Renormalization-group analysis of grand unified theories in curved spacetime

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We consider a class of grand unified theories (GUT's) based on the Georgi-Glashow model in curved spacetime. We are particularly concerned with the coupling constants involving the curvature. These include the cosmological and gravitational constants, as well as coupling constants appearing in terms quadratic in the curvature and in terms which link the Higgs bosons to the scalar curvature. For asymptotically free theories, we use the renormalization group to obtain expressions for these effective coupling constants at high curvature (between the GUT and Planck scales). We discuss the role of the effective coupling constants in the gravitational field equations. These results may be of importance for cosmology.

I. INTRODUCTION

One of the areas of recent interest is the role played by grand unified theories in the evolution of the early universe. This leads to a closer examination of interacting quantum field theories in curved spacetime. Of concern is the behavior of the theories at high curvature, since this is where any effects peculiar to curved spacetime would be expected to have the most dramatic consequences. The high-curvature limit is of relevance to the final stages of evaporation of a black hole, as well as to the early universe.

In this paper, we use renormalization-group methods to study the high-curvature limit of a class of grand unified theories (GUT's). We focus in particular on the effective coupling constants which multiply curvature invariants in the action. These couplings include the cosmological constant and the gravitational constant, as well as couplings between the Higgs bosons and the scalar curvature. The behavior of these running coupling constants at high curvature is of significance for symmetry breaking and for the dynamics of cosmological models. We find that the values of these effective couplings at high curvature may be quite different from their present values.

The first question which arises for interacting quantum field theory in curved spacetime concerns renormalizability. Various scalar field theories have been investigated and shown to be renormalizable in curved spacetime if they are also renormalizable in flat spacetime.[1] The main new feature which arises is the possibility that additional local counterterms involving the curvature (and therefore vanishing in flat spacetime) may be needed. The renormalization of gauge theories in curved spacetime has also received some attention.[2–9] For gauge theories it is also found that the renormalizability is unaffected by spacetime curvature. Although no one has yet studied the renormalization of a gauge theory as complicated as those which we shall consider here, it is extremely likely that these theories are renormalizable in curved spacetime just as for the theories in Refs. 2–9.

The particular models which we shall study are the SU(5) model of Georgi and Glashow,[10] and a generalization of this theory due to Chang, Das, and Perez-Mercader[11] which has the same low-energy predictions, but which is totally asymptotically free. It is well known from results of 't Hooft,[12] Gross, and Wilczek,[13] and Politzer[14] that in the absence of too many fermions the gauge coupling constant is asymptotically free for non-Abelian gauge theories; however, in models with Higgs scalars, the quartic self-couplings of the scalar fields are not asymptotically free.[15] Results which involve nonasymptotically free couplings based upon perturbation theory may become unreliable at high energy. Chang, Das, and Perez-Mercader[11] (see also Fradkin and Kalashnikov[16]) proposed a version of the Georgi-Glashow model in which the Higgs self-couplings and the Yukawa couplings were proportional to a power of the gauge coupling constant. This leads to a one-coupling-constant theory in which all of the coupling constants are asymptotically free. As a consequence, predictions based upon perturbation theory are reliable at high momentum.

In flat spacetime the renormalization group allows the behavior of the Green's functions to be studied at high momentum. In a general curved spacetime there is no natural definition of momentum space so that it is not immediately clear how to proceed. (One possibility might be to use the momentum-space approach of Bunch and Parker.[17]) Nelson and Panangaden[18] have argued that the natural analog of the usual flat-spacetime procedure of scaling the external momenta[19] is to look at the behavior of the Green's functions under a rescaling $g_{\mu\nu} \rightarrow s^{-2} g_{\mu\nu}$ of the background metric.[20] In flat spacetime, increasing s gives the short-distance, or high-momentum limit. In curved spacetime, consideration of the curvature invariants shows that increasing s corresponds to the high-curvature limit. One of us has shown[21] how this analysis may be applied to the effective action, and this serves as the starting point for our discussion.

We have already remarked above that the renormalization of gauge theories should only be affected by the addition of extra local counterterms involving the curvature. As a consequence, the renormalization counterterms for all coupling constants appearing in the flat-spacetime theory are unchanged. In particular, the existence of asymptotic freedom is unaffected by curvature (or nontrivial spacetime topology). (See discussions in Refs. 7 and 21.) For theories which include scalar fields ϕ, it is necessary to add terms of the form $R\phi^2$ to the action, as well as terms which involve only the curvature. For ϕ^3 theory in six dimensions, which is also asymptotically free, it was demonstrated in Ref. 22 that the coupling constant of the $R\phi^2$ term had an ultraviolet fixed point given by its conformal value of $\frac{1}{5}$. Previous calculations of the high-curvature limit of the coupling constants which multiply pure gravitational terms are contained in Refs. 21 and 23. Other calculations of renormalization-group functions and applications of renormalization-group techniques to quantum field theory in curved spacetime include Refs. 24—35. The purpose of this paper is to use the curved-spacetime renormalization-group method to study the high-curvature limit of a realistic grand unified theory.

The results we obtain for the running coupling constants are most relevant between the GUT scale (at which full symmetry is restored) and the Planck scale. That is because we have worked with the fully symmetric phase. In addition, we have not quantized the gravitational field itself. As we shall see, the effective cosmological constant in the theories under consideration may become large and positive at high curvature (even if it is zero today). The effective coupling constants linking the Higgs bosons to the the scalar curvature approach the conformal value of $\frac{1}{6}$. Furthermore, if one considers the limit of arbitrarily large curvature, then the running gravitational constant (and hence the effective Planck time) may conceivably approach zero. We will discuss the magnitudes of these effects more fully elsewhere.[36]

The outline of our paper is the following. In Sec. II we present a short discussion of the background-field method and show how it may be used to compute the effective action. The extra terms in the action necessary for renormalizability in curved spacetime are written out in Eq. (2.1). The remainder of the action involving the Georgi-Glashow part as well as the additional terms required for total asymptotic freedom is given in Sec. III. We also list the equations satisfied by the various propagators in the theory. The evaluation of that part of the one-loop effective action relevant for obtaining the renormalization counterterms for the coupling constants involving $R\phi^2$-type terms for the Higgs fields is given in Sec. IV. Results for both the Georgi-Glashow model[10] and the generalization of Chang, Das, and Perez-Mercader[11] are presented. The actual counterterms are obtained in Sec. V. In Sec. VI we give a renormalization-group analysis of the running coupling constants in the $R\phi^2$ terms. The counterterms for the constants appearing in the generalized Einstein-Hilbert gravitational action are computed in Sec. VII, with a renormalization-group analysis of these terms given in Sec. VIII. In Sec. IX, we discuss the relation of the renormalization group to the high-curvature limit. The behavior of the effective Einstein equations at high curvature is obtained. The final section presents our conclusions. A number of technical details which are involved in our calculations are contained in the Appendices. Appendix A contains our curvature and group conventions. In Appendix B, we present an account of spinors in curved spacetime and our notation for two-component spinors. It is shown in Appendix C how the curved-spacetime momentum-space technique of Bunch and Parker[17] may be used to obtain a number of results which are necessary in order to calculate the one-loop counterterms. In Appendix D, the relation of our notation to that of Chang, Das, and Perez-Mercader[11] is given and the expressions for the coupling constants in the fully asymptotically free theory are quoted.

II. EFFECTIVE-ACTION, BACKGROUND-FIELD METHOD

As noted in the Introduction, the bare Lagrangian L is the sum of the generalized Einstein-Hilbert Lagrangian which involves only gravitational terms and the generalization to curved spacetime of a totally asymptotically free SU(5) grand unified theory. We first outline our procedure. There are two types of Higgs fields, denoted by Φ (24 components) and H (5 components), respectively. The bare couplings of the Higgs bosons to the curvature and the terms involving only the curvature are of the form

$$L_{\text{curv}} = -\xi_\phi R\,\text{tr}(\Phi^2) - \xi_H R H^\dagger H + \Lambda + \kappa R + \alpha_1 R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + \alpha_2 R^{\mu\nu} R_{\mu\nu} + \alpha_3 R^2 \,, \tag{2.1}$$

where ξ_ϕ, ξ_H, Λ, κ, and the α's are bare coupling constants [Λ and κ being related to the cosmological constant Λ_c and gravitational constant G by $\kappa = (16\pi G)^{-1}$ and $\Lambda = -(8\pi G)^{-1}\Lambda_c$]. In order to obtain the renormalization-group equations for these couplings, we must find the way in which these bare couplings split up into the sum of finite renormalized couplings and counterterms (which cancel the infinities of the quantized theory). For that purpose it is convenient to work with the effective action, using the background-field method.[37—40] Dimensional regularization[41—43] will be used to obtain the pole part of the relevant one-loop terms in the effective action. For the totally asymptotically free theory, the one-loop contributions dominate and are thus sufficient to determine the limiting values of the effective renormalized coupling constants. It will also be necessary to take into account certain features of the two-loop contributions to the renormalization-group equations for ξ_ϕ and ξ_H.

For grand unified theories, such as the Georgi-Glashow theory, which are only asymptotically free in the gauge coupling, it may still be possible to use the one-loop renormalization-group equations to predict the behavior of the running coupling constants in the relevant range of energies between the GUT and the Planck scales (if the nonasymptotically free effective couplings do not grow too large). We also remark that with a background gravitational field no terms of higher order in the curvature are induced by renormalization, so that Eq. (2.1) includes all the bare couplings involving the curvature.

The effective action Γ is obtained from the vacuum persistence amplitude or generating functional for disconnected graphs

$$Z[J;\hat{q}] = \int d\mu[q] \exp(iI[q+\hat{q}] + iJq) \,, \tag{2.2}$$

where a J-independent normalization has been ignored, and the fields, denoted collectively by $q(x)$, have been shifted by the addition of the background fields $\hat{q}(x)$. Here the measure $d\mu[q]$ includes the ghost and gauge-fixing factors,[38,44] and I denotes the classical action:

$$I=\int dv_x L \ , \qquad (2.3)$$

where dv_x is the invariant volume element. Fermion fields are treated as anticommuting quantities in (2.2). As is well known, the quantity $W=-i\ln Z$ generates connected graphs. The generating functional for single-particle irreducible graphs is obtained from W by a Legendre transformation[45,46]

$$\Gamma[u;\hat{q}]=W[J;\hat{q}]-\int dv_x u(x)J(x) \ , \qquad (2.4)$$

where

$$u(x)=\delta W[J;\hat{q}]/\delta J(x) \ . \qquad (2.5)$$

(The functional dependence of u on the background field has been suppressed.) Finally, the effective action $\Gamma[\hat{q}]$ is obtained by evaluating (2.4) at $u=0$:

$$\Gamma[\hat{q}]=\Gamma[u=0;\hat{q}] \ . \qquad (2.6)$$

As in Refs. 47 and 48, one can evaluate the terms in the loop expansion of the effective action as follows. Set $J=0$ in Eq. (2.2). Expand $I[q+\hat{q}]$ in powers of q, dropping the term linear in q. Thus,

$$I[q+\hat{q}]=I[\hat{q}]+\tfrac{1}{2}qI_2q+O(q^3) \ , \qquad (2.7)$$

where all indices and a spacetime integration have been suppressed in the second term on the right, and I_2 is a differential operator which depends on the background fields $\hat{q}$. Then

$$Z[\hat{q}]=\exp(iI[\hat{q}])\int d\mu[q]\exp[iqI_2q/2+O(q^3)] \ . \qquad (2.8)$$

The effective action is the single-particle irreducible part of Z. The loop expansion is

$$\Gamma[\hat{q}]=\Gamma^{(0)}[\hat{q}]+\Gamma^{(1)}[\hat{q}]+\cdots \ . \qquad (2.9)$$

Here $\Gamma^{(0)}[\hat{q}]=I[\hat{q}]$ and

$$\Gamma^{(1)}[\hat{q}]=-i\ln Z^{(1)}[\hat{q}] \ , \qquad (2.10)$$

where

$$Z^{(1)}[\hat{q}]=\int d\mu[q]\exp(iqI_2q/2) \qquad (2.11)$$

is the one-loop contribution of Z. $\Gamma^{(1)}$ is evaluated by summing only the connected graphs in $Z^{(1)}$.

For the renormalization of Λ, κ, and the α's in Eq. (2.1) we can set all of the background fields $\hat{q}$ to zero, while for the renormalization of ξ_ϕ and ξ_H we only require that the background Higgs fields $\hat{\phi}$ and $\hat{H}$ be nonzero. In that way we obtain all one-loop pole terms having a form analogous to (2.1) when written in terms of the nonvanishing background fields.

