Vacuum $\langle \phi^2 \rangle$ in Schwarzschild spacetime

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For a conformally coupled scalar field we calculate the renormalized value of $\langle \phi^2 \rangle$ for the Hartle-Hawking vacuum in Schwarzschild spacetime for the region exterior to the horizon. We find that the mode-sum expression for $\langle \phi^2 \rangle_{\rm ren}$ separates naturally into two parts, a part that has a simple analytic form coinciding with the approximate expression of Whiting and Page, and a remainder that is small. We evaluate the remainder numerically and also provide an analytic approximation to it. Our results agree with, but are substantially more accurate than, those previously given by Fawcett.

I. INTRODUCTION

In this article we examine, for the case of a conformally coupled scalar field, the renormalized value of $\langle \phi^2 \rangle$ in the Hartle-Hawking vacuum for the region exterior to a Schwarzschild black hole. We regard this calculation as an exercise preparatory to the computation of the renormalized vacuum expectation value of the stress-energy tensor, to which we shall return elsewhere, but which is also interesting in its own right since the expression we obtain for $\langle \phi^2 \rangle$ divides naturally into two parts:

$$\langle \phi \rangle^2 = \frac{1}{12(8\pi M)^2} \frac{[1-(2M/r)^4]}{(1-2M/r)} + \frac{\Delta(r)}{(8\pi M)^2} , \quad (1.1)$$

where the first part, as has been noted by Page,[1] may be written in the interesting form

$$\frac{1}{12(8\pi M)^2} \frac{[1-(2M/r)^4]}{(1-2M/r)} = \tfrac{1}{12}({T_{\rm loc}}^2 - {T_{\rm acc}}^2) , \quad (1.2)$$

where

$$T_{\rm loc} = \frac{1}{8\pi M} \left[1-\frac{2M}{r}\right]^{-1/2} \quad (1.3)$$

is the local temperature at radius r of the Hawking blackbody radiation and

$$T_{\rm acc} = \frac{1}{2\pi}\frac{M}{r^2} \left[1-\frac{2M}{r}\right]^{-1/2} \quad (1.4)$$

is the Unruh acceleration temperature appropriate to an observer whose proper acceleration has the value

$$\frac{M}{r^2}\left[1-\frac{2M}{r}\right]^{-1/2}$$

required to maintain a constant radius r.

Equation (1.2) provides an appealing physical interpretation for the first term on the right-hand side (RHS) of Eq. (1.1). The term $\frac{1}{12} {T_{\rm loc}}^2$, having the same form as $\langle \phi^2 \rangle_{\rm ren}$ would have in Minkowski spacetime at a temperature T, may be thought of as being the contribution from the Hawking blackbody radiation. The term $-\frac{1}{12} {T_{\rm acc}}^2$ may be compared to the form of $\langle \phi^2 \rangle_{\rm ren}$ in Minkowski spacetime for the case of an accelerated (Fulling) vacuum.[2] This interpretation provides a surprisingly clear division into a "real-particle" contribution and a pure "vacuum-polarization" contribution to $\langle \phi^2 \rangle$. This might not have been anticipated since the dominant wavelengths contributing to $\langle \phi^2 \rangle_{\rm ren}$ are of order M and hence are of the same order as the length over which the geometry varies. Curvature terms might have been expected to make this splitting meaningless. Instead they make a contribution which is embodied in $\Delta(r)$, the second term on the RHS of Eq. (1.1), which nowhere exceeds 1% of the first term.

II. THE RENORMALIZATION OF $\langle \phi^2 \rangle$

We shall calculate $\langle \phi^2 \rangle$ by taking a coincidence limit of the Hartle-Hawking propagator

$$G_H(x,x') = i\langle H \mid T\phi(x)\phi(x') \mid H \rangle . \quad (2.1)$$

It is shown in Ref. 3 that, for imaginary values of the Schwarzschild "time" coordinate $t=-i\tau$, this propagator may be expressed in the form

$$G_H(-i\tau,r,\theta,\phi;-i\tau',r',\theta',\phi') = \frac{i}{32\pi^2M^2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)P_l(\xi_<)Q_l(\xi_>)$$
$$+\frac{i}{32\pi^2M^2}\sum_{n=1}^{\infty}\frac{1}{n}\cos n\kappa(\tau-\tau')\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)p_l^n(\xi_<)q_l^n(\xi_>) , \quad (2.2)$$

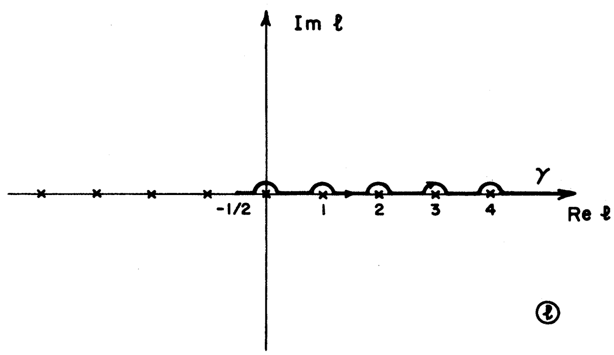

FIG. 1. The curve γ used in converting the l sum to a contour integral.

