring devices and processes as dynamical processes. Up to now there are only examples which show the following: instruments for Q_f and P_g, if they simultaneously interact with the observed object, get mutually influenced so that the resulting composite instrument $\underline{I}_{f,g}$ measures a joint observable $A_{f,g}$ and thus necessarily respects the uncertainty relation. It is in this sense that the uncertainty relation may be interpreted to show up an aspect of complementarity: instruments corresponding to complementary observables exclude each other not only in the sense described in Sec. III but also in that they disturb each other (if simultaneously applied) to the extent given by the uncertainty relation.

Only for Fourier couples $Q_f(X),P_g(Y)$ there exists a joint observable $A_{f,g}(X\times Y)=E$. We shall describe now the possibilities of determinative and preparatory measurements ϕ_E going beyond the features displayed in Sec. VII B.

1. Determinative character

As there is a full instrument $\underline{I}_{f,g}$, or joint observable $A_{f,g}$, for the effect-valued measures Q_f,P_g, one may employ this single instrument $\underline{I}_{f,g}$ for a determination of the statistics

$$\{\mathrm{tr}[\alpha A_{f,g}(X\times Y)]{:}X,Y\in\underline{B}(\underline{R})\} \tag{7.3}$$

including the full statistics of the marginal observables

$$\{\mathrm{tr}[\alpha Q_f(X)]{:}X\in\underline{B}(\underline{R})\},\quad \{\mathrm{tr}[\alpha P_g(Y)]{:}\ Y\in\underline{B}(\underline{R})\}\ . \tag{7.4}$$

In the case of complementary pairs Q_{f_0},P_{g_0} one has to use two different instruments $\underline{I}_{f_0}$ and $\underline{I}_{g_0}$. As pointed out by Ali and Prugovecki[26] there are informationally, or probabilistically complete joint observables for which the sets (7.3) uniquely determine the state which can never be achieved by means of the sets (7.4). That is, the determinative possibilities of instruments $\underline{I}_{f,g}$ for joint measurements of Fourier couples Q_f,P_g are, with respect to state determinations, even better than those of, e.g., instruments $\underline{I}_{f_0},\underline{I}_{g_0}$ corresponding to complementary Q_{f_0},P_{g_0}.

2. Preparatory features

According to conclusion (6.7) there is no possibility of performing preparatory joint measurements leading to objective sharp position and momentum values (or better, value sets). However, the models of Busch[27,29] show that there are probabilistically preparatory measurements of some $A_{f,g}$ which provide a preparation of unsharp position and momentum values. Assume that A_{fg} is given by $\alpha_{qp}=|\varphi_{qp}\rangle\langle\varphi_{qp}|$ [see Eq. (6.3d)]. Let $Z\in\underline{B}(\underline{R}^2)$ take the form $Z=X_0\times Y_0$ with $X_0=[q_0-\delta q,q_0+\delta q]$, $Y_0=[p_0-\delta p,p_0+\delta p]$. For sufficiently small $\delta q\ll\Delta f$ and $\delta p\ll\Delta g$ the system can be found after measurement in a state β which (up to normalization) is approximately given by $|\varphi_{q_0p_0}\rangle\langle\varphi_{q_0p_0}|$. That is, to the measured value set $X_0\times Y_0$, or unsharp value $(q_0,\Delta f)\times(p_0,\Delta g)$ there corresponds an object state $\beta\cong|\varphi_{q_0p_0}\rangle\langle\varphi_{q_0p_0}|$ characterized by

$$\begin{aligned}&\sum\lambda_i|\psi_i(q)|^2\cong f(q-q_0),\quad \mathrm{tr}[\beta Q]=\sum\lambda_i(\psi_i,Q\psi_i)\cong q_0,\quad \Delta(Q,\beta)\cong\Delta f\ ,\\ &\sum\lambda_i|\tilde{\psi}_i(p)|^2|^2\cong g(p-p_0),\quad \mathrm{tr}[\beta P]=\sum\lambda_i(\psi_i,P\psi_i)\cong p_0,\quad \Delta(P,\beta)\cong\Delta g\ .\end{aligned} \tag{7.5}$$

Let us call a probabilistically preparatory measurement *quasipreparatory* if it serves to prepare objective unsharp position and/or momentum values as described in (7.5). An example of such quasipreparatory instruments is given by

$$\underline{I}_{f,g}(Z)=\frac{1}{2\pi}\int_Z dq\,dp\,(\varphi_{qp'},\alpha\varphi_{qp})\,|\varphi_{qp}\rangle\langle\varphi_{qp}|$$

(cf. also Schroeck[30]; in the work of Busch[18] somewhat different expressions have been used for the same notion).