First consider the renormalization of ξ_ϕ and ξ_H. We can write

$$\tfrac{1}{2}qI_2q=I_2^{(0)}+I_{\rm int}^{(1)}+I_{\rm int}^{(2)} \ , \qquad (2.12)$$

where the superscripts denote the powers of the background Higgs fields which occur in each term. $I_{\rm int}=I_{\rm int}^{(1)}+I_{\rm int}^{(2)}$ will be treated as an interaction, while $I_2^{(0)}$ will yield the propagators to be used on the internal lines. Expanding $\exp(iI_{\rm int})=\sum i^n(I_{\rm int})^n/n!$, we need only retain the terms quadratic in the background Higgs fields in order to obtain the pole terms which must be canceled by renormalization of ξ_ϕ and ξ_H. These quadratic terms are $i^2(I_{\rm int}^{(1)})^2/2+iI_{\rm int}^{(2)}$. It follows from Eqs. (2.10) and (2.11) that the part of the one-loop effective action which is quadratic in the background fields is

$$\Gamma_{\rm quad}^{(1)}=\langle I_{\rm int}^{(2)}\rangle+(i/2)\langle (I_{\rm int}^{(1)})^2\rangle \ . \qquad (2.13)$$

Here the angular brackets are to be evaluated using the Wick reduction formula, keeping only the single-particle irreducible parts. Thus, the calculation of the pole terms in ξ_ϕ and ξ_H reduces to finding $I_{\rm int}^{(1)}$ and $I_{\rm int}^{(2)}$, and then evaluating $\Gamma_{\rm quad}^{(1)}$.

To renormalize Λ, κ, and the α's, we set the background fields $\hat{q}$ to zero in Eqs. (2.10) and (2.11), obtaining

$$\Gamma^{(1)}[0]=-i\ln\int d\mu[q]\exp(iI_2^{(0)}) \ . \qquad (2.14)$$

This Gaussian integral yields the trace of a determinant. We will use the proper-time or heat-kernel expansion to obtain the pole part of $\Gamma^{(1)}[0]$, which contains the various counterterms. This analysis is carried out in Sec. VII.

III. PROPAGATOR EQUATIONS

The various propagators will satisfy equations determined by the part of the Lagrangian which contributes to $I_2^{(0)}$. For generality, we work with an SU(n)-invariant Lagrangian in calculating the counterterms, and only specialize later to $n=5$. The full Lagrangian L consists of three main parts:

$$L=L_1+L_2+L_{\rm curv}+(\text{ghost terms}) \ , \qquad (3.1)$$

where $L_{\rm curv}$ was given in Eq. (2.1) and

$$\begin{aligned} L_1=&-\tfrac{1}{4}F^i_{\mu\nu}F^{i\mu\nu}-\frac{1}{2\omega}(\nabla_\mu A^{i\mu})^2+\mathrm{tr}[(D_\mu\Phi)(D^\mu\Phi)]+(D_\mu H)^\dagger(D^\mu H)\\ &+\tfrac{1}{2}[\bar{\psi}_{Lab}{}^{\dot{A}}\sigma^\mu{}_{\dot{A}B}(D_\mu\psi_L{}^B)^{ab}+\bar{\chi}_L{}^{a\dot{A}}\sigma^\mu{}_{\dot{A}B}(D_\mu\chi_L{}^B)_a+\text{H.c.}]\\ &+(\gamma\chi_{LaA}\psi_L{}^{abA}H_b^*+\Gamma\epsilon_{abcde}\psi_L{}^{ab}{}_A\psi_L{}^{cdA}H^e+\text{H.c.})+\mu_\phi{}^2\mathrm{tr}(\Phi^2)-a[\mathrm{tr}(\Phi^2)]^2-2b\,\mathrm{tr}(\Phi^4)\\ &+\tfrac{1}{2}\mu_H{}^2H^\dagger H-\frac{\lambda}{4}(H^\dagger H)^2-2\alpha H^\dagger H\,\mathrm{tr}(\Phi^2)-2\beta H^\dagger\Phi^2H \ , \end{aligned} \qquad (3.2)$$

$$L_2=\bar{B}_i\gamma^\mu(D_\mu B)^i+\bar{\Theta}_a\gamma^\mu(D_\mu\Theta)^a+i\sqrt{2}\{k_2\bar{\Theta}_a\Theta^b\phi^i(F^i)^a{}_b+[k_4\bar{B}_i(F^i)^a{}_b\Theta^b H_a^*+\text{H.c.}]$$
$$+2k_5\bar{B}_i(F^i)^a{}_bB^j(F^j)^b{}_c\phi^k(F^k)^c{}_a+2k_6\bar{B}_i(F^i)^b{}_cB^j(F^j)^a{}_b\phi^k(F^k)^c{}_a\}\ . \qquad (3.3)$$

Here L_1 is the Georgi-Glashow Lagrangian[10] and L_2 is the part of the Lagrangian added by Chang, Das, and Perez-Mercader.[11] The $(F_i)^a{}_b$ are the matrix elements of the n^2-1 generators of the Lie algebra of SU(n) in the fundamental representation. The multiplets of Higgs bosons are denoted by $\Phi=\phi^iF^i$ and H, where Φ is Hermitian and transforms under the adjoint representation, and H is complex and transforms under the n representation of SU(n). The gauge-fixing parameter is denoted by ω. Because the gauge-fixing term is independent of the Higgs fields, the ghost terms in L will not contribute to the renormalization of ξ_ϕ and ξ_H and can be ignored. For the fermion fields ψ and χ in L, we are using two-component spinor notation with capital roman spinor indices (except for subscript L which denotes left-handedness). The $\psi_L{}^{ab}$ transform as the antisymmetric tensor product of two n representations of SU(n), and the $\chi_L{}^a$ transform under the n^* representation. The gauge-covariant derivative D_μ and the space- and time-dependent matrices σ^μ and γ^μ are the appropriate ones for curved spacetime. The gauge coupling g appears as usual in $F^i_{\mu\nu}$ and in D_μ. Here

$$F^i_{\mu\nu}=\partial_\mu A^i_\nu-\partial_\nu A^i_\mu+gf^{ijk}A^j_\mu A^k_\nu\ . \qquad (3.4)$$

The form of D_μ appropriate to the various fields is listed in Appendices A and B. Our notation is similar to that in the review by Langacker,[49] although some numerical factors differ because of slightly different conventions. A more complete discussion of our notation, including two-component spinors, is given in Appendices A and B. Fermion fields are treated as anticommuting quantities.

The Dirac spinors B^i and Θ^a are the heavy fermions [transforming under the adjoint and the n representation of SU(n), respectively] introduced by Chang, Das, and Perez-Mercader[11] in order to obtain a consistent theory in which the coupling constants appearing in L_1 and L_2 are proportional to positive powers of the gauge coupling g, and hence are asymptotically free. (The relation of our notation to that of Chang, Das, and Perez-Mercader[11] is given in Appendix D.) Generation indices have been suppressed.

The minimal Georgi-Glashow model is obtained by omitting L_2 and setting the number N of light-fermion generations in L_1 to 3 and n to 5. The first model in Ref. 11 of Chang, Das, and Perez-Mercader is obtained by including one generation of the heavy fermions in L_2 and setting the number of light-fermion generations in L_1 to $N=7$. In their second model of Ref. 11, the number of light generations is $N=3$.

The part of L which contributes to $I_2^{(0)}$ consists of those terms which are quadratic in the quantum fields. This part of the Lagrangian is seen to be the same as one would obtain by setting to zero in Eqs. (3.1)–(3.3) the following couplings: g, γ, Γ, a, b, λ, α, β, and the k's. From this "kinetic" or "noninteracting" part of L follows the equations for the propagators used in calculating the counterterms for ξ_ϕ and ξ_H.

Scalar propagators. The scalar propagators

$$\langle\phi^i(x)\phi^j(x')\rangle=i\delta^{ij}\Delta_\phi(x,x') \qquad (3.5)$$

and

$$\langle H^a(x)H_b^*(x')\rangle=i\delta^a_b\Delta_H(x,x') \qquad (3.6)$$

satisfy

$$(\Box+\xi_\phi R-\mu_\phi{}^2)\Delta_\phi(x,x')=-\delta(x,x') \qquad (3.7a)$$

and

$$(\Box+\xi_H R-\mu_H{}^2/2)\Delta_H(x,x')=-\delta(x,x')\ . \qquad (3.7b)$$

Recall that the ϕ^i are related to the Φ in L by $\Phi=\phi^iF^i$, where the F^i are the generators ($i=1$ to n^2-1) of SU(n) in the fundamental (n-dimensional) representation. The group index on H^a runs from 1 to n.

Spinor propagators. The two-point function of the left-handed Weyl spinor $\chi_L{}^{aA}$, where a ($=1$ to n) is the group index and A ($=1$ to 2) is the spinor index, is

$$\langle\chi_L{}^{aA}(x)\bar{\chi}_L{}^{b\dot{B}}(x')\rangle=i\delta^{ab}S^{A\dot{B}}(x,x')\ , \qquad (3.8)$$

where

$$\sigma^\mu{}_{\dot{A}C}\nabla_\mu S^{C\dot{B}}(x,x')=\delta_{\dot{A}}{}^{\dot{B}}\delta(x,x')\ . \qquad (3.9)$$

Here ∇_μ denotes the spacetime-covariant derivative (for its action on a two-component spinor see Appendix B).

Similarly, for the fermions $\psi_L{}^{abB}$ one finds

$$\langle\psi_{Lab}{}^A(x)\bar{\psi}_{Lcd}{}^{\dot{B}}(x')\rangle=i\tfrac{1}{2}(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc})S^{A\dot{B}}(x,x')\ , \qquad (3.10)$$

where $S^{A\dot{B}}$ satisfies Eq. (3.9).

Defining $G_{\dot{A}}{}^{\dot{B}}$ by

$$S^{A\dot{B}}(x,x')=\sigma^{\nu A\dot{C}}\nabla_\nu G_{\dot{C}}{}^{\dot{B}}(x,x')\ , \qquad (3.11)$$

one has, upon use of identity (B50),

$$(\Box+\tfrac{1}{4}R)G_{\dot{C}}{}^{\dot{B}}(x,x')=\delta_{\dot{C}}{}^{\dot{B}}\delta(x,x')\ . \qquad (3.12)$$

It should be noted that in this equation the operator $\Box=\nabla^\mu\nabla_\mu$ is acting on a bispinor.

The Lagrangian L_2 contains Dirac spinors B^i and Θ^a, where i and a are group indices. Let

$$\langle B^i(x)\bar{B}_j(x')\rangle=i\delta^i_jS(x,x')\ , \qquad (3.13)$$

$$\langle\Theta^a(x)\bar{\Theta}_b(x')\rangle=i\delta^a_bS(x,x')\ . \qquad (3.14)$$

Then $S(x,x')$ is the Dirac propagator satisfying

$$\gamma^{\hat{a}}\nabla_{\hat{a}}S(x,x')=\delta(x,x')\ , \qquad (3.15)$$

where $\hat{a}$ denotes a vierbein index, and should not be confused with a group index. As for the Weyl spinors, if we define G by

$$S(x,x')=\gamma^{\hat{a}}\nabla_{\hat{a}}G(x,x')\ , \tag{3.16}$$

then G satisfies [using identity (B60)]

$$(\Box+\tfrac{1}{4}R)G(x,x')=\delta(x,x')\ . \tag{3.17}$$

Again, it should be noted that $\Box$ operates on the bispinor G.

Vector propagator. We work with the gauge-fixing parameter ω arbitrary, in order to check gauge independence. The gauge two-point function

$$\langle A^i_\mu(x)A^j_\nu(x')\rangle=i\delta^{ij}G_{\mu\nu}(x,x') \tag{3.18}$$

satisfies

$$[\delta^\mu{}_\lambda\Box+R^\mu{}_\lambda-(1-\omega^{-1})\nabla^\mu\nabla_\lambda]G^\lambda{}_\nu(x,x')=\delta^\mu{}_\nu\delta(x,x')\ . \tag{3.19}$$

As before, ∇_μ is the spacetime-covariant derivative and $\Box=\nabla_\mu\nabla^\mu$.

The above equations will be used to obtain the pole parts of the various propagators and related quantities appearing in the one-loop effective action. This is done in Appendix C.

IV. EVALUATION OF THE ONE-LOOP EFFECTIVE ACTION

Following the procedure described in Sec. II, we must now find $I^{(1)}_{\rm int}$ and $I^{(2)}_{\rm int}$, the parts of the action quadratic in the quantum fields, but linear and quadratic, respectively, in the background fields. As we are interested here in the renormalization of ξ_ϕ and ξ_H, only Higgs background fields $\hat{\phi}^i$ and $\hat{H}^a$ will be introduced. Clearly, there are no contributions to $I_{\rm int}$ coming from $L_{\rm curv}$ of Eq. (2.1). (However, $L_{\rm curv}$ does contribute to $I^{(0)}_2$ and hence to the propagators used in the calculation.) Because it may be of independent interest, we first obtain the result for the Georgi-Glashow theory. Then we give the result for the fully asymptotically free model.