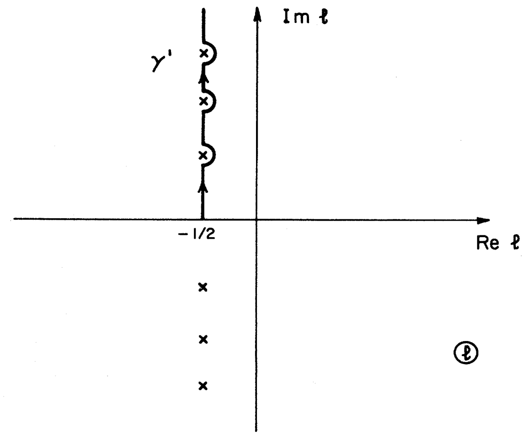

FIG. 2. The curve γ' used in evaluating $\mathscr{I}_n(\xi)$. The crosses correspond to the poles of the function $q_l^n(\xi)$.

where

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')\ ,$$

$$\xi=\frac{r}{M}-1\ ,$$

and $\xi_<$ and $\xi_>$ denote the smaller and the greater of ξ and ξ'. P_l and Q_l are Legendre functions and p_l^n and q_l^n are solutions of the radial equation

$$\left[\frac{d}{d\xi}(\xi^2-1)\frac{d}{d\xi}-l(l+1)-\frac{n^2}{16}\frac{(1+\xi)^4}{(\xi^2-1)}\right]R(\xi)=0 \qquad (2.3)$$

specified by the requirements that, for $n>0$, $p_l^n(\xi)$ is the solution that remains bounded as $\xi\to 1$ and $q_l^n(\xi)$ is the solution that tends to zero as $\xi\to\infty$. These solutions are normalized by requiring

$$p_l^n(\xi)\sim(\xi-1)^{n/2}\ ,$$
$$q_l^n(\xi)\sim(\xi-1)^{-n/2}\quad\text{as } \xi\to 1\ . \qquad (2.4)$$

The geodesic point-splitting scheme of DeWitt[4] and Christensen[5] yields the renormalized value of $\langle\phi^2\rangle$ as the limit

$$\langle H\,|\,\phi^2(x)\,|\,H\rangle_{\rm ren}=\lim_{x'\to x}\left[-iG_H(x,x')-\frac{1}{8\pi^2\sigma(x,x')}\right]\ , \qquad (2.5)$$

where $\sigma(x,x')$ denotes the geodetic interval between x and x'.

We wish to set $x=(-i\tau,r,\theta,\phi)$ and $x'=(-i(\tau+\epsilon),r,\theta,\phi)$ and to take the limit $\epsilon\to 0$. The representation (2.2) for the propagator is not immediately amenable to this process, since the partial coincidence limit is represented as

$$G_H(-i\tau,r,\theta,\phi;-i(\tau+\epsilon),r,\theta,\phi)=\frac{i}{32\pi^2M^2}\sum_{l=0}^{\infty}(2l+1)P_l(\xi)Q_l(\xi)+\frac{i}{32\pi^2M^2}\sum_{n=1}^{\infty}\frac{1}{n}\cos n\kappa\epsilon\sum_{l=0}^{\infty}(2l+1)p_l^n(\xi)q_l^n(\xi)\ . \qquad (2.6)$$

The problem is that each of the l sums diverges even though the left-hand side (LHS) of this equation is finite and unambiguous for each nonzero ϵ. A simple resolution of this difficulty is afforded by the observation that for each *nonzero* ϵ

$$\sum_{n=1}^{\infty}\cos n\kappa\epsilon=-\tfrac{1}{2}\ . \qquad (2.7)$$

This enables us to rewrite (2.6) as a convergent sum,

$$G_H(-i\tau,r,\theta,\phi;-i(\tau+\epsilon),r,\theta,\phi)=\frac{i}{32\pi^2M^2}\sum_{n=1}^{\infty}\cos n\kappa\epsilon\sum_{l=0}^{\infty}(2l+1)\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-2P_l(\xi)Q_l(\xi)\right]$$
$$=\frac{i}{32\pi^2M^2}\sum_{n=1}^{\infty}\cos n\kappa\epsilon\sum_{l=0}^{\infty}(2l+1)\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]\ , \qquad (2.8)$$

where in passing to the final form we have made use of the identity

$$\sum_{l=0}^{\infty}\left[(2l+1)P_l(\xi)Q_l(\xi)-\frac{1}{(\xi^2-1)^{1/2}}\right]=0\ , \qquad (2.9)$$

which is established in the Appendix.