It seems that the deeper reason for the validity of the

TABLE I. Comparison of the various types of (joint) measurements.

	(Q_f, P_g)		
		Second-type Jauch couple	
Linguistic possibilities	First-type Jauch couple	Fourier couple	non-Fourier couple
Formal: types of statements	Complementarity: no joint measurements at all First extreme view	Joint measurements within limits of uncertainty relation Standard view	Joint measurements with arbitrary accuracy Second extreme view
Material: content of measuring statements	Preparatory measurements: objective sharp values	Quasipreparatory joint measurements: objective unsharp values	Probabilistically preparatory joint measurements: objective lower bound estimations for probabilities, no information about values
	Determinative measurements: statistical information about sharp values and their probabilities	Determinative joint measurements: statistical information about unsharp values and their probabilities	Determinative joint measurements: statistical estimative information only about probabilities, not about values
Similarity to classical language	——— formally: type of statements ——→ ←——— materially: content of statements ———		

conjectures (7.1) and (7.2) evaluating the significance of the uncertainty relation lies in the existence of quasipreparatory measurements among the minimal probabilistically preparatory measurements of Q_f and P_g, respectively: Combination of two such instruments cannot destroy that quasipreparatory character (only the preparatory nature gets lost if it was there) so that in the final states both unsharp position and momentum values are objectified. At this point the probabilistic uncertainty relation for states and the individualistic uncertainty relation for unsharp values come into close connection (cf. four pages earlier).

To conclude this section we shall summarize the most important distinguishing features of the various types of pairs Q_f, P_g and their measurements (see Table I). By introducing the extended frame certain formal restrictions contained in the language of the standard frame can be weakened, namely, the complementarity of, e.g., position and momentum observables; language becomes more similar to classical language in this respect as one may talk about joint measurements of (probabilistically) complementary observables. However, there is a price to be paid which consists in a loss of informational content of the measuring statements: In case of joint measurements it is no longer possible to speak of objective sharp values but only of objective unsharp joint values (respecting the uncertainty relation) or even of estimative probabilities. Thus the nonclassical nature of the quantum language remains preserved.

Finally we give a formulation of the three views which shows that they may be interpreted consistently to point out three different aspects of complementarity, or of the nonclassical nature of the quantum language:

(1) First extreme view: Instruments for first-type Jauch couples are mutually exclusive; there are no joint measurements which could prepare both sharp position and momentum values.

(2) Second extreme view: It is possible to perform joint measurements of position and momentum with arbitrarily small inaccuracies, namely, by means of second-type Jauch couples; but such measurements at best are probabilistically preparatory and neither preparatory nor quasipreparatory.

(3) Standard view: If instruments for (quasi-)preparatory position and momentum measurements are combined then the resulting measurement is a quasipreparatory joint measurement obeying the uncertainty relation.

If formulated in such detail there obviously is no longer any contradiction between the three viewpoints.

VIII. AN EXAMPLE AND CONCLUDING REMARKS

To show how a measurement of a joint effect $A_{f_0,g_0}(X_0 \times Y_0)$ connected with a Fourier couple (Q_{f_0}, P_{g_0}) may be interpreted as joint measurement of a non-Fourier second-type Jauch couple $Q_f(X), P_g(Y)$ we take f_0, g_0 and f, g as

$$f_0(q) = |\varphi(q)|^2, \quad g_0(p) = |\tilde{\varphi}(p)|^2 = g(p) \tag{8.1}$$

with φ having bounded support $[-q_0, +q_0]$. Further, let f be a distribution function with $\text{supp}(f) = \underline{R}$, e.g., a Gaussian. Then a simple estimation shows that to any bounded $X \in \underline{B}(\underline{R})$ there exists a $X_0 \subseteq X$, $X_0 \in \underline{B}(\underline{R})$ such that

$$\nu_{X_0}^{f_0}(q) \le \nu_X^f(q) \quad \text{for all } q \in \underline{R} \ . \tag{8.2}$$