Contribution of L_I. Replacing ϕ^i by $\phi^i+\hat{\phi}^i$ and H^a by $H^a+\hat{H}^a$ in L_1, and collecting the terms having the appropriate powers of the background fields and quadratic in the quantum fields, we obtain

$$\begin{aligned}I^{(1)}_{\rm int}=\int dv_x[\,&gf^{ijk}(\partial_\mu\hat{\phi}^i)A^{j\mu}\phi^k+gf^{ijk}(\partial_\mu\phi^i)A^{j\mu}\hat{\phi}^k-ig(\partial_\mu\hat{H}^*_a)A^{i\mu}(F^i)^a{}_bH^b\\&-ig(\partial_\mu H^*_a)A^{i\mu}(F^i)^a{}_b\hat{H}^b+ig\hat{H}^*_aA^{i\mu}(F^i)^a{}_b(\partial_\mu H^b)+igH^*_aA^{i\mu}(F^i)^a{}_b(\partial_\mu\hat{H}^b)\\&+\gamma\chi_{LaA}\psi_L{}^{abA}\hat{H}^*_b+\gamma^*\bar{\chi}_L{}^a{}_{\dot{A}}\bar{\psi}_{Lab}{}^{\dot{A}}\hat{H}^b+\Gamma\epsilon_{abcde}\psi_L{}^{ab}{}_A\psi_L{}^{cdA}\hat{H}^e+\Gamma^*\epsilon^{abcde}\bar{\psi}_{Lab\dot{A}}\bar{\psi}_{Lcd}{}^{\dot{A}}\hat{H}^*_e]\ .\end{aligned} \tag{4.1}$$

Similarly, the relevant terms quadratic in the background fields yield

$$\begin{aligned}I^{(2)}_{\rm int}=\int dv_x[\,&\tfrac{1}{2}g^2f^{ijk}f^{ilm}\hat{\phi}^k\hat{\phi}^mA^j_\mu A^{l\mu}+g^2\hat{H}^*_aA^i_\mu(F^i)^a{}_bA^{j\mu}(F^j)^b{}_c\hat{H}^c-a\hat{\phi}^i\hat{\phi}^j\phi^i\phi^j-\tfrac{1}{2}a\hat{\phi}^i\hat{\phi}^i\phi^j\phi^j\\&-8b\hat{\phi}^i\hat{\phi}^j\phi^k\phi^l{\rm tr}(F^iF^jF^kF^l)-4b\hat{\phi}^i\hat{\phi}^k\phi^j\phi^l{\rm tr}(F^iF^jF^kF^l)-(\lambda/2)(\hat{H}^*_a\hat{H}^a)(H^*_bH^b)\\&-(\lambda/2)(\hat{H}^*_aH^a)(H^*_b\hat{H}^b)-\alpha\hat{H}^*_a\hat{H}^a\phi^i\phi^i-\alpha H^*_aH^a\hat{\phi}^i\hat{\phi}^i-2\beta H^*_a\hat{\phi}^i\hat{\phi}^j(F^iF^j)^a{}_bH^b-2\beta\hat{H}^*_a(F^iF^j)^a{}_b\hat{H}^b\phi^i\phi^j],\end{aligned} \tag{4.2}$$

where we have omitted terms which will not contribute to $\langle I^{(2)}_{\rm int}\rangle$ because they contain vanishing two-point functions [as, for example, $\langle H^a(x)H^b(x')\rangle$ and $\langle H^a(x)\phi^i(x')\rangle$].

It is now a straightforward calculation to evaluate $\langle(I^{(1)}_{\rm int})^2\rangle$ using Wick's theorem and the definitions of the propagators in Eqs. (3.5), (3.6), and (3.8). The identities (A7), (A9), and (A10) as well as

$$\epsilon_{abcde}\epsilon^{abcdf}=(n-1)!\delta^f_e \tag{4.3}$$

are used to obtain the expression

$$\begin{aligned}\langle(I^{(1)}_{\rm int})^2\rangle=-\int dv_x\int dv_{x'}\Bigg[\,&ng^2\partial_\mu\hat{\phi}^i(x)\partial'_\nu\hat{\phi}^i(x')G^{\mu\nu}(x,x')\Delta_\phi(x,x')-ng^2\partial_\mu\hat{\phi}^i(x)\hat{\phi}^i(x')G^{\mu\nu}(x,x')\partial'_\nu\Delta_\phi(x,x')\\&-ng^2\hat{\phi}^i(x)\partial'_\nu\hat{\phi}^i(x')G^{\mu\nu}(x,x')\partial_\mu\Delta_\phi(x,x')+ng^2\hat{\phi}^i(x)\hat{\phi}^i(x')G^{\mu\nu}(x,x')\partial_\mu\partial'_\nu\Delta_\phi(x,x')\\&-\left[\frac{n^2-1}{n}\right]g^2\hat{H}^*_a(x)\partial'_\nu\hat{H}^a(x')G^{\mu\nu}(x,x')\partial_\mu\Delta_H(x,x')+\left[\frac{n^2-1}{n}\right]g^2\hat{H}^*_a(x)\hat{H}^a(x')G^{\mu\nu}(x,x')\\&\times\partial_\mu\partial'_\nu\Delta_H(x,x')+\left[\frac{n^2-1}{n}\right]g^2\partial'_\nu\hat{H}^*_a(x')\partial_\mu\hat{H}^a(x)G^{\mu\nu}(x,x')\Delta_H(x,x')\\&-\left[\frac{n^2-1}{n}\right]g^2\partial_\mu\hat{H}^*_a(x)\hat{H}^a(x')G^{\mu\nu}(x,x')\partial'_\nu\Delta_H(x,x')-(n-1)|\gamma|^2\hat{H}^*_a(x)\hat{H}^a(x')\\&\times S_{A\dot{B}}(x,x')S^{A\dot{B}}(x,x')-4(n-1)!|\Gamma|^2\hat{H}^*_a(x')\hat{H}^a(x)S_{A\dot{B}}(x,x')S^{A\dot{B}}(x,x')\Bigg]\ .\end{aligned} \tag{4.4}$$

The pole part of this expression is now obtained with the use of Eqs. (C33)—(C37) and gives

$$\frac{i}{2}\text{P.P.}[\langle (I_{\text{int}}^{(1)})^2\rangle]=\epsilon^{-1}\int dv_x\{(3-\omega)ng^2\partial^\mu\hat{\phi}^i\partial_\mu\hat{\phi}^i$$
$$+[(3-\omega)n^{-1}(n^2-1)g^2-(n-1)|\gamma|^2-4(n-1)!|\Gamma|^2]\partial^\mu\hat{H}^*_a\partial_\mu\hat{H}^a$$
$$+ng^2(\omega\xi_\phi+\omega/12-\tfrac{1}{4})R\hat{\phi}^i\hat{\phi}^i$$
$$+[n^{-1}(n^2-1)g^2(\omega\xi_H+\omega/12-\tfrac{1}{4})+\tfrac{1}{6}(n-1)|\gamma|^2+\tfrac{2}{3}(n-1)!|\Gamma|^2]R\hat{H}^*_a\hat{H}^a\}\ . \tag{4.5}$$

A similar procedure is used to evaluate $\langle I_{\text{int}}^{(2)}\rangle$ from Eq. (4.2). We find

$$\langle I_{\text{int}}^{(2)}\rangle=i\int dv_x\left[\tfrac{1}{2}ng^2\hat{\phi}^i\hat{\phi}^iG^\mu{}_\mu(x,x)+\left[\frac{n^2-1}{2n}\right]g^2\hat{H}^*_a\hat{H}^aG^\mu{}_\mu(x,x)-\tfrac{1}{2}(n^2+1)a\hat{\phi}^i\hat{\phi}^i\Delta_\phi(x,x)\right.$$
$$-\left[\frac{2n^2-3}{n}\right]b\hat{\phi}^i\hat{\phi}^i\Delta_\phi(x,x)-\tfrac{1}{2}(n+1)\lambda\hat{H}^*_a\hat{H}^a\Delta_H(x,x)-(n^2-1)\alpha\hat{H}^*_a\hat{H}^a\Delta_\phi(x,x)$$
$$\left.-n\alpha\hat{\phi}^i\hat{\phi}^i\Delta_H(x,x)-\left[\frac{n^2-1}{n}\right]\beta\hat{H}^*_a\hat{H}^a\Delta_\phi(x,x)-\beta\hat{\phi}^i\hat{\phi}^i\Delta_H(x,x)\right]\ . \tag{4.6}$$

From the results in Eqs. (C21) and (C24) it follows that

$$\text{P.P.}[\langle I_{\text{int}}^{(2)}\rangle]=-\epsilon^{-1}\int dv_x\{[(n/12)(\omega+3)g^2+(n^2+1)(\xi_\phi-\tfrac{1}{6})a+2(2n^2-3)n^{-1}(\xi_\phi-\tfrac{1}{6})b$$
$$+2n(\xi_H-\tfrac{1}{6})\alpha+2(\xi_H-\tfrac{1}{6})\beta]R\hat{\phi}^i\hat{\phi}^i$$
$$+[\tfrac{1}{12}(\omega+3)n^{-1}(n^2-1)g^2+(n+1)(\xi_H-\tfrac{1}{6})\lambda$$
$$+2(n^2-1)(\xi_\phi-\tfrac{1}{6})\alpha+2n^{-1}(n^2-1)(\xi_\phi-\tfrac{1}{6})\beta]R\hat{H}^*_a\hat{H}^a\}\ . \tag{4.7}$$

Finally, Eq. (2.13) yields the part of the one-loop effective action which is quadratic in the background Higgs fields and comes from the Georgi-Glashow Lagrangian L_1:

$$\text{P.P.}(\Gamma_{GG}^{(1)})=\epsilon^{-1}\int dv_x\{(3-\omega)ng^2\partial^\mu\hat{\phi}^i\partial_\mu\hat{\phi}^i$$
$$+[(3-\omega)n^{-1}(n^2-1)g^2-N(n-1)|\gamma|^2-4N(n-1)!|\Gamma|^2]\partial^\mu\hat{H}^*_a\partial_\mu\hat{H}^a$$
$$+[(\omega\xi_\phi-\tfrac{1}{2})ng^2-(n^2+1)(\xi_\phi-\tfrac{1}{6})a-2n^{-1}(2n^2-3)(\xi_\phi-\tfrac{1}{6})b$$
$$-2n(\xi_H-\tfrac{1}{6})\alpha-2(\xi_H-\tfrac{1}{6})\beta]R\hat{\phi}^i\hat{\phi}^i$$
$$+[(\omega\xi_H-\tfrac{1}{2})n^{-1}(n^2-1)g^2-(n+1)(\xi_H-\tfrac{1}{6})\lambda-2(n^2-1)(\xi_\phi-\tfrac{1}{6})\alpha$$
$$-2n^{-1}(n^2-1)(\xi_\phi-\tfrac{1}{6})\beta+\tfrac{1}{6}N(n-1)|\gamma|^2+\tfrac{2}{3}N(n-1)!|\Gamma|^2]R\hat{H}^*_a\hat{H}^a\}\ , \tag{4.8}$$

where N, which multiplies the terms proportional to $|\gamma|^2$ and $|\Gamma|^2$, denotes the number of generations of light fermions.

Result for the fully asymptotically free model

Because L_2 in Eq. (3.3) is linear in the Higgs fields it contributes only to $I_{\text{int}}^{(1)}$. In addition, it is clear that there will be no cross terms with (4.1) in $\langle (I_{\text{int}}^{(1)})^2\rangle$. The required two-point functions were defined in Eqs. (3.13) and (3.14). Using Eqs. (A9), (A10), (A13), and (A14), we find in addition to (4.4) the following contribution to $\langle (I_{\text{int}}^{(1)})^2\rangle$:

$$-2\int dv_x\int dv_{x'}\left[\tfrac{1}{2}k_2{}^2\hat{\phi}^i(x)\hat{\phi}^i(x')+\left[\frac{n^2-1}{n}\right]k_4{}^2\hat{H}^*_a(x)\hat{H}^a(x')\right.$$
$$\left.+\left[\frac{n^2-2}{2n}\right](k_5{}^2+k_6{}^2)\hat{\phi}^i(x)\hat{\phi}^i(x')-\frac{2}{n}k_5k_6\hat{\phi}^i(x)\hat{\phi}^i(x')\right]\text{tr}[S(x,x')S(x',x)]\ . \tag{4.9}$$

The additional contribution to $(i/2)$P.P.$[\langle (I_{\text{int}}^{(1)})^2\rangle]$ follows upon use of Eq. (C38) as

$$4\epsilon^{-1}\int dv_x\{[-\tfrac{1}{2}k_2^2+(2/n)k_5k_6-(2n)^{-1}(n^2-2)(k_5^2+k_6^2)]\partial^\mu\hat\phi^i\partial_\mu\hat\phi^i-n^{-1}(n^2-1)k_4^2\partial^\mu\hat H^*_a\partial_\mu\hat H^a$$
$$+[\tfrac{1}{12}k_2^2-(3n)^{-1}k_5k_6+(12n)^{-1}(n^2-2)(k_5^2+k_6^2)]R\hat\phi^i\hat\phi^i+(6n)^{-1}(n^2-1)k_4^2R\hat H^*_a\hat H^a\}\ . \qquad (4.10)$$

If there are N_H generations of heavy fermions (B and Θ) with the same coupling constants, then this result would be multiplied by N_H. However, as both models of Chang, Das, and Perez-Mercader[11] have only one generation of heavy fermions, we take $N_H=1$.