The subtraction term takes the form

$$\frac{1}{8\pi^2\sigma(x,x')}=\frac{1}{4\pi^2}\left[\left[\frac{\xi+1}{\xi-1}\right]\frac{1}{\epsilon^2}+\frac{1}{12M^2(\xi+1)^2(\xi^2-1)}\right]+O(\epsilon^2)$$
$$=-\frac{\kappa^2}{4\pi^2}\left[\frac{\xi+1}{\xi-1}\right]\left\{\sum_{n=1}^{\infty}n\cos n\kappa\epsilon+\frac{1}{12}\left[1-\left[\frac{2}{\xi+1}\right]^4\right]\right\}+O(\epsilon^2)\,, \tag{2.10}$$

where, in passing to the last equality, we have set $\kappa=(4M)^{-1}$ and have employed the identity

$$\frac{1}{\epsilon^2}=-\kappa^2\sum_{n=1}^{\infty}n\cos n\kappa\epsilon-\frac{\kappa^2}{12}+O(\epsilon^2)\,. \tag{2.11}$$

Performing the subtraction indicated in (2.5) we obtain

$$\langle H\,|\,\phi^2(x)\,|\,H\rangle_{\text{ren}}=\frac{1}{12(8\pi M)^2}\frac{[1-(2M/r)^4]}{(1-2M/r)}+\frac{\Delta(r)}{(8\pi M)^2}\,, \tag{2.12}$$

where

$$\Delta(r)=2\sum_{n=1}^{\infty}\left\{\sum_{l=0}^{\infty}\left[(2l+1)\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]+\frac{n}{2}\left[\frac{\xi+1}{\xi-1}\right]\right\}. \tag{2.13}$$

We shall now show that this sum converges. The l sum presents no problems. For large l we see, from the WKB approximants, that the term in square brackets is $O(l^{-2})$. The n sum requires a more detailed analysis. We shall show that, in virtue of certain surprising cancellations, the term in curly brackets is $O(n^{-5})$ for large n.

In order to discuss the convergence of the n sum it is convenient to convert the l sum to a contour integral. We write

$$\sum_{l=0}^{\infty}\left[(2l+1)\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]=-\text{Re}\left[\frac{1}{\pi i}\int_\gamma dl\pi\cot\pi l\left[(2l+1)\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]\right] \tag{2.14}$$

with γ the contour of Fig. 1. We further write

$$\cot\pi l=-i+\frac{2i}{1-e^{-2\pi i l}} \tag{2.15}$$

and thereby divide the integral into two terms. The first of these may be taken to run along the real axis. The analytic properties of the functions $p_l^n(\xi)$ and $q_l^n(\xi)$ qua functions of l, specifically that $p_l^n(\xi)$ is analytic on the entire l plane and that $q_l^n(\xi)$ is analytic on the l plane apart from isolated simple poles that lie on the line $\text{Re}l=-\frac{1}{2}$,[6] permit the second integral to be rotated to the contour γ' of Fig. 2. Thus, we find

$$\sum_{l=0}^{\infty}\left[(2l+1)\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]=\mathscr{I}_n(\xi)+\mathscr{J}_n(\xi)\,, \tag{2.16}$$

where

$$\mathscr{I}_n(\xi)=\int_{-1/2}^{\infty}dl\left[(2l+1)\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right] \tag{2.17}$$

and

$$\mathscr{J}_n(\xi)=4\mathscr{P}\int_0^{\infty}\frac{d\lambda\lambda}{e^{2\pi\lambda}+1}\frac{1}{n}p^n_{-1/2+i\lambda}(\xi)q^n_{-1/2+i\lambda}(\xi) \tag{2.18}$$

with $\mathscr{P}$ denoting the principal value.

In order to exhibit the large-n behavior of $\mathscr{I}_n(\xi)$ we subtract and add again the third-order WKB approximant to the product $(1/n)p_l^n(\xi)q_l^n(\xi)$ (the successive WKB approximants are derived in the Appendix and presented in Table II):

$$\mathscr{I}_n(\xi)=\int_{-1/2}^{\infty}dl(2l+1)[p_l^n(\xi)q_l^n(\xi)-W_l^{(1)n}(\xi)-W_l^{(2)n}(\xi)-W_l^{(3)n}(\xi)]$$
$$+\int_{-1/2}^{\infty}dl\left[(2l+1)[W_l^{(1)n}(\xi)+W_l^{(2)n}(\xi)+W_l^{(3)n}(\xi)]-\frac{2}{(\xi^2-1)^{1/2}}\right]. \tag{2.19}$$

The second integral in (2.19) may be evaluated explicitly. Thus, we find

$$\mathscr{I}_n(\xi)=\int_{-1/2}^{\infty}dl(2l+1)\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-W_l^{(1)n}(\xi)-W_l^{(2)n}(\xi)-W_l^{(3)n}(\xi)\right]$$
$$-\frac{n}{2}\left[\frac{\xi+1}{\xi-1}\right]-\frac{1}{3n(\xi+1)^2}-\frac{(\xi-1)(17\xi-63)}{15n^3(\xi+1)^6}\ . \tag{2.20}$$

We observe that the term linear in n will cancel against the last term in the curly brackets in (2.13).