Together with the representation

$$Q_f(X) = \int_{\underline{R}} dq\, f(q) Q(X+q) = \nu_X^f(Q) \quad (8.3)$$

this gives

$$Q_{f_0}(X_0) \le Q_f(X) \ . \quad (8.4)$$

Moreover, for arbitrary bounded $Y_0, Y \in \underline{B}(\underline{R})$, $Y_0 \subseteq Y$, we have

$$P_{g_0}(Y_0) \le P_g(Y) \ . \quad (8.5)$$

Taken together

$$A_{f_0,g_0}(X_0 \times Y_0) \in \textit{lower bound}\{Q_f(X), P_g(Y)\} \ . \quad (8.6)$$

Although a probabilistically preparatory measurement of $A_{f_0,g_0}(X_0 \times Y_0)$ may be quasipreparatory with respect to the Fourier couple Q_{f_0}, P_{g_0} it never can be so for *both* members of the non-Fourier second-type Jauch couple Q_f, P_g. In particular $\Delta f \Delta g$ may be chosen arbitrarily small below $h/4\pi$.

Finally, we should mention an attempt by Park and Margenau[6] to prove the second extreme view. They employ the time-of-flight method as an example for a joint measurement of position and momentum with arbitrary accuracy. This attempt has been refuted by Jauch[5] as well as by de Muynck *et al.*[31] We try to give a formal reconstruction of the argument in order to show up the point where it fails.

Take a vector state φ which is localized in some bounded region of space, that is, there exists a $d > 0$ such that $\varphi(x) = 0$ for $|x| > d$. Let $U_t = \exp[-(i2\pi/h)(1/2m)P^2 t]$ be the free time evolution of operator, and denote $\varphi_t = U_t \varphi$. Then

$$(\varphi, P((p_1,p_2))\varphi) = (\varphi_t, P((p_1,p_2))\varphi_t) = \lim_{t\to\infty} \left(\varphi_t, Q\left[\left[\frac{p_1 t}{m}, \frac{p_2 t}{m} \right] \right] \varphi_t \right) \quad (8.7)$$

holds. In this sense the observable $F(Q) \equiv (m/t)Q$ approximates the observable P for $t \to \infty$. Now Park and Margenau claim that, since in quantum mechanics only probabilistic predictions can be made, no further justification than (8.7) is needed for the interpretation of the measured values of $F(Q)$ as momentum values. But then one would have simultaneous arbitrarily sharp values for position and momentum. However, essentially for the same reason (i.e., the probabilistic nature of quantum mechanics) Jauch[5] and de Muynck *et al.*[31] argue that it is not possible to identify the values of $F(Q)$ as momentum values. This paradoxical situation—contradictory conclusions from one and the same premise—can be explained by taking into account the different interpretational background of the two parties. For Park and Margenau the measured values of observables have nothing to do with possible properties of physical systems which implies that their notion of measurement is weaker than that used in the present paper: Only the determinative aspect is relevant and allows Park and Margenau's conclusion. However, if one takes into account the preparatory aspect, that is, a stronger concept of measurement which still seems to be consistent with the quantum-mechanical formalism then, as emphasized by Jauch and de Muynck *et al.*, it is not admissible to speak of joint position and momentum values as possible properties of a physical system. Thus according to the view taken also in the present investigation the time-of-flight method fails to serve as an example of the second extreme view.

We wish to emphasize further that the existence of a vector state $\varphi \in \underline{H}_1$ for which $(\varphi, Q(X)\varphi) = 1$ and $(\varphi, P(Y)\varphi) \neq 0$ (for given bounded value sets X and Y), as demonstrated by Park and Margenau, does not contradict with the fact that $Q(X) \wedge P(Y) = 0$ [for any two bounded $X, Y \in \underline{B}(\underline{R})$] which, in our approach excludes the joint measurements. Furthermore, it is also fully consistent with probabilistic complementarity (3.1) of position Q and momentum P.

To see which observables according to our approach really become approximately equal, we write (8.7) in a modified form. Let $X = [-d, +d]$. Then

$$\operatorname{tr}\left[\alpha Q(X) U_t^+ Q\left[\left[\frac{p_1 t}{m}, \frac{p_2 t}{m} \right] \right] U_t Q(X) \right] \underset{t\to\infty}{\longrightarrow} \operatorname{tr}[\alpha Q(X) P((p_1,p_2)) Q(X)] \quad (8.8)$$

for all α in $T_s(\underline{H})_1^+$, that is,

$$Q(X) U_t^+ Q\left[\left[\frac{p_1 t}{m}, \frac{p_2 t}{m} \right] \right] U_t Q(X) \to Q(X) P((p_1,p_2)) Q(X) \quad (8.9)$$

as $t \to \infty$, weakly. On one hand, from this it becomes clear how the time-of-flight method represents an intuitive illustration of the concept of momentum, or velocity. However, on the other hand, it is seen that strictly speaking there is no approximation of momentum by means of some function of position.