Adding the contributions from L_1 and L_2 contained in Eqs. (4.5), (4.7), and (4.10), we obtain the pole part of $\Gamma^{(1)}_{\text{quad}}$, the one-loop effective action quadratic in the Higgs fields [see Eq. (2.13)]:

$$\text{P.P.}(\Gamma^{(1)})=\epsilon^{-1}\int dv_x\{[(3-\omega)ng^2-2k_2^2+(8/n)k_5k_6-2n^{-1}(n^2-2)(k_5^2+k_6^2)]\partial^\mu\hat\phi^i\partial_\mu\hat\phi^i$$
$$+[(3-\omega)n^{-1}(n^2-1)g^2-N(n-1)|\gamma|^2-4N(n-1)!|\Gamma|^2-4n^{-1}(n^2-1)k_4^2]\partial^\mu\hat H^*_a\partial_\mu\hat H^a$$
$$+[(\omega\xi_\phi-\tfrac{1}{2})ng^2-(n^2+1)(\xi_\phi-\tfrac{1}{6})a-(2/n)(2n^2-3)(\xi_\phi-\tfrac{1}{6})b$$
$$-2n(\xi_H-\tfrac{1}{6})\alpha-2(\xi_H-\tfrac{1}{6})\beta+\tfrac{1}{3}k_2^2+(3n)^{-1}(n^2-2)(k_5^2+k_6^2)-4(3n)^{-1}k_5k_6]R\hat\phi^i\hat\phi^i$$
$$+[(\omega\xi_H-\tfrac{1}{2})n^{-1}(n^2-1)g^2-(n+1)(\xi_H-\tfrac{1}{6})\lambda-2(n^2-1)(\xi_\phi-\tfrac{1}{6})\alpha$$
$$-2n^{-1}(n^2-1)(\xi_\phi-\tfrac{1}{6})\beta+(N/6)(n-1)|\gamma|^2+\tfrac{2}{3}N(n-1)!|\Gamma|^2$$
$$+\tfrac{2}{3}n^{-1}(n^2-1)k_4^2]R\hat H^*_a\hat H^a\}\ . \qquad (4.11)$$

The counterterms required for renormalization of ξ_ϕ and ξ_H follow from this expression.

V. RENORMALIZATION OF ξ_ϕ AND ξ_H

The pole part of $\Gamma[\hat q]$ in Eq. (2.9) must vanish. Therefore, the pole part of the one-loop effective action $\Gamma^{(1)}[\hat q]$ must be canceled by the poles in the bare fields and coupling constants which appear in $\Gamma^{(0)}[\hat q]=I[\hat q]$. Let $\hat\phi_B{}^i=Z_\phi{}^{1/2}\hat\phi^i$, $\xi_{\phi B}=\xi_\phi+\delta\xi_\phi$, and analogous equations for the H fields, where the subscript B labels the bare quantities and Z_ϕ is the field renormalization factor $Z_\phi=1+\delta Z_\phi$. (Because the ξ_B's are dimensionless, independent of the dimension of the spacetime, no mass parameter appears in the above equations.) Writing the bare action $I[\hat q]$ in terms of the renormalized quantities and the pole terms $\delta\xi$ and δZ, one has (since the only nonzero background fields are the Higgs fields)

$$I[\hat q]=\int dv_x[\tfrac{1}{2}Z_\phi\partial^\mu\hat\phi^i\partial_\mu\hat\phi^i+Z_H\partial^\mu\hat H^*_a\partial_\mu\hat H^a-\tfrac{1}{2}Z_\phi(\xi_\phi+\delta\xi_\phi)R\hat\phi^i\hat\phi^i-Z_H(\xi_H+\delta\xi_H)R\hat H^*_a\hat H^a]\ . \qquad (5.1)$$

Neglecting terms of order $\delta\xi\delta Z$ and using

$$\text{P.P.}\{I[\hat q]+\Gamma^{(1)}[\hat q]\}=0\ , \qquad (5.2)$$

we obtain the one-loop counterterms

$$\delta Z_\phi=-2\epsilon^{-1}[(3-\omega)ng^2-2k_2^2+(8/n)k_5k_6-2n^{-1}(n^2-2)(k_5^2+k_6^2)]\ , \qquad (5.3)$$

$$\delta Z_H=-\epsilon^{-1}[(3-\omega)n^{-1}(n^2-1)g^2-(n-1)N|\gamma|^2-4N(n-1)!|\Gamma|^2-4n^{-1}(n^2-1)k_4^2] \qquad (5.4)$$

and

$$\delta\xi_\phi=\epsilon^{-1}\bar\xi_\phi[6ng^2-2(n^2+1)a-4n^{-1}(2n^2-3)b-4k_2^2-4n^{-1}(n^2-2)(k_5^2+k_6^2)+16k_5k_6]+\epsilon^{-1}\bar\xi_H(-4n\alpha-4\beta)\ , \qquad (5.5)$$

$$\delta\xi_H=\epsilon^{-1}\bar\xi_H[3n^{-1}(n^2-1)g^2-(n+1)\lambda-(n-1)N|\gamma|^2-4(n-1)!N|\Gamma|^2-4n^{-1}(n^2-1)k_4^2]$$
$$+\epsilon^{-1}\bar\xi_\phi[-2(n^2-1)\alpha-2n^{-1}(n^2-1)\beta]\ , \qquad (5.6)$$

where we have defined (with appropriate subscripts)

$$\bar\xi=\xi-\tfrac{1}{6}\ . \qquad (5.7)$$

These results are for SU(n) with n arbitrary, and N generations of light fermions. The arbitrary gauge-fixing parameter ω has dropped out of (5.5) and (5.6), making these results manifestly gauge invariant.. It is also noteworthy that the ξ's appear in the one-loop counterterms only in the form of $(\xi-\frac{1}{6})$.

The counterterms for the Georgi-Glashow Lagrangian are obtained from the above expressions by setting all the k's to zero. In that theory, the full set of renormalization-group equations are known to result in the running couplings λ, γ, Γ, α, β, a, and b not being asymptotically free in the high-momentum limit. As pointed out in Refs. 50 and 51, if those couplings are assumed to not grow large in the energy range where g is small, one can still make use of the asymptotic freedom of the gauge coupling.[52] In that case, the behavior of the running couplings ξ_ϕ and ξ_H in the high-curvature limit would depend on the particular values of the other couplings at high energy. Therefore, it is of interest to consider a totally asymptotically free theory[11,16] in which the previous couplings all become small at high energy. For such a theory, one can reach definite conclusions concerning the behavior of the ξ's.

Chang, Das, and Perez-Mercader[11] have shown that for the Lagrangian of Eq. (3.1) in flat spacetime there is a solution of the full set of renormalization-group equations in which the above couplings are proportional to positive powers of the gauge coupling g. For the case of SU(5), $n=5$, they calculated the numerical values of the proportionality constants. The values of those couplings for their two theories (with $N=3$ and 7, respectively), are listed in Appendix D. Those values hold also in curved spacetime because couplings to the curvature do not appear in the relevant renormalization-group equations.

For each of the fully asymptotically free models, Eqs. (5.5) and (5.6) take the form

$$\delta\xi_\phi=\epsilon^{-1}g^2(c_{11}\bar{\xi}_\phi+c_{12}\bar{\xi}_H)\ , \tag{5.8}$$

$$\delta\xi_H=\epsilon^{-1}g^2(c_{21}\bar{\xi}_\phi+c_{22}\bar{\xi}_H)\ , \tag{5.9}$$

where the c_{ij} are numerical constants calculated by setting $n=5$ in Eqs. (5.5) and (5.6) and substituting the appropriate values for N and the various proportionality constants from Appendix D. For the model with $N=3$, we find $c_{11}=-13.42$, $c_{12}=-1.695$, $c_{21}=-4.067$, and $c_{22}=-21.41$. The model with $N=7$ has $c_{11}=-27.21$, $c_{12}=-4.215$, $c_{21}=-10.12$, and $c_{22}=-37.48$. We are now in a position to obtain the renormalization-group equations for ξ_ϕ and ξ_H.

VI. RENORMALIZATION-GROUP EQUATIONS FOR ξ_ϕ AND ξ_H

It is well known that in the process of renormalization, a new mass parameter μ appears. In dimensional regularization, μ is introduced[53] in order to keep the action dimensionless for all values of d, the number of dimensions of the spacetime. The renormalized coupling "constants" will depend on the value of μ (the so-called renormalization point). The renormalization-group equations, from which that μ dependence follows, are an expression of the fact that the bare coupling constants are independent of μ (which only appears in the process of regularization). The renormalization of the gauge coupling constant g is unaffected by the curvature to one-loop order. Consequently, the renormalized coupling g satisfies

$$\mu\, dg/d\mu=(32\pi^2)^{-1}\epsilon g-2z(4\pi)^{-2}g^3\ , \tag{6.1}$$

where $\epsilon=(4\pi)^2(d-4)$, and the constant z depends on n and N. The number z must be positive for g to be asymptotically free.[12-15] At dimension $d=4$, the solution is

$$g^2(\mu)=g_0{}^2[1+z(2\pi)^{-2}g_0{}^2\ln(\mu/\mu_0)]^{-1}\ , \tag{6.2}$$

where μ_0 is an arbitrary fixed unit of mass, and g_0 is the value of g at $\mu=\mu_0$.

The equation $\mu\, d\xi_{\phi B}/d\mu=0$ implies that

$$\begin{aligned}0=\ &\mu\, d\xi_\phi/d\mu+\mu(dg/d\mu)\partial(\delta\xi_\phi)/\partial g\\&+\mu(d\xi_\phi/d\mu)\partial(\delta\xi_\phi)/\partial\xi_\phi\\&+\mu(d\xi_H/d\mu)\partial(\delta\xi_\phi)/\partial\xi_H\ ,\end{aligned} \tag{6.3}$$

where it is clear from Eqs. (5.8) and (5.9) that the $\delta\xi$'s depend on μ through g and the ξ's. One has also the analogous equation involving $\mu\, d\xi_H/d\mu$. Substituting Eq. (6.1) into Eq. (6.3) and defining $\delta\xi_i^{(1)}$ as the coefficient of the simple pole in the expansion $\delta\xi_i=\epsilon^{-1}\delta\xi_i^{(1)}+O(\epsilon^2)$, for $i=\phi$ and H, the terms of order ϵ^0 give

$$\mu\, d\xi_\phi/d\mu=-(32\pi^2)^{-1}g\partial(\delta\xi_\phi^{(1)})/\partial g\ . \tag{6.4}$$

One also has the same equation with ϕ replaced by H. The one-loop contributions to the quantities $\delta\xi_\phi^{(1)}$ and $\delta\xi_H^{(1)}$ are the coefficients of ϵ^{-1} in Eqs. (5.8) and (5.9). Then Eq. (6.4) and the corresponding equation for ξ_H can be written in the matrix form

$$\mu\, d\bar{\xi}/d\mu=-(4\pi)^{-2}g^2C\bar{\xi}\ , \tag{6.5}$$

where $\bar{\xi}$ is a column matrix having components $\xi_i-\frac{1}{6}$ with $i=\phi$ and H, and C is a 2×2 matrix having components c_{ij} defined in Eqs. (5.8) and (5.9).

The above renormalization-group equation for the ξ's is readily solved by diagonalizing the matrix C. The eigenvalues are

$$\lambda_i=\tfrac{1}{2}\{\text{tr}C\pm[(\text{tr}C)^2-4\det C]^{1/2}\}\ , \tag{6.6}$$

where $i=1$ (2) corresponds to the + (−) sign. The solution of Eq. (6.5) is then

$$\begin{aligned}\bar{\xi}_\phi(\mu)=-c_{12}[&(c_{11}-\lambda_1)^{-1}\xi_{10}f_1(\mu)\\&+(c_{11}-\lambda_2)^{-1}\xi_{20}f_2(\mu)]\ ,\end{aligned} \tag{6.7}$$

$$\bar{\xi}_H(\mu)=\xi_{10}f_1(\mu)+\xi_{20}f_2(\mu)\ , \tag{6.8}$$

where ξ_{10} and ξ_{20} are constants, and

$$f_i(\mu)=[1+z(4\pi^2)^{-1}g_0{}^2\ln(\mu/\mu_0)]^{-(\lambda_i/12)}\ (i=1,2)\ . \tag{6.9}$$

We see that $\bar{\xi}_\phi$ and $\bar{\xi}_H$ will approach zero as μ approaches infinity, only if the eigenvalues λ_i are both positive [they must be real so that the $\xi(\mu)$'s are real for all values of μ]. For a general theory of the present type, these requirements are met if and only if both tr$C$ and det$C$ are positive and $(\text{tr}C)^2<4\det C$. For each of the two theories of Chang, Das, and Perez-Merader,[11] the values of the c_{ij} [given after Eq. (5.9)] do not satisfy these conditions. Therefore, it would appear that the $\bar{\xi}(\mu)$ diverge as μ approaches infinity, or that $\bar{\xi}(\mu)$ is constant at the value 0 [which is also a solution of Eq. (6.5)]. However, it can be shown that the effect of two-loop contributions to the renormalization-group equations will make $\bar{\xi}(\mu)=0$ an unstable fixed point, leaving only the divergent solution.

This divergence is analogous to that of other couplings of the theory. It is natural to search for a solution of the same type as for the other couplings; that is, to see if a solution exists in which the coupling constants for large μ are proportional to a positive power of the gauge coupling. It turns out to be possible to find such a solution, if it is assumed that $\bar{\xi}$ (but not ξ) for large μ is proportional to an appropriate power of g. Thus, in our previous matrix notation, suppose that for large μ,

$$\bar{\xi}(\mu)=g^2(\mu)r , \tag{6.10}$$

where r is a column matrix with constant dimensionless components r_1 and r_2.

If Eq. (6.10) is substituted into Eq. (6.5) and Eq. (6.1) with $\epsilon=0$ is used, then one finds that a solution seems to exist only if $\det(C-2zI)=0$. That would require C to have degenerate eigenvalues equal to $2z$, a condition which is clearly not met by the theories under consideration.