Turning now to $\mathscr{J}_n(\xi)$ we note that in view of the exponential factor in (2.18) the dominant contribution to the integral arises in the neighborhood of $\lambda=0$; in order to identify the leading-n behavior we write

$$\mathscr{J}_n(\xi)=4\mathscr{P}\int_0^{\infty}\frac{d\lambda\lambda}{e^{2\pi\lambda}+1}\left[\frac{1}{n}p^n_{-1/2+i\lambda}(\xi)q^n_{-1/2+i\lambda}(\xi)-W^{(1)n}_{-1/2}(\xi)-\left[W^{(2)n}_{-1/2}(\xi)+\frac{\lambda^2}{2}\frac{\partial^2}{\partial\lambda^2}W^{(1)n}_{-1/2}(\xi)\right]\right]$$
$$+4\int_0^{\infty}\frac{d\lambda\lambda}{e^{2\pi\lambda}+1}\left[W^{(1)n}_{-1/2}(\xi)+\left[W^{(2)n}_{-1/2}(\xi)+\frac{\lambda^2}{2}\frac{\partial^2}{\partial\lambda^2}W^{(1)n}_{-1/2}(\xi)\right]\right]\ , \tag{2.21}$$

where we have separated off terms corresponding to an expansion of the second-order WKB approximant about $\lambda=0$. This isolates the $O(n^{-1})$ and $O(n^{-3})$ terms in the second integral which may be evaluated explicitly:

$$4\int_0^{\infty}\frac{d\lambda\lambda}{e^{2\pi\lambda}+1}\left[W^{(1)n}_{-1/2}(\xi)+\left[W^{(2)n}_{-1/2}(\xi)+\frac{\lambda^2}{2}\frac{\partial^2}{\partial\lambda^2}W^{(1)n}_{-1/2}(\xi)\right]\right]=\frac{1}{3n(\xi+1)^2}+\frac{(\xi-1)(17\xi-63)}{15n^3(\xi+1)^6}\ . \tag{2.22}$$

Note that these terms exactly cancel the corresponding terms in (2.20). The cancellation of the $O(n^{-1})$ term was to be expected since otherwise the n sum in Eq. (2.13) would not converge. However, the cancellation of the $O(n^{-3})$ we find surprising. Assembling these various results we have

$$\Delta(r)=2\sum_{n=1}^{\infty}[\bar{\mathscr{I}}_n(\xi)+\bar{\mathscr{J}}_n(\xi)]\ , \tag{2.23}$$

where

$$\bar{\mathscr{I}}_n(\xi)=\int_{-1/2}^{\infty}dl(2l+1)\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-W_l^{(1)n}(\xi)-W_l^{(2)n}(\xi)-W_l^{(3)n}(\xi)\right] \tag{2.24}$$

and

$$\bar{\mathscr{J}}_n(\xi)=4\mathscr{P}\int_0^{\infty}\frac{d\lambda\lambda}{e^{2\pi\lambda}+1}\left[\frac{1}{n}p^n_{-1/2+i\lambda}(\xi)q^n_{-1/2+i\lambda}(\xi)-W^{(1)n}_{-1/2}(\xi)-\left[W^{(2)n}_{-1/2}(\xi)+\frac{\lambda^2}{2}\frac{\partial^2}{\partial\lambda^2}W^{(1)n}_{-1/2}(\xi)\right]\right]\ . \tag{2.25}$$

Our analysis has shown that both $\bar{\mathscr{I}}_n(\xi)$ and $\bar{\mathscr{J}}_n(\xi)$ are $O(n^{-5})$ for large n. Thus far our results are exact. We may at this point approximate $\bar{\mathscr{I}}_n(\xi)$ and $\bar{\mathscr{J}}_n(\xi)$ by replacing the integrands in (2.24) and (2.25) by WKB terms of one higher order. This amounts to approximating $\bar{\mathscr{I}}_n(\xi)$ and $\bar{\mathscr{J}}_n(\xi)$ by the $O(n^{-5})$ terms and yields the asymptotic form, valid for r large,

$$\Delta(r)\sim\Delta_5(r)\ , \tag{2.26}$$

where

$$\Delta_5(r)=\tfrac{9}{28}(\xi-1)^3\left[\frac{2}{\xi+1}\right]^{11}\zeta(5)\ . \tag{2.27}$$

It might be expected, on the basis of the fact that the criterion for the validity of the WKB approximation to the solutions of the radial equation holds both as $r\to\infty$ and as $r\to 2M$, that $\Delta_5(r)$ would also asymptotically approximate $\Delta(r)$ near the horizon. This is not the case as evidenced by the fact that $\Delta_5(r)$ has a vanishing derivative at $r=2M$ whereas it is known from the work of Fawcett[7] and Page[1] that the derivative of $\Delta(r)$ at $r=2M$ cannot be zero. The origin of this breakdown in WKB approximation seems to be associated with the fact that the radial function $q_l^n(\xi)$ contains a logarithm which is not present in its WKB approximation.[8] The higher WKB terms do not remedy this fact. The next-order approximation to $\Delta(r)$, due to the $O(n^{-7})$ terms in (2.23), is

$$\Delta_7(r)=\tfrac{3}{4}(\xi-1)^3\left[\frac{2}{\xi+1}\right]^{15}(4\xi^2-18\xi+17)\zeta(7)\ . \tag{2.28}$$

Near the horizon $\Delta_7(r)$ is again $O((\xi-1)^3)$ but with a larger coefficient than $\Delta_5(r)$. It seems that this behavior persists in all higher terms.