An argument similar to that of Park and Margenau has been given by Ballentine[7] in terms of the slit experiment which essentially is the same type of experiment as the time-of-flight experiment: A localization (slit), followed by waiting, then again localization (screen). For the same reasons as above the assignment of some calculated momentum value is *ad hoc* and has nothing to do with a momentum measurement.

So we are left with the type of example given in the first part of this section: According to quantum theory there are (probably) no joint measurements violating the uncertainty relations which give sharp values but only such measurements which yield probability estimates for pairs of sharp values.

Note added in proof: After a correction in the paper by Busch (Ref. 18, to be published in J. Math. Phys.), statement (6.6b) can no longer be held up. However, the line of argument in Secs. VI to VIII in the present paper remains true if second-type Jauch couples (Q_f, P_g) are defined as follows: there exist bounded $X, Y \in \underline{B}(\underline{R})$ such that

$$\operatorname{ran}[Q_f(X)^{1/2}] \cap \operatorname{ran}[P_g(Y)^{1/2}] \neq \{0\} .$$

[1]W. Heisenberg, Z. Phys. 43, 172 (1927).
[2]W. Pauli, *General Principles of Quantum Mechanics* (Springer, Berlin, 1980). (Translated from the 1958 edition of *Die allgemeinen Prinzipien der Wellenmechanik* of 1933.)
[3]N. Bohr, Phys. Rev. 48, 696 (1935).
[4]P. Suppes, Philos. Sci. 22, 378 (1961).
[5]J. M. Jauch, in *Logic and Probability in Quantum Mechanics,* edited by P. Suppes (Reidel, Dordrecht, 1976), pp. 123–146. [Originally published in *Synthese* 29, 331 (1974).]
[6]J. Park and H. Margenau, Int. J. Theor. Phys. 1, 211 (1968).
[7]L. E. Ballentine, Rev. Mod. Phys. 42, 358 (1970).
[8]B. Mielnik, Commun. Math. Phys. 15, 1 (1969).
[9]E. B. Davies and J. T. Lewis, Commun. Math. Phys. 17, 239 (1970).
[10]C. Edwards, Commun. Math. Phys. 16, 207 (1970).
[11]E. B. Davies, *Quantum Theory of Open Systems* (Academic, London, 1976).
[12]E. G. Beltrametti and G. Cassinelli, *The Logic of Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1981).
[13]K. Kraus, Ann. Phys. (N.Y.) 64, 311 (1971).
[14]G. Süssmann, Abh. Bayer. Akad. Wiss. 88, (1958).
[15]P. J. Lahti, Int. J. Theor. Phys. 19, 789 (1980).
[16]P. J. Lahti, Int. J. Theor. Phys. 22, 911 (1983).
[17]P. J. Lahti, Report No. Turku-FTL-R43, 1983 (unpublished).
[18]P. Busch, J. Math. Phys. (to be published).
[19]P. Busch and P. J. Lahti, (unpublished).
[20]M. Jammer, Philos. Sci. 49, 478 (1982).
[21]M. Jammer, *The Conceptual Development of Quantum Mechanics* (McGraw-Hill, New York, 1966), p. 330.
[22]W. Heisenberg, *The Physical Principles of the Quantum Theory* (Dover, New York, 1949), pp. 26 and 27. (Originally published by the University of Chicago Press in 1930.)
[23]N. Bohr, in *Albert Einstein: Philosopher-Scientist* (The Library of Living Philosophers, Evanston, Illinois, 1949), pp. 218–224.
[24]S. Ali and G. Emch, J. Math. Phys. 15, 176 (1974).
[25]E. Prugovecki, J. Math. Phys. 17, 1673 (1976).
[26]S. Ali and E. Prugovecki, J. Math. Phys. 18, 219 (1977).
[27]P. Busch (unpublished).
[28]G. Ludwig, in *Structure and Approximation in Physical Theories,* edited by A. Hartkämper and H.-J. Schmidt (Plenum, New York, 1982), pp. 7–19.
[29]P. Busch, Doctoral dissertation, Köln (1982).
[30]F. Schroeck, Found. Phys. 12, 825 (1982).
[31]W. de Muynck, P. Janssen, and A. Santman, Found. Phys. 9, 71 (1979).