However, the renormalization-group equation in (6.5) ignores two-loop contributions. When the latter contributions are included, it becomes clear that a solution of the form of Eq. (6.10) does exist. When two-loop diagrams are included, one expects that the generalization of Eq. (6.5) will have the form

$$d\xi_\phi/dt = c_{11}\bar{\xi}_\phi g^2+c_{12}\bar{\xi}_H g^2 +b_1(\xi_\phi-p_1)g^4+b_2(\xi_H-p_2)g^4 \tag{6.11}$$

and

$$d\xi_H/dt = c_{21}\bar{\xi}_\phi g^2+c_{22}\bar{\xi}_H g^2 +b_3(\xi_\phi-p_3)g^4+b_4(\xi_H-p_4)g^4 , \tag{6.12}$$

where b_i,p_i are given numbers, and $t=\ln(\mu/\mu_0)$. The two-loop generalization of Eq. (6.1) at dimension 4 ($\epsilon=0$) will have the form

$$dg/dt=-\beta_0 g^3+\beta_1 g^5+\cdots , \tag{6.13}$$

where β_0 is given in Eq. (6.1).

Then, substituting Eqs. (6.10) and (6.13) into Eqs. (6.11) and (6.12), we find that the resulting equations can be written in matrix form as

$$Mr+\text{terms of order } g^2=v , \tag{6.14}$$

where $M=C+2\beta_0 I$, r is a column matrix with components r_1, r_2, and v is a column matrix with components $v_1=-b_1(\frac{1}{6}-p_1)-b_2(\frac{1}{6}-p_2)$ and $v_2=-b_3(\frac{1}{6}-p_3)-b_4(\frac{1}{6}-p_4)$. We do not expect the p_i to equal $\frac{1}{6}$ because, even in the example of $\lambda\phi^4$ interaction with additional coupling to the scalar curvature, the corresponding constants[54,55] do not equal $\frac{1}{6}$ (although at the one-loop level one does have factors of $\xi-\frac{1}{6}$ as here). For large μ or t, the terms of order g^2 in Eq. (6.14) can be neglected. Then one has the solution

$$r=M^{-1}v , \tag{6.15}$$

provided that $\det M$ is not zero. The latter condition is clearly satisfied for the values of the c_{ij} and β_0 in the present theories, so that a solution having the form of Eq. (6.10) for large μ does exist. Thus, in the asymptotically free theories under consideration, there exists a solution to the renormalization-group equations in which the $\bar{\xi}$'s approach zero and the ξ's approach $\frac{1}{6}$ (the so-called conformal value) in the limit of large μ.

VII. GRAVITATIONAL COUNTERTERMS

In order to renormalize the gravitational and cosmological constant, as well as the couplings α_i to the quadratic curvature terms, we start with Eq. (2.14),

$$\Gamma^{(1)}=-i\ln\int d\mu[q]\exp(iI_2^{(0)}) , \tag{7.1}$$

where the background fields are zero, and $I_2^{(0)}$ is the part of the action which is quadratic in the quantum fields. From the Lagrangian of Eq. (3.1), it follows that $I_2^{(0)}$ can be written as a sum of terms of the general form

$$\int dv_x q^\dagger H q , \tag{7.2}$$

where H is a differential operator (indices have been suppressed, and a factor of $\frac{1}{2}$ should be present if the quantum field q is real). To within additive constants, Eq. (7.1) integrates to

$$\Gamma^{(1)}=Ki\,\mathrm{Tr}\ln H , \tag{7.3}$$

where K is 1 for complex boson fields, $\frac{1}{2}$ for real boson fields, and -1 for complex fermion fields, and Tr includes integration over the spacetime indices. Now,

$$\delta\Gamma^{(1)}=-Ki\delta\left[\mathrm{Tr}\int_0^\infty ds\, s^{-1}\exp(-isH)\right] , \tag{7.4}$$

so that one can let[37,38]

$$\Gamma^{(1)}=-Ki\int dv_x\int_0^\infty ds\, s^{-1}\mathrm{tr}\langle x\,|\exp(-isH)|\,x\rangle , \tag{7.5}$$

where tr is over indices other than spacetime indices.

The coincidence limit of the proper-time expansion is[38]

$$\langle x\,|\exp(-isH)|\,x\rangle =\exp(-im^2s)i(4\pi is)^{-d/2}F(x,x;is) , \tag{7.6}$$

where d is the spacetime dimension, and

$$F(x,x;is)=1+isf_1(x,x)+(is)^2f_2(x,x)+\cdots . \tag{7.7}$$

Substituting this into Eq. (7.5), one finds [see, for example, Eq. (5.69) of Ref. 56, but with the sign conventions of our Appendix A] that the pole part of $\Gamma^{(1)}$ is

$$\mathrm{P.P.}(\Gamma^{(1)})=-\epsilon^{-1}2K\int dv_x \mathrm{tr}E_2 , \tag{7.8}$$

where

$$E_2=f_2-m^2f_1+\tfrac{1}{2}m^4 . \tag{7.9}$$

Here, m refers to a mass term of the usual sign. For the Higgs fields, the m^2 term appears with the opposite sign. In Eq. (7.8), the dimension d has been set equal to 4 except in the pole involving $\epsilon=(4\pi)^2(d-4)$.

Typically, the operator H has the form

$$H=\delta^i_j\Box+Q^i{}_j , \tag{7.10}$$

where i,j denote the appropriate spacetime or spinor indices, depending on the type of field under consideration. Let

$$[\nabla_\mu,\nabla_\nu]=W_{\mu\nu} . \tag{7.11}$$

The form of $W_{\mu\nu}$, of course, depends on the type of field under consideration. It can be shown that, in general,[38,57] for an operator H of the above form

$$\mathrm{tr}E_2(H)=\mathrm{tr}\{(360)^{-1}[(-12\Box R+5R^2-2R^{\mu\nu}R_{\mu\nu}+2R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})I+30W^{\mu\nu}W_{\mu\nu}+180Q^2-60RQ+60\nabla^\mu\nabla_\mu Q]\}\ . \tag{7.12}$$

It should be noted that in the case of spinors, ∇_μ includes the appropriate spin connection. In Eqs. (7.11) and (7.12), indices i,j have been suppressed.

For the real scalars (ϕ) and complex scalars (H), the operators H appear in Eqs. (3.7a) and (3.7b), respectively. In these cases, $W_{\mu\nu}=0$, and one obtains from Eqs. (7.8) and (7.10)—(7.12) the results

$$\mathrm{P.P.}(\Gamma_\phi^{(1)})=-\epsilon^{-1}N_0^R\int dv_x[\tfrac{1}{2}\mu_\phi{}^4-(\xi_\phi-\tfrac{1}{6})\mu_\phi{}^2R+\tfrac{1}{2}(\xi_\phi-\tfrac{1}{6})^2R^2-\tfrac{1}{180}R^{\mu\nu}R_{\mu\nu}+\tfrac{1}{180}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}] \tag{7.13}$$

and

$$\mathrm{P.P.}(\Gamma_H^{(1)})=-\epsilon^{-1}2N_0^C\int dv_x[\tfrac{1}{8}\mu_H{}^4-(\xi_H-\tfrac{1}{6})\tfrac{1}{2}\mu_H{}^2R+\tfrac{1}{2}(\xi_H-\tfrac{1}{6})^2R^2-\tfrac{1}{180}R^{\mu\nu}R_{\mu\nu}+\tfrac{1}{180}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}]\ , \tag{7.14}$$

where N_0^R is the number of real scalar fields ϕ_i, and N_0^C is the number of complex scalar fields H^a. For SU(n), $N_0^R=n^2-1$ and $N_0^C=n$. In the above, the terms involving $\Box R$ do not contribute because they integrate to zero.

The operator H for the vector fields A_μ^i appears in Eq. (3.19). Any physically relevant quantity should be independent of the gauge-fixing parameter ω. This holds, in particular, for the components of the expectation value of the stress tensor, since they are measurable at least in principle. But in fact P.P.($\Gamma^{(1)}$) is given by the same trE_2 coefficient which determines the trace anomaly,[58] and so the expression for trE_2 must be independent of ω, which implies that the gravitational counterterms necessary to cancel P.P.($\Gamma^{(1)}$) are independent[59] of ω. We may therefore work in the Feynman gauge $\omega=1$ so that H has the form of Eq. (7.10) with Q corresponding to $R^\mu{}_\nu$. Then tr$Q=R$ and Tr(Q^2)$=R^{\mu\nu}R_{\mu\nu}$. Also

$$[\nabla_\mu,\nabla_\nu]A^\lambda=-R_{\mu\nu}{}^\lambda{}_\rho A^\rho\ , \tag{7.15}$$

so that

$$(W_{\mu\nu})^\lambda{}_\rho=-R_{\mu\nu}{}^\lambda{}_\rho\ , \tag{7.16}$$

and $\mathrm{tr}(W^{\mu\nu}W_{\mu\nu})=-R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. Then from Eq. (7.12) we obtain for each vector field

$$\mathrm{tr}E_2^V=\tfrac{1}{360}(12\Box R-40R^2+172R^{\mu\nu}R_{\mu\nu}-22R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})\ . \tag{7.17}$$

(Note that here tr$I=\delta^\mu{}_\mu=4$.) For each vector field, we must also include a ghost field, which is a complex scalar field with $H=\Box$. For each scalar ghost field, one has

$$\mathrm{tr}E_2^G=\tfrac{1}{360}(-12\Box R+5R^2-2R^{\mu\nu}R_{\mu\nu}+2R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})\ . \tag{7.18}$$

Because the ghost fields are complex scalar anticommuting quantities, the constant K in Eq. (7.10) is -1 for each ghost field. Therefore, including a complex ghost field with each vector field A_μ^i, we have for the contribution of the vector fields to the pole part of the effective action

$$\mathrm{P.P.}(\Gamma_{\rm vector}^{(1)})=-\epsilon^{-1}N_1\int dv_x(\mathrm{tr}E_2^V-2\,\mathrm{tr}E_2^G)=-\epsilon^{-1}N_1\int dv_x(-\tfrac{5}{36}R^2+\tfrac{22}{45}R^{\mu\nu}R_{\mu\nu}-\tfrac{13}{180}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})\ , \tag{7.19}$$

where N_1 is the number of vector fields. For SU(n), $N_1=n^2-1$.

For the Dirac spinors, the operator H appears on the left-hand side (LHS) of Eq. (3.15). It can be shown[38] that in this case

$$\mathrm{tr}\ln H=\tfrac{1}{2}\,\mathrm{Tr}\ln(H')\ , \tag{7.20}$$

where $H'=\Box+\frac{1}{4}R$ is the operator appearing in Eq. (3.17). [The derivation just uses Eq. (B60).] This operator is of the form of Eq. (7.10), so that Eqs. (7.11), (7.12), and (7.8) can be used. Applying (7.11) to a Dirac spinor, one finds [see Eqs. (B55) and (B58)]

$$W_{\mu\nu}=-\tfrac{1}{4}R_{\mu\nu\hat a\hat b}\gamma^{\hat a}\gamma^{\hat b}\ , \tag{7.21}$$

where the caret indicates a vierbein index, and $\gamma^{\hat a}$ refers to the ordinary γ matrices. Then

$$\mathrm{tr}(W_{\mu\nu}W^{\mu\nu})=-\tfrac{1}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\ , \tag{7.22}$$

and $Q=\frac{1}{4}RI$, where I is the 4×4 unit matrix. Hence, Eq. (7.12) yields

$$\mathrm{tr}E_2(H')=\tfrac{1}{360}(12\Box R+5R^2-8R^{\mu\nu}R_{\mu\nu}-7R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})\ , \tag{7.23}$$

and, with $K=-1$ and the factor of $\frac{1}{2}$ from Eq. (7.20), Eq. (7.8) gives, for $N_{1/2}^D$ Dirac spinors,

$$\mathrm{P.P.}(\Gamma_{\rm Dirac}^{(1)})=\epsilon^{-1}N_{1/2}^D\int dv_x(\tfrac{1}{72}R^2-\tfrac{1}{45}R^{\mu\nu}R_{\mu\nu}-\tfrac{7}{360}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})\ . \tag{7.24}$$

The argument is similar for the left-handed Weyl (two-component) spinors. Now the operator H appears on the LHS of Eq. (3.9) and the operator H' appears on the LHS of Eq. (3.12). Equation (7.20) is also valid in this case. We find [see Eq. (B48) for $W_{\mu\nu}$] that

$$\mathrm{tr}(W_{\mu\nu}W^{\mu\nu})=-\tfrac{1}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\ . \tag{7.25}$$

Also, $Q=\frac{1}{4}RI$, where now I is the 2×2 unit matrix, so that tr$I=2$, tr$Q=R/2$, and tr(Q^2)$=R^2/8$. Then

$$\mathrm{tr}E_2(H')=\tfrac{1}{360}(6\Box R+\tfrac{5}{2}R^2-4R^{\mu\nu}R_{\mu\nu}-\tfrac{7}{2}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})\ . \tag{7.26}$$

This is half the result for a Dirac spinor, in agreement with Refs. 60 and 61. Therefore, if there are $N^W_{1/2}$ Weyl spinors,

$$\mathrm{P.P.}(\Gamma^{(1)}_{\mathrm{Weyl}})=\epsilon^{-1}N^W_{1/2}\int dv_x(\tfrac{1}{144}R^2-\tfrac{1}{90}R^{\mu\nu}R_{\mu\nu}-\tfrac{7}{720}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})\ . \tag{7.27}$$