III. NUMERICAL EVALUATION OF $\Delta(r)$

The integral expression (2.23) for $\Delta(r)$, while displaying clearly the large-n behavior of the contributions to $\Delta(r)$, is not amenable to numerical evaluation. In order to bring $\Delta(r)$ into a more suitable form we return to the sum (2.13) from which we subtract and then add the third-order WKB approximant to $p_l^n(\xi)q_l^n(\xi)$ thereby obtaining

TABLE I. Values of $\langle H \mid \phi^2 \mid H \rangle$ and $\Delta(r)$.

ξ	$(8\pi M)^2\langle\phi^2\rangle$	$\Delta(r)$	$\Delta_5(r)$	ξ	$(8\pi M)^2\langle\phi^2\rangle$	$\Delta(r)$	$\Delta_5(r)$
1.0	0.33333	0.00000	0.00000	4.1	0.13417	0.00032	0.00033
1.1	0.31087	0.00060	0.00019	4.2	0.13275	0.00030	0.00030
1.2	0.29156	0.00099	0.00092	4.3	0.13141	0.00029	0.00026
1.3	0.27485	0.00125	0.00193	4.4	0.13012	0.00026	0.00024
1.4	0.26027	0.00140	0.00287	4.5	0.12890	0.00024	0.00021
1.5	0.24749	0.00149	0.00357	4.6	0.12774	0.00022	0.00019
1.6	0.23620	0.00152	0.00401	4.7	0.12664	0.00021	0.00017
1.7	0.22617	0.00151	0.00421	4.8	0.12558	0.00019	0.00015
1.8	0.21723	0.00149	0.00421	4.9	0.12458	0.00018	0.00013
1.9	0.20920	0.00143	0.00407	5.0	0.12363	0.00017	0.00012
2.0	0.20199	0.00137	0.00385	5.1	0.12271	0.00016	0.00011
2.1	0.19547	0.00131	0.00357	5.2	0.12183	0.00015	0.00010
2.2	0.18953	0.00122	0.00327	5.3	0.12099	0.00014	0.00009
2.3	0.18417	0.00117	0.00296	5.4	0.12019	0.00013	0.00008
2.4	0.17925	0.00110	0.00266	5.5	0.11941	0.00012	0.00007
2.5	0.17474	0.00103	0.00238	5.6	0.11867	0.00011	0.00006
2.6	0.17060	0.00096	0.00212	5.7	0.11795	0.00010	0.00006
2.7	0.16679	0.00090	0.00188	5.8	0.11726	0.00009	0.00005
2.8	0.16327	0.00084	0.00166	5.9	0.11661	0.00009	0.00005
2.9	0.16000	0.00078	0.00147	6.0	0.11597	0.00008	0.00004
3.0	0.15697	0.00072	0.00130	6.1	0.11536	0.00008	0.00004
3.1	0.15417	0.00068	0.00115	6.2	0.11477	0.00007	0.00004
3.2	0.15154	0.00063	0.00101	6.3	0.11420	0.00007	0.00003
3.3	0.14910	0.00059	0.00089	6.4	0.11365	0.00006	0.00003
3.4	0.14680	0.00054	0.00079	6.5	0.11312	0.00006	0.00003
3.5	0.14466	0.00051	0.00070	6.6	0.11260	0.00005	0.00002
3.6	0.14266	0.00049	0.00061	6.7	0.11211	0.00005	0.00002
3.7	0.14148	0.00043	0.00054	6.8	0.11163	0.00005	0.00002
3.8	0.13895	0.00040	0.00048	6.9	0.11116	0.00004	0.00002
3.9	0.13727	0.00037	0.00042	7.0	0.11072	0.00004	0.00002
4.0	0.13568	0.00035	0.00038				

$$\Delta(r)=2\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}(2l+1)\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-W_l^{(1)n}(\xi)-W_l^{(2)n}(\xi)-W_l^{(3)n}(\xi)\right]+2\sum_{n=1}^{\infty}[U_n(\xi)+V_n(\xi)]\;, \tag{3.1}$$

where we have defined

$$U_n(\xi)=\sum_{l=0}^{\infty}\left[(2l+1)W_l^{(1)n}(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]+\frac{n}{2}\left[\frac{\xi+1}{\xi-1}\right] \tag{3.2}$$

and

$$V_n(\xi)=\sum_{l=0}^{\infty}(2l+1)[W_l^{(2)n}(\xi)+W_l^{(3)n}(\xi)]\;. \tag{3.3}$$

The first sum in the expression for $\Delta(r)$ now converges rapidly and we shall return to its numerical evaluation. We turn now, however, to a consideration of the contribution U_n which is defined by Eq. (3.2) in terms of an l sum which converges only slowly, the summand being $O(l^{-2})$ for large l. It is therefore convenient to replace the l sum by a contour integral in a manner analogous to the process that leads to Eq. (2.16). The integral along the real axis precisely cancels the third term in the definition of U_n. Thus, we find

$$U_n(\xi)=\frac{4}{(\xi^2-1)^{1/2}}\int_0^{\Lambda}\frac{d\lambda\lambda}{(\Lambda^2-\lambda^2)^{1/2}(e^{2\pi\lambda}+1)}\;, \tag{3.4}$$

where

$$\Lambda_n(\xi)=\frac{n(1+\xi)^2}{4(\xi^2-1)^{1/2}}\;, \tag{3.5}$$

which may be evaluated without difficulty. There is also a term (see Table I),

$$W_l^{(2a)n}(\xi)=\frac{1}{8[\chi_l^n(\xi)]^3}\;, \tag{3.6}$$

which caused slow convergence in the sum (3.3). It is convenient, therefore, to deal separately with this contribution by means of a contour integral; after deforming the contour to γ'' (see Fig. 3) we find

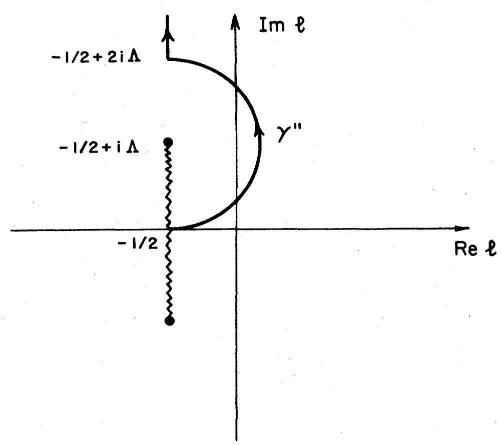

FIG. 3. The curve γ'' used to evaluate Eq. (3.7). The branch cut is associated with the zeros of $\chi_l^n(\xi)$.

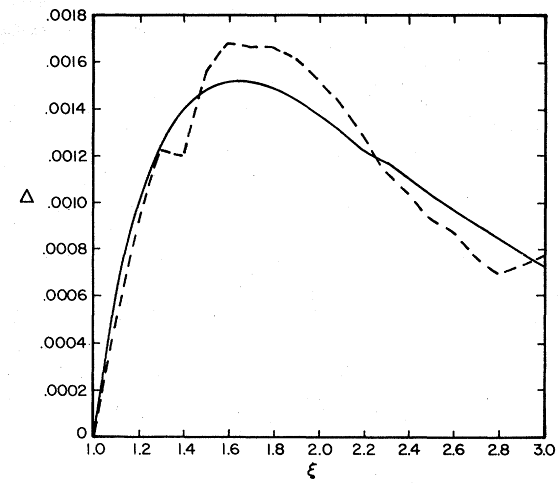

FIG. 4. Δ as a function of ξ. The dashed line represents Fawcett's results.

$$\sum_{l=0}^{\infty}(2l+1)W_l^{(2a)n}(\xi)=\frac{1}{n(\xi+1)^2(\xi^2-1)}+\frac{1}{\Lambda(\xi^2-1)^{3/2}}\mathrm{Re}\int_0^{\pi}\frac{d\theta\sin\theta/2}{(2-e^{i\theta})^{3/2}\{1+\exp[2\pi\Lambda(1-e^{i\theta})]\}}\ , \tag{3.7}$$

which, despite the appearance of the integral, is also very easily evaluated. The remaining parts of $W_l^{(2)n}(\xi)$ are $O(l^{-5})$ for large l as is $W_l^{(3)n}(\xi)$. Their contribution to $V_n(\xi)$ may be rapidly evaluated by direct summation.

Returning to the evaluation of the sum

$$\sum_{l=0}^{\infty}(2l+1)\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-W_l^{(1)n}(\xi)-W_l^{(2)n}(\xi)-W_l^{(3)n}(\xi)\right]$$

we observe that convergence presents no problems since the summand is $O(l^{-6})$ for large l. It is important, however, to evaluate the product $p_l^n(\xi)q_l^n(\xi)$ accurately in view of the significant cancellation against the WKB approximant.