Adding together the previous results, we find that for a grand unified theory with N_0^R real scalars, N_0^C complex scalars, N_1 vectors, $N^W_{1/2}$ Weyl spinors, and $N^D_{1/2}$ Dirac spinors, the pole part of the one-loop effective action is

$$\begin{aligned}\mathrm{P.P.}(\Gamma^{(1)})=\epsilon^{-1}\int dv_x\{&-\tfrac{1}{2}\mu_\phi{}^4N_0^R-\tfrac{1}{4}\mu_H{}^4N_0^C+[N_0^R(\xi_\phi-\tfrac{1}{6})\mu_\phi{}^2+N_0^C(\xi_H-\tfrac{1}{6})\mu_H{}^2]R\\&+[-\tfrac{1}{2}N_0^R(\xi_\phi-\tfrac{1}{6})^2-N_0^C(\xi_H-\tfrac{1}{6})^2+\tfrac{5}{36}N_1+\tfrac{1}{144}N^W_{1/2}+\tfrac{1}{72}N^D_{1/2}]R^2\\&+[\tfrac{1}{180}N_0^R+\tfrac{1}{90}N_0^C-\tfrac{22}{45}N_1-\tfrac{1}{90}N^W_{1/2}-\tfrac{1}{45}N^D_{1/2}]R^{\mu\nu}R_{\mu\nu}\\&+[-\tfrac{1}{180}N_0^R-\tfrac{1}{90}N_0^C+\tfrac{13}{180}N_1-\tfrac{7}{720}N^W_{1/2}-\tfrac{7}{360}N^D_{1/2}]R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\}\ .\end{aligned} \tag{7.28}$$

Then

$$\mathrm{P.P.}(I_B+\Gamma^{(1)})=0 \tag{7.29}$$

yields the counterterms for the coupling constants Λ, κ, and the α_i appearing in L_{curv} of Eq. (2.1). Here, I_B is the action containing the bare coupling constants, such as $\Lambda_B=\Lambda+\delta\Lambda$, where Λ denotes the renormalized coupling. We thus obtain

$$\delta\Lambda=\epsilon^{-1}(\tfrac{1}{2}N_0^R\mu_\phi{}^4+\tfrac{1}{4}N_0^C\mu_H^4)\ , \tag{7.30}$$

$$\delta\kappa=-\epsilon^{-1}[N_0^R(\xi_\phi-\tfrac{1}{6})\mu_\phi{}^2+N_0^C(\xi_H-\tfrac{1}{6})\mu_H{}^2]\ , \tag{7.31}$$

$$\delta\alpha_1=(720\epsilon)^{-1}(4N_0^R+8N_0^C-52N_1+7N^W_{1/2}+14N^D_{1/2})\ , \tag{7.32}$$

$$\delta\alpha_2=(180\epsilon)^{-1}(-N_0^R-2N_0^C+88N_1+2N^W_{1/2}+4N^D_{1/2})\ , \tag{7.33}$$

and

$$\delta\alpha_3=(144\epsilon)^{-1}[72(\xi_\phi-\tfrac{1}{6})^2N_0^R+144(\xi_H-\tfrac{1}{6})^2N_0^C-20N_1-N^W_{1/2}-2N^D_{1/2}]\ . \tag{7.34}$$

For minimal SU(5), one has $N_0^R=24$, $N_0^C=5$, $N_1=24$, $N^D_{1/2}=0$, and $N^W_{1/2}=N(5+10)$, where N is the number of generations of light fermions. In the fully asymptotically free SU(5) theory, all the above numbers are the same, except that $N^D_{1/2}=(5+24)$.

VIII. RENORMALIZATION-GROUP EQUATIONS FOR GRAVITATIONAL COUPLINGS

The counterterms $\delta\Lambda$ and $\delta\kappa$ involve $\mu_\phi{}^2$ and $\mu_H{}^2$. Therefore, we must obtain the dependence of the Higgs masses on the renormalization-group parameter μ. The renormalization-group equations for the Higgs masses are given for the $N=3$ model in the second paper of Ref. 11. They use the relation

$$\mu_H{}^2(t)=-0.927\,207\mu_\phi{}^2(t)\ , \tag{8.1}$$

where $t=\ln(\mu/\mu_0)$. Then the renormalization-group equations that they write reduce to

$$16\pi^2d(\mu_\phi{}^2)/dt=z_1g^2\mu_\phi{}^2 \tag{8.2}$$

with

$$z_1=12.634\,34\ . \tag{8.3}$$

Using Eq. (6.2), the solution is found to be

$$\mu_\phi{}^2(t)=\mu_\phi{}^2(0)[1+z(2\pi)^{-2}g_0{}^2t]^{z_1/4z}\ . \tag{8.4}$$

For the $N=3$ SU(5) theory, one has $z=3$, so that $z_1/4z=1.052\,86$.

At arbitrary dimension d, we have

$$\begin{aligned}\kappa_B&=\mu^{d-4}(\kappa+\delta\kappa)\ ,\\\Lambda_B&=\mu^{d-4}(\Lambda+\delta\Lambda)\ ,\\\alpha_{iB}&=\mu^{d-4}(\alpha_i+\delta\alpha_i)\ ,\end{aligned} \tag{8.5}$$

where κ has mass dimension 2, Λ has mass dimension 4, and the α_i have dimension 0. The bare and renormalized Higgs masses, of course, have dimensions of mass for all d. Let us consider first the renormalization-group equation for κ. From Eq. (7.31), at one loop, $\delta\kappa$ is a function of the $\bar{\xi}$'s and the Higgs masses. Then a standard renormalization-group analysis, as in Sec. VI, gives

$$d\kappa/dt=-(4\pi)^{-2}\epsilon\kappa+\beta_\kappa \tag{8.6}$$

with

$$\beta_\kappa=-(4\pi)^{-2}\delta\kappa^{(1)}\ , \tag{8.7}$$

where $\delta\kappa^{(1)}$ is defined as the coefficient of ϵ^{-1} in Eq. (7.31). Setting $d=4$ ($\epsilon=0$) and using Eqs. (6.10) and (8.1) then gives

$$d\kappa/dt=(4\pi)^{-2}Kg^2\mu_\phi{}^2 \tag{8.8}$$

with

$$K\equiv N_0^Rr_1-N_0^Cr_2(0.927\,207)\ . \tag{8.9}$$

[See Eq. (6.10) for the definition of r_1 and r_2.]

In view of Eq. (8.2), this can be integrated at once to give

$$\kappa(\mu)=Kz_1{}^{-1}\mu_\phi{}^2(t)+\mathrm{const}\ , \tag{8.10}$$

where it will be recalled that $t=\ln(\mu/\mu_0)$. Then for large μ/μ_0, the quantity $\kappa(\mu)$ is proportional to the square of the Higgs mass.

Similarly, we find from Eq. (7.30) that

$$d\Lambda/dt=-(4\pi)^{-2}\epsilon\Lambda+\beta_\Lambda \quad (8.11)$$

with

$$\beta_\Lambda=-(4\pi)^{-2}L\mu_\phi{}^4 , \quad (8.12)$$

where

$$L\equiv\tfrac{1}{2}N_0^R+\tfrac{1}{4}N_0^C(0.927\,207)^2 . \quad (8.13)$$

At $d=4$, Eq. (8.11) has the solution

$$\Lambda(\mu)=-\frac{L}{2g_0{}^2(2z+z_1)}\mu_\phi{}^4(0)\times\left[\left[1+\frac{z}{4\pi^2}g_0{}^2t\right]^{1+z_1/2z}-1\right]+\Lambda_0 , \quad (8.14)$$

where Λ_0 is a constant of integration. The constant L is positive, so that $\Lambda(\mu)\rightarrow-\infty$ as $\mu\rightarrow\infty$.

For α_i $(i=1,2,3)$ we find

$$d\alpha_i/dt=-(4\pi)^{-2}\epsilon\alpha_i-(4\pi)^{-2}\delta\alpha_i^{(1)} , \quad (8.15)$$

where $\delta\alpha_i^{(1)}$ is the coefficient of ϵ^{-1} in the equation for $\delta\alpha_i$ [i.e., Eqs. (7.32) through (7.34)]. At $d=4$, the solutions for α_1 and α_2 are

$$\alpha_i(\mu)=\alpha_{i0}-(4\pi)^{-2}a_it \quad \text{for } i=1,2 , \quad (8.16)$$

where

$$a_1=(720)^{-1}(4N_0^R+8N_0^C-52N_1+7N_{1/2}^W+14N_{1/2}^D) , \quad (8.17)$$

and

$$a_2=(180)^{-1}(-N_0^R-2N_0^C+88N_1+2N_{1/2}^W+4N_{1/2}^D) . \quad (8.18)$$

After using Eq. (6.10), the renormalization-group equation for α_3 at $d=4$ has the form

$$d\alpha_3/dt=-(4\pi)^{-2}[(c_1r_1{}^2+c_2r_2{}^2)g^4+c_3] , \quad (8.19)$$

where

$$c_1=\tfrac{1}{2}N_0^R,\ \ c_2=N_0^C , \quad (8.20)$$

and

$$c_3=-(144)^{-1}(20N_1+N_{1/2}^W+2N_{1/2}^D) . \quad (8.21)$$

The solution is

$$\alpha_3(\mu)=\alpha_{30}-(4\pi)^{-2}c_3t+(4z)^{-1}(c_1r_1{}^2+c_2r_2{}^2)g_0{}^2\times\{[1+(4\pi^2)^{-1}zg_0{}^2t]^{-1}-1\} . \quad (8.22)$$

The coefficient c_3 is negative so that $\alpha_3(\mu)$ approaches $+\infty$ as $\mu\rightarrow\infty$. The physical interpretation of these results will be taken up in Sec. IX.

IX. RELATION OF THE RENORMALIZATION GROUP TO THE HIGH-CURVATURE LIMIT

In this section we wish to discuss the physical interpretation of the results which have been obtained from our renormalization-group analysis. It was asserted in Refs. 18, 21, and 23 that the curved-spacetime renormalization group is related to the high-curvature limit of the theory. Here we give a detailed argument which shows under what circumstances the renormalization group yields the high-curvature behavior of the theory. Our discussion will bring out some important points not previously noticed, and will show more precisely how the rescaled values of the running couplings are related to the high-curvature limit.

The effective action Γ is a dimensionless (in units with $\hbar=c=1$) functional of the renormalized background fields, coupling constants, and masses. Let us denote those quantitites collectively by q_i (we omit the caret on background fields). The metric, which may be regarded as a background field, will be denoted separately as $g_{\mu\nu}$. Consider a family of metrics $g_{\mu\nu}(s)$ parametrized by a dimensionless parameter s. We will work throughout this section at spacetime dimension $d=4$. Suppose that q_i has mass dimension δ_i (i.e., $\dim q_i=\text{mass}^{\delta_i}$). We can write

$$\Gamma[q_i(\mu),\mu,g_{\mu\nu}(s)]=F[\mu^{-\delta_i}q_i(\mu),\mu^2g_{\mu\nu}(s)] , \quad (9.1)$$

where μ is the renormalization-group parameter of dimension mass. Because Γ is dimensionless, it must be possible to write it as a functional of the dimensionless combinations appearing as the arguments of F. The metric $g_{\mu\nu}(s)$ has not been quantized, so that it has no dependence on the renormalization-group parameter μ. The effective action Γ is also independent of μ, as the bare quantities from which it was originally formed can have no μ dependence.

The high-curvature limit is that in which $g_{\mu\nu}(s)$ approaches zero as s approaches infinity. It is convenient to let $g_{\mu\nu}(s)$ approach zero in such a way that

$$s^2g_{\mu\nu}(s)\equiv g_{\mu\nu} \quad (9.2)$$

remains constant. Then $g^{\mu\nu}(s)$ is proportional to s^2 and $R_{\mu\nu}(s)$ is independent of s. It follows that $R(s)$ is proportional to s^2 and $R^{\mu\nu}(s)R_{\mu\nu}(s)$ and $R^{\mu\nu\rho\sigma}(s)R_{\mu\nu\rho\sigma}(s)$ are proportional to s^4. Thus, the invariants formed from $R^\alpha{}_{\beta\gamma\delta}(s)$ approach infinity in the large-s limit.