As a consequence of the Wronskian relation between $p_l^n(\xi)$ and $q_l^n(\xi)$ we have the integral relation

$$q_l^n(\xi)=2np_l^n(\xi)\int_{\xi}^{\infty}\frac{dx}{(x^2-1)[p_l^n(x)]^2}\ . \tag{3.8}$$

For values of its argument less than 3.0 $p_l^n(\xi)$ may be evaluated by summing its series representation and for values of its argument greater than 3.0 $p_l^n(\xi)$ is obtained by a fourth-order Runge-Kutta integration procedure. Transformation of the integral to a finite range and an extended six-point integration routine efficiently achieves an accuracy of eight significant figures.

In virtue of the rapid convergence of the n sum established in Sec. II we find it suffices to evaluate numerically only the contributions corresponding to $n=1$ and $n=2$ and to approximate the remaining terms by their contribution to $\Delta_5(r)$.

The method outlined above allows $\langle\phi^2\rangle$ to be calculated to five-figure accuracy while using only two minutes of C.P.U. time. Our results are presented in Table I and depicted in Figs. 4 and 5.

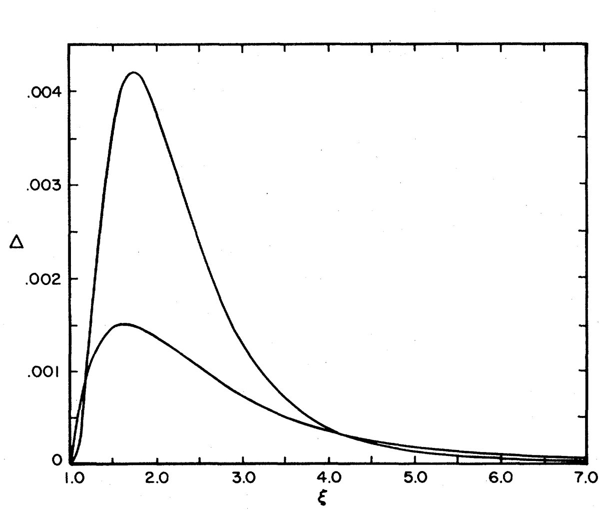

FIG. 5. Δ_5 and Δ as a function of ξ. Δ_5 is the curve with the higher peak.

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APPENDIX

We present, in this appendix, a procedure for evaluating the WKB approximants to the product

TABLE II. The WKB approximants.

$$\chi^2=(l+\tfrac{1}{2})^2(\xi^2-1)+\frac{n^2}{16}(1+\xi)^4,\quad \chi_0=\frac{n}{4}(1+\xi)^2$$

$$W^{(1)}=\frac{1}{\chi}$$

$$W^{(2)}=\frac{1}{8\chi^3}-\frac{\chi_0^{\,2}}{4\chi^5}(2\xi^2-6\xi+7)+\frac{5\chi_0^{\,4}}{8\chi^7}(\xi-2)^2$$

$$W^{(3)}=\frac{16\xi^2+11}{128\chi^5}-\frac{\chi_0^{\,2}}{32\chi^7}(16\xi^4-60\xi^3+88\xi^2-70\xi+171)$$
$$+\frac{7\chi_0^{\,4}}{64\chi^9}(56\xi^4-320\xi^3+773\xi^2-1020\xi+666)-\frac{231\chi_0^{\,6}}{32\chi^{11}}(2\xi^2-6\xi+7)(\xi-2)^2+\frac{1155\chi_0^{\,8}}{128\chi^{13}}(\xi-2)^4$$

$$W^{(4)}=\frac{1}{1024\chi^7}(128\xi^4+824\xi^2+173)$$
$$-\frac{\chi_0^{\,2}}{512\chi^9}(256\xi^6-1008\xi^5+1344\xi^4-776\xi^3+2058\xi^2-12510\xi+11901)$$
$$+\frac{3\chi_0^{\,4}}{1024\chi^{11}}(10112\xi^6-66976\xi^5+195648\xi^4-337216\xi^3+392289\xi^2-382324\xi+333472)$$
$$-\frac{11\chi_0^{\,6}}{256\chi^{13}}(6304\xi^6-53172\xi^5+198348\xi^4-430770\xi^3+599925\xi^2-536904\xi+248042)$$
$$+\frac{429\chi_0^{\,8}}{1024\chi^{15}}(1968\xi^4-11488\xi^3+28963\xi^2-37948\xi+23448)(\xi-2)^2$$
$$-\frac{255255\chi_0^{\,10}}{512\chi^{17}}(\xi-2)^4(2\xi^2-6\xi+7)+\frac{425425\chi_0^{\,12}}{1024\chi^{19}}(\xi-2)^6$$

$$\frac{1}{n}p_l^n(\xi)q_l^n(\xi)$$

together with a proof of the identity (2.9).

1. The WKB approximants

We set

$$\alpha^2(\xi)=\frac{1}{n}p_l^n(\xi)q_l^n(\xi)\ , \tag{A1}$$

choose a new radial variable z such that

$$\frac{d}{dz}=(\xi^2-1)\frac{d}{d\xi}\ , \tag{A2}$$

and write

$$\chi^2(\xi)=(l+\tfrac{1}{2})^2(\xi^2-1)+\frac{n^2}{16}(1+\xi)^4\ . \tag{A3}$$

It follows from the radial equation (2.3) and the Wronskian relation

$$p_l^n(\xi)\frac{d}{d\xi}q_l^n(\xi)-q_l^n(\xi)\frac{d}{d\xi}p_l^n(\xi)=-\frac{2n}{\xi^2-1} \tag{A4}$$

that α satisfies the nonlinear equation

$$\frac{d^2\alpha}{dz^2}-[h^2\chi^2-\tfrac{1}{4}(\xi^2-1)]\alpha+\frac{1}{\alpha^3}=0 \tag{A5}$$

with h an expansion parameter that will ultimately be set to unity. It is convenient to rewrite the differential equation in the form

$$\alpha=\chi^{-1/2}\left[1-\frac{(1/\alpha)d^2\alpha/dz^2+\frac{1}{4}(\xi^2-1)}{h^2\chi^2}\right]^{-1/4} \tag{A6}$$

and to solve iteratively, taking $\chi^{-1/2}$ as a first approximation. This procedure yields

$$\alpha^2\sim\sum_{k=1}^{\infty}\frac{W_l^{(k)n}(\xi)}{h^{2k-2}}\ . \tag{A7}$$

The first four $W^{(k)}$ are displayed in Table II.

2. The identity (2.9)

We shall here establish the relation

$$\sum_{l=0}^{\infty}\left[(2l+1)(-1)^nP_l^{-n}(\xi)Q_l^n(\xi)-\frac{1}{(\xi^2-1)^{1/2}}\right]=-\frac{n}{(\xi^2-1)^{1/2}} \tag{A8}$$

of which (2.9) is a special case. We first note the standard identity[9]

$$Q_l(\xi\xi'-(\xi^2-1)^{1/2}(\xi'^2-1)^{1/2}\cos\psi)=P_l(\xi_<)Q_l(\xi_>)+2\sum_{n=1}^{\infty}(-1)^n\cos(n\psi)P_l^{-n}(\xi_<)Q_l^n(\xi_>) . \tag{A9}$$

Taking the Fourier inverse of this expansion permits us to express a product of Legendre functions in terms of a single Legendre function

$$P_l^{-n}(\xi_<)Q_l^n(\xi_>)=(-1)^n\int_0^{2\pi}\frac{d\psi}{2\pi}\cos(n\psi)Q_l(\xi\xi'-(\xi^2-1)^{1/2}(\xi'^2-1)^{1/2}\cos\psi) . \tag{A10}$$

This relation together with Heine's formula

$$\sum_{l=0}^{\infty}(2l+1)P_l(\mu)Q_l(\eta)=\frac{1}{\eta-\mu} , \tag{A11}$$

which, in particular, holds for real values of μ and η such that $\eta>\mu\geq 1$ allows us to evaluate the following sum:

$$(-1)^n\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)P_l^{-n}(\xi_<)Q_l^n(\xi_>)=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\psi\cos n\psi}{(\xi\xi'-\cos\gamma)-(\xi^2-1)^{1/2}(\xi'^2-1)^{1/2}\cos\psi}$$

$$=\frac{2e^{-nu}}{(\xi^2+\xi'^2-2\xi\xi'\cos\gamma-\sin^2\gamma)^{1/2}} , \tag{A12}$$

where, in the last equality,

$$u=\cosh^{-1}\left[\frac{\xi\xi'-\cos\gamma}{(\xi^2-1)^{1/2}(\xi'^2-1)^{1/2}}\right] . \tag{A13}$$

If we now set $\xi'=\xi$ and let $\gamma\to 0$ we find

$$(-1)^n\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)P_l^{-n}(\xi)Q_l^n(\xi)$$

$$=\frac{1}{\gamma(\xi^2-1)^{1/2}}-\frac{n}{\xi^2-1}+O(\gamma) . \tag{A14}$$

Finally we note that

$$\sum_{l=0}^{\infty}P_l(\cos\gamma)=\frac{1}{2\sin(\gamma/2)} . \tag{A15}$$

By multiplying this identity by $(\xi^2-1)^{-1/2}$, subtracting from (A14), and letting $\gamma\to 0$ we establish the desired result (A8).

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[1] D. N. Page, Phys. Rev. D 25, 1499 (1982).
[2] P. Candelas and D. Deutsch, Proc. R. Soc. London A354, 79 (1979). The quantity calculated in this article is $\langle T_{\mu\nu}\rangle$; the form of $\langle\phi^2\rangle_{\text{ren}}$ follows straightforwardly.
[3] P. Candelas, Phys. Rev. D 21, 2185 (1980).
[4] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).
[5] S. M. Christensen, Phys. Rev. D 14, 2490 (1976).
[6] P. Candelas (paper in preparation).
[7] M. S. Fawcett, Ph.D. thesis, University of Cambridge, Cambridge, England.
[8] We are grateful to B. Whiting for impressing this fact upon us.
[9] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. R. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953).