Replacing μ by μs in Eq. (9.1), and using the μ independence of Γ, we obtain

$$\Gamma[q_i(\mu),\mu,g_{\mu\nu}(s)]=F[\mu^{-\delta_i}s^{-\delta_i}q_i(\mu s),\mu^2g_{\mu\nu}] , \quad (9.3)$$

where Eq. (9.2) has been used. Thus, comparing the right-hand sides (RHS's) of (9.3) and (9.1), we find

$$\Gamma[q_i(\mu),\mu,g_{\mu\nu}(s)]=\Gamma[s^{-\delta_i}q_i(\mu s),\mu,g_{\mu\nu}] . \quad (9.4)$$

To find the relation between the behavior of $q_i(\mu s)$ and the effective couplings at high curvature, consider the action

$$I[q_{iB},g_{\mu\nu}]=\int dv_x[\Lambda_B+\kappa_BR+\alpha_{1B}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}+\alpha_{2B}R^{\mu\nu}R_{\mu\nu}+\alpha_{3B}R^2+L_Q(q_{iB},g_{\mu\nu})+L_C(p,\rho,g_{\mu\nu},u^\lambda)] , \quad (9.5)$$

where L_Q is the Lagrangian containing the bare quantities q_{iB}. We have also included a classical Lagrangian L_C to facilitate the subsequent discussion. We can, for example, take L_C to correspond to a perfect fluid,

$$L_C(p,\rho,g_{\mu\nu},u^\lambda)=\tfrac{1}{2}(\rho+p)u^\mu u^\nu g_{\mu\nu}-(\rho+3p)\ , \qquad (9.6)$$

which leads, upon variation of the action with respect to $g_{\mu\nu}$ to the energy-momentum tensor

$$T_C^{\mu\nu}=(\rho+p)u^\mu u^\nu-pg^{\mu\nu}\ . \qquad (9.7)$$

(The constraint $g_{\mu\nu}u^\mu u^\nu=1$ has been imposed after variation.) Because p and ρ are classical they do not depend on μ, and there is no distinction between these bare and renormalized quantities at $d=4$. We regard p and ρ as parametrized by s in such a way as to make $g_{\mu\nu}(s)$ a solution of the equations of motion [but for brevity we will write p for $p(s)$ and ρ for $\rho(s)$].

Working to one-loop order (the argument is readily extended to arbitrary order), the counterterms δq_i are chosen to cancel the pole part of $\Gamma^{(1)}$. We have

$$\Gamma[q_i(\mu),\mu,g_{\mu\nu}]=I[q_i(\mu),g_{\mu\nu}]+I^{(1)}[q_i(\mu),\mu,g_{\mu\nu}]\ , \qquad (9.8)$$

where

$$I[q_i(\mu),g_{\mu\nu}]=I[q_{iB},g_{\mu\nu}]+\text{P.P.}\{\Gamma^{(1)}[q_{iB},g_{\mu\nu}]\} \qquad (9.9)$$

and

$$I^{(1)}[q_i(\mu),\mu,g_{\mu\nu}]=\Gamma^{(1)}[q_{iB},g_{\mu\nu}]-\text{P.P.}\{\Gamma^{(1)}[q_{iB},g_{\mu\nu}]\}\ . \qquad (9.10)$$

The first term on the RHS of Eq. (9.8) has the same functional form as I in Eq. (9.5), but with the bare quantities q_{iB} replaced by the renormalized quantities $q_i(\mu)$. Because $I[q_{iB},g_{\mu\nu}]$ is dimensionless, it must satisfy the units scaling relation $I[v^{\delta_i}q_{iB},v^{-2}g_{\mu\nu}]=I[q_{iB},g_{\mu\nu}]$, where v is any dimensionless number. Therefore, we must also have

$$I[v^{\delta_i}q_i(\mu),v^{-2}g_{\mu\nu}]=I[q_i(\mu),g_{\mu\nu}]\ , \qquad (9.11)$$

as can be explicitly verified. Alternatively, because μ appears in $q_i(\mu)$ in a dimensionless combination, one can write Eq. (9.11) directly from dimensional considerations. However, the final term of Eq. (9.8), and hence the total effective action, need not obey the analogous scaling relation.

From Eqs. (9.4) and (9.8) with $g_{\mu\nu}$ replaced by $g_{\mu\nu}(s)$, we have

$$\Gamma[q_i(\mu),\mu,g_{\mu\nu}(s)]=I[s^{-\delta_i}q_i(\mu s),g_{\mu\nu}]+I^{(1)}[s^{-\delta_i}q_i(\mu s),\mu,g_{\mu\nu}]\ . \qquad (9.12)$$

Using Eq. (9.11) with $v=s$, this becomes

$$\Gamma[q_i(\mu),\mu,g_{\mu\nu}(s)]=I[q_i(\mu s),g_{\mu\nu}(s)]+I^{(1)}[s^{-\delta_i}q_i(\mu s),\mu,g_{\mu\nu}]\ . \qquad (9.13)$$

Only the q_i which are in L_Q appear in $I^{(1)}$ because it arises from renormalization of quantum fields.

Varying Eq. (9.13) with respect to $g_{\mu\nu}(s)$, and using the explicit functional form of I in Eq. (9.5) with the appropriate arguments, we obtain

$$\frac{-\delta\Gamma[q_i(\mu),\mu,g_{\mu\nu}(s)]}{\delta g_{\mu\nu}(s)}=\sqrt{-g(s)}[-\tfrac{1}{2}\Lambda(\mu s)g^{\mu\nu}(s)+\kappa(\mu s)G^{\mu\nu}(s)+\alpha_3(\mu s)^{(1)}H^{\mu\nu}(s)$$
$$+\alpha_2(\mu s)^{(2)}H^{\mu\nu}(s)+\alpha_1(\mu s)H^{\mu\nu}(s)+\tfrac{1}{2}T_Q^{\mu\nu}(q_i(\mu s),g_{\mu\nu}(s))$$
$$+\tfrac{1}{2}T_C^{\mu\nu}(p,\rho,g_{\mu\nu}(s),u^\lambda(s))+\tfrac{1}{2}s^6T^{(1)\mu\nu}(s^{-\delta_i}q_i(\mu s),\mu,g_{\mu\nu})]\ , \qquad (9.14)$$

where $T_C^{\mu\nu}(p,\rho,g_{\mu\nu}(s),u^\lambda(s))$ is given by Eq. (9.7) with $g_{\mu\nu}$ and u^λ replaced by $g_{\mu\nu}(s)$ and $u^\lambda(s)$, respectively, and

$$T_Q^{\mu\nu}(q_i(\mu s),g_{\mu\nu}(s))=-2[-g(s)]^{-1/2}\delta I_Q[q_i(\mu s),g_{\mu\nu}(s)]/\delta g_{\mu\nu}(s) \qquad (9.15)$$

with

$$I_Q[q_i(\mu s),g_{\mu\nu}(s)]=\int dv_x(s)L_Q(q_i(\mu s),g_{\mu\nu}(s))\ .$$

Also

$$T^{(1)\mu\nu}[s^{-\delta_i}q_i(\mu s),\mu,g_{\mu\nu}]=-2(-g)^{-1/2}\delta I^{(1)}[s^{-\delta_i}q_i(\mu s),\mu,g_{\mu\nu}]/\delta g_{\mu\nu}\ . \qquad (9.16)$$

The factor of s^6 in the last term of (9.14) appears because (9.13) was varied with respect to $g_{\mu\nu}(s)$ and $[-g(s)]^{1/2}$ was factored out. We have defined[62]

$${}^{(1)}H^{\mu\nu}=-(-g)^{-1/2}\frac{\delta}{\delta g_{\mu\nu}}\int dv_x R^2\ , \qquad (9.17)$$

$${}^{(2)}H^{\mu\nu}=-(-g)^{-1/2}\frac{\delta}{\delta g_{\mu\nu}}\int dv_x R^{\rho\sigma}R_{\rho\sigma}\ , \qquad (9.18)$$

$$H^{\mu\nu}=-(-g)^{-1/2}\frac{\delta}{\delta g_{\mu\nu}}\int dv_x R^{\lambda\tau\rho\sigma}R_{\lambda\tau\rho\sigma}\ . \qquad (9.19)$$

($G^{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}Rg^{\mu\nu}$ is the usual Einstein tensor.) As a consequence of the Gauss-Bonnet theorem in four dimensions the following relation holds:

$$H^{\mu\nu}=-{}^{(1)}H^{\mu\nu}+4\,{}^{(2)}H^{\mu\nu}\ . \qquad (9.20)$$

The gravitational field equation is obtained by setting Eq. (9.14) equal to zero [recall that p and ρ are parametrized by s in such a way that $g_{\mu\nu}(s)$ is a solution of the field equation].

In the absence of quantum fields, $T_Q^{\mu\nu}$ and $T^{(1)\mu\nu}$ would not be present and Λ, κ, and the α_i in Eq. (9.14) would be independent of s. In that case the α_i could be chosen to equal zero, and one would recover the usual classical theory of Einstein, with $\kappa=(16\pi G)^{-1}$ and $\Lambda=-(8\pi G)^{-1}\Lambda_c$ where Λ_c is the cosmological constant. When quantum fields are taken into account, the picture is greatly altered at high curvature. The coefficients $\alpha_i(\mu s)$ of the quadratic curvature terms are in general nonzero and grow in magnitude with increasing curvature, as do $\Lambda(\mu s)$ and $\kappa(\mu s)$. The functional forms of these running couplings are given by Eqs. (8.16)–(8.18), (8.22), (8.14), and (8.10) with μ replaced by μs [thus, in those equations we put $t=\ln(s\mu/\mu_0)$, where μ/μ_0 is a dimensionless constant]. We are free to set $\mu=\mu_0$ if we like; in this case $t=\ln s$. It should be noted that s is given in terms of the ratio of two curvature invariants, for example, $s=[R^{\mu\nu}(s)R_{\mu\nu}(s)/R^{\rho\sigma}R_{\rho\sigma}]^{1/4}$. When the effective action is written in terms of the rescaled coupling constants, it may appear that s may introduce a metric dependence into them, so that they will give additional contributions to (9.14) when varied. However, we are *guaranteed* by the method that s is in fact a constant. Thus, the running coupling constants appearing in the effective action really do have no metric dependence. The tensor $T_Q^{\mu\nu}$ has the same form as the classical energy-momentum tensor of the particle fields, with the background fields replacing the classical fields. The dynamical equations obeyed by the background fields are obtained by variation of the effective action with respect to the background fields. In the fully symmetric vacuum state, which is assumed to be the relevant ground state in the high-curvature limit, the background fields vanish. In that state, $T_Q^{\mu\nu}(q_i(\mu s),g_{\mu\nu}(s))=0$.

It is clear from Eq. (9.14) that the effective gravitational constant $G(\mu s)$ and cosmological constant $\Lambda_c(\mu s)$ depend on s. In the absence of further significant renormalization effects coming from $T^{(1)\mu\nu}$, those effective couplings are given at high curvature by

$$G(\mu s)=[16\pi\kappa(\mu s)]^{-1} \tag{9.21}$$

and

$$\Lambda_c(\mu s)=-\Lambda(\mu s)[2\kappa(\mu s)]^{-1}\ . \tag{9.22}$$

The behavior of the couplings and masses in this theory is reminiscent of variable-mass and variable-G classical theories.[63]

Suppose we take s to be sufficiently large so that the $\ln(s\mu/\mu_0)$ terms in Eq. (9.21) are dominant. From Eqs. (8.4) and (8.10) it follows that, if the constant K defined in Eq. (8.10) is positive, then G approaches zero like $(\ln s)^{-z_1/4z}\approx(\ln s)^{-1}$. In this case the effective Planck time (or length) would approach zero. On the other hand, if K is negative, then before κ can go to $-\infty$ it must pass through zero. As κ passes through zero, G first approaches $+\infty$ and then jumps to $-\infty$. However, quantum gravitational effects are likely to become important before this jump in the value of G occurs. In any event, in these theories the role of the usual Planck scale seems to become obscure.

Similarly, from Eqs. (8.14) and (9.22) and the behavior already discussed for κ, it follows that the cosmological constant Λ_c becomes large and positive. This is clearly true when K is non-negative which ensures that G always remains positive. If K is negative, then the growth of G as κ approaches zero will force Λ_c to become large and positive.

Finally, we note that for arbitrarily large values of s the dominant terms in the gravitational field equation (9.14) can be given explicitly. In a fully asymptotically free theory, the running gauge, Yukawa, and Higgs couplings (which all have $\delta_i=0$) approach zero, so that $T^{(1)\mu\nu}(s^{-\delta_i}q_i(\mu s),\mu,g_{\mu\nu})$ approaches its form for a free-field theory. Although the effective Higgs masses grow large as s increases [see Eq. (8.4)], it is the quantities $s^{-2}\mu_\phi{}^2(s)$ which enter $T^{(1)\mu\nu}$, and these expressions approach zero at large s. Therefore, we may neglect the Higgs masses in $T^{(1)\mu\nu}$ in the high-curvature limit. In addition, we have shown that $\xi_\phi(\mu s)$ and $\xi_H(\mu s)$ go over to their conformal values of $\frac{1}{6}$. Hence, for arbitrarily large values of s the expression for $T^{(1)\mu\nu}(s^{-\delta_i}q_i(\mu s),\mu,g_{\mu\nu})$ goes over into that which would be obtained from a free massless, conformally coupled theory (with the appropriate number of fields). Let us denote this contribution by $T_{\rm fc}^{(1)\mu\nu}(g_{\mu\nu})$, where fc refers to the free conformal value. Also in the limit of arbitrarily large s, the quadratic curvature terms ${}^{(i)}H^{\mu\nu}(s)$, which scale as s^6, will dominate in Eq. (9.14) over the terms involving $G^{\mu\nu}(s)$ and $g^{\mu\nu}(s)$, which scale as s^4 and s^2, respectively (the logarithmic scaling of the couplings does not alter the conclusion). Thus, in the limit of arbitrarily large s, the gravitational field equation in the symmetric phase reduces to (in four dimensions)

$$[\alpha_3(\mu s)-\alpha_1(\mu s)]\,{}^{(1)}H^{\mu\nu}(s)+[\alpha_2(\mu s)+4\alpha_1(\mu s)]\,{}^{(2)}H^{\mu\nu}(s)+\tfrac{1}{2}T_C^{\mu\nu}(p,\rho,g_{\mu\nu}(s),u^\lambda(s))+\tfrac{1}{2}s^6T_{\rm fc}^{(1)\mu\nu}(g_{\mu\nu})=0\ . \tag{9.23}$$

Note that Eq. (9.23) only depends on the number and types of fields present and not on the details of a particular asymptotically free grand unified theory.

Using the results for the α_i obtained in Eqs. (8.16)–(8.22), we find that in the limit of large s (or large μ) the first two terms of Eq. (9.23) become

$$(\alpha_3-\alpha_1)^{(1)}H^{\mu\nu}+(\alpha_2+4\alpha_1)^{(2)}H^{\mu\nu}=A(-\tfrac{2}{3}{}^{(1)}H^{\mu\nu}+2^{(2)}H^{\mu\nu})\ , \tag{9.24}$$

where

$$A=-(4\pi)^{-2}\ln\left[\frac{\mu s}{\mu_0}\right](120)^{-1}(N_0^R+2N_0^C+12N_1+3N_{1/2}^W+6N_{1/2}^D)\ . \tag{9.25}$$

Noting that the square of the Weyl tensor is

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}=R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}-2R^{\alpha\beta}R_{\alpha\beta}+\tfrac{1}{3}R^2\ ,$$

we observe using the definitions in Eqs. (9.17) and (9.18) that

$$-\tfrac{2}{3}{}^{(1)}H^{\mu\nu}+2{}^{(2)}H^{\mu\nu}$$

$$=-(-g)^{-1/2}\frac{\delta}{\delta g_{\mu\nu}}\int dv_x C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}\ . \qquad (9.26)$$

[The Gauss-Bonnet identity (9.20) has been used here.]

The result in Eq. (9.26) implies that the gravitational part of the Lagrangian at very high curvature must be a linear combination of $C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$ and the quantity

$$G=R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}-4R^{\alpha\beta}R_{\alpha\beta}+R^2$$

which is the integrand of the Euler characteristic, and therefore a topological invariant. In fact, it is easy to see that in this limit (keeping only the dominant terms)

$$L_{\text{curv}}=AC^{\alpha\beta\gamma\delta}(s)C_{\alpha\beta\gamma\delta}(s)+BG(s)\ , \qquad (9.27)$$

where

$$B=(4\pi)^{-2}\ln\left[\frac{\mu s}{\mu_0}\right](720)^{-1}(2N_0^R+4N_0^C+124N_1+11N_{1/2}^W+22N_{1/2}^D)\ . \qquad (9.28)$$

This result for L_{curv} is valid quite generally, provided that ξ_ϕ and ξ_H approach $\frac{1}{6}$ in the high-curvature limit (as in asymptotically free GUT's), or else that scalar fields are not present at all. In such theories, the high-curvature limit will have field equations which are invariant under conformal transformations $[g_{\mu\nu}\to\Omega^2(x)g_{\mu\nu}]$. It is also of interest that B is positive [see Eq. (9.28)] and A is negative [see Eq. (9.25)] regardless of the numbers of fields of each type that are present in the GUT.

Lagrangians of the general form of Eq. (9.27) have been considered as candidates for a possible fundamental gravitational Lagrangian in induced gravity. (See, for example, S. L. Adler in Ref. 67.) If Eq. (9.27) were to be adopted as a fundamental gravitational action, then the negativity of A would ensure that the generating functional converges for metrics of Riemannian signature.

The case of conformal flatness requires special consideration. In conformally flat spacetimes such as the Robertson-Walker universes, one has (see p. 183 of Ref. 61)

$$T^{(1)\mu\nu}_{(\text{fc})}(g_{\mu\nu})=-a_1{}^{(1)}H^{\mu\nu}+a_3{}^{(2)}H^{\mu\nu}\ , \qquad (9.29)$$

where a_1 and a_3 are known numerical constants which depend on the numbers of fields of each spin present, and

$${}^{(3)}H^{\mu\nu}=\tfrac{1}{12}R^2g^{\mu\nu}-R_{\alpha\beta}R^{\alpha\mu\beta\nu}\ . \qquad (9.30)$$

${}^{(3)}H^{\mu\nu}$ is a locally conserved tensor in conformally flat spacetimes, but cannot be obtained from the variation of a geometrical term in the action. The quantity ${}^{(2)}H^{\mu\nu}$ does not appear explicitly in Eq. (9.29) because in conformally flat spacetimes the identity

$${}^{(2)}H^{\mu\nu}=\tfrac{1}{3}{}^{(1)}H^{\mu\nu} \qquad (9.31)$$

holds.

As a consequence of this last identity, Eq. (9.24) vanishes. Therefore the logarithmically growing terms present in $\alpha_i(\mu s)$ cancel in Eq. (9.23). For the asymptotically free theory, it follows from Eqs. (8.16)—(8.22) that at very high curvature in conformally flat spacetimes, we are left with Eq. (9.23), but with the coefficients of ${}^{(1)}H^{\mu\nu}(s)$ and ${}^{(2)}H^{\mu\nu}(s)$ constants involving the α_{i0}. Because the ${}^{(i)}H^{\mu\nu}(s)$ are quadratic in the curvature, we have

$$S^6T^{(1)\mu\nu}_{(\text{fc})}(g_{\mu\nu})=T^{(1)\mu\nu}_{(\text{fc})}(g_{\mu\nu}(s)) \qquad (9.32)$$

for conformally flat spacetimes.

The identity given in Eq. (9.31) will also hold in many nonconformally flat spacetimes. For example, in any Einstein space ($R_{\mu\nu}=\lambda g_{\mu\nu}$ with constant λ) both ${}^{(1)}H^{\mu\nu}$ and ${}^{(2)}H^{\mu\nu}$ vanish identically. In such cases, the terms involving $\ln(\mu s/\mu_0)$ in Eq. (9.23) will cancel, even if ${}^{(1)}H^{\mu\nu}$ and ${}^{(2)}H^{\mu\nu}$ are nonzero.

Nevertheless, it is important to point out that in a general spacetime the logarithmically growing terms in the $\alpha_i(\mu s)$ will be present in Eq. (9.23) and give the dominant contribution to the effective coupling constants. It would require a detailed analysis to reveal the relative importance of the various effective coupling constants at the GUT scale. It seems likely that the running couplings $\Lambda(\mu s)$, $\kappa(\mu s)$, $\alpha_i(\mu s)$, and $\xi(\mu s)$ all play an important role in the evolution of cosmological models containing grand unified fields.

X. DISCUSSION AND CONCLUSIONS

In the preceding sections we have discussed how the curved-spacetime renormalization-group method could be used to analyze the high-curvature behavior of a gauge theory. We were particularly interested in the behavior of the coupling constants in Eq. (2.1) which are not present in flat spacetime. We considered gauge theories which are totally asymptotically free. A similar renormalization-group analysis based on perturbation theory should be applicable to other gauge theories, such as the Georgi-Glashow theory, if as is usually assumed, the nonasymptotically free couplings do not grow too large. We studied a particular class of totally asymptotically free theories based on the gauge group SU(5) which are due to Chang, Das, and Perez-Mercader,[11] although we believe that the results which have been obtained are fairly general.

In Sec. VI, the effective couplings ξ_ϕ and ξ_H were calculated. We discussed two possible ways which these constants could approach the conformal value of $\frac{1}{6}$ in the high-curvature limit in an asymptotically free theory. The first is that the eigenvalue conditions imposed on the coupling constants in the flat-spacetime part of the theory which ensure asymptotic freedom could naturally lead to solutions of the renormalization-group equations with $\frac{1}{6}$ as an ultraviolet limit. This was the condition that λ_1,λ_2 in Eq. (6.6) be both positive. Although this was not true for the particular examples which we examined, it is possible that other models could have this behavior. A second way, which is in the spirit of a one-coupling-constant theory, is to look for solutions to the renormalization-group equations which are proportional to some power of the gauge coupling. We argued that such solutions should always exist and are such that the ξ parameters approach $\frac{1}{6}$ in the high-curvature limit. In addition, because of asymptotic freedom, the other interactions approach zero. It should be noted that in gauge theories with this behavior, particle creation will be suppressed in isotropically expanding universes.[64,56,26] This may, however, not be a generic feature of gauge theories, since nonconformal values of ξ as well as interactions can give particle production.

The behavior under the renormalization group of Λ, κ, and α_i appearing in Eq. (2.1) was discussed in Sec. IX. Although zero or very small today, the effective cosmological constant Λ_c can become large and positive at high curvature. This may have implications for inflation[65] in the early universe. Similarly, the $\alpha_i(\mu s)$ coupling constants, which multiply terms in the action which are quadratic in the curvature, are expected to make a significant contribution to the effective Einstein equations at early times, even if their values today are small. The behavior of the effective gravitational constant requires a more detailed analysis. However, if our result for κ is used in the limit of arbitrarily large curvature, then the effective value of G would be significantly altered from its present value. In order to quantitatively study the detailed behavior of the above effective coupling constants in the early universe, it would be necessary to start from the low-energy limit of the theory with the known values of the coupling constants and to integrate the renormalization-group equations up to and beyond the GUT scale. We are presently investigating this.

The behavior found here is similar to the basic idea behind induced gravity[66,67] in which the Λ and κR terms are not present in the initial gravitational action, but are induced by quantum effects. The main difference between the renormalization-group method and that of induced gravity is that we examine the high-curvature limit of the theory, whereas it is the low-curvature limit which is obtained in induced gravity. (Also, to obtain unique predictions for the low-curvature values of Λ_c and G, Higgs scalars are generally excluded.)

The necessity of including curvature-squared terms in the bare Lagrangian in Eq. (2.1) for purposes of renormalizability was first pointed out by Utiyama and DeWitt.[68] There have been several studies of the effects that nonzero values of these constants can have in cosmology.[69–74] We discussed in Sec. IX that in the limit of arbitrarily large curvature, the effective action involved only the combination $AC^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}+BG$. [See Eqs. (9.27), (9.28), and (9.25).] This leads to field equations with interesting conformal properties. The α_i would lead to no effects in the gravitational field equations if the effective gravitational action were to approach only the topological invariant involving G. However, this can never occur in any grand unified model since all of the particle species present contribute to A in Eq. (9.25) with the same sign.

A number of cosmological consequences of superheavy fermions in the asymptotically free GUT theory of Fradkin and Kalashnikov[16] have been investigated by Kalashnikov and Khlopov.[75] These superheavy fermions will also influence the effect of an initially anisotropic expansion of the universe on the baryon-to-entropy ratio of the universe. The effect of initial anisotropy on that ratio was studied by one of us,[76] and the discussion can be extended to include superheavy fermions. In that discussion,[76] it was assumed that the particles entering into the GUT mechanism for generating baryon number were created by the gravitational field of the expanding universe at about the Planck temperature. Our present results indicate that the effective coupling constants at that time (in the asymptotically free GUT theory of Chang, Das, and Perez-Mercader[11]) may be values which suppress the production of Higgs and gauge bosons in isotropically expanding universes. As the decay of those bosons are principal sources of baryon-antibaryon asymmetry, an initially anisotropic expansion of the universe may be required in order to produce the observed value of the baryon-to-entropy ratio.

We have not considered what effects there might be coming from graviton loops, since quantum gravity is not renormalizable or asymptotically free, at least within the standard perturbative context. One possibility is to include quantum-gravity effects in our analysis, and in the spirit of the fully asymptotically free theories, look for a solution of the renormalization-group equations in which κ, α_i, and Λ involve some appropriate power of the gauge coupling. We also note that because there are large numbers of matter fields present in grand unified theories it may be possible to use the $1/N$ expansion to justify the neglect of graviton loops.[77–80]

Interesting work of Fradkin and Tseytlin[81] has indicated that quantization of a theory which includes curvature-squared terms in addition to the basic Einstein-Hilbert gravitational action may result in an asymptotically free theory of gravity. They also suggest that the inclusion of such quantum gravitational effects may change the behavior of the nongauge couplings in GUT's, which do not normally exhibit asymptotic freedom to be asymptotically free. A renormalization-group analysis of the gravitational coupling constants is presented, but without the curved-spacetime interpretation of the renormalization group which we considered here. In addition, we have examined in detail the effects of matter interactions in a realistic GUT, and exhibited explicit results on the ξ coupling constants. It would be of interest to see whether or not the quantization of gravity would result in the full asymptotic freedom of the Georgi-Glashow model (or any other GUT) in the manner suggested by Fradkin and Tseytlin.

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APPENDIX A: SUMMARY OF GROUP NOTATION

Our metric and curvature conventions are taken to be $(---)$ in the notation of Ref. 82. Explicitly, our metric has signature -2, and the Riemann curvature tensor is defined in terms of the Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$ (in a coordinate basis) by

$$R^{\lambda}{}_{\mu\nu\sigma}=\Gamma^{\lambda}_{\mu\nu,\sigma}-\Gamma^{\lambda}_{\mu\sigma,\nu}+\Gamma^{\lambda}_{\sigma\rho}\Gamma^{\rho}_{\mu\nu}-\Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\mu\sigma}\ . \qquad (A1)$$

The Ricci tensor is defined by $R_{\mu\nu}=R^{\lambda}{}_{\mu\lambda\nu}$.

The generators for the Lie algebra of the gauge group are denoted by T^i and are chosen to be Hermitian and satisfy

$$[T^i,T^j]=if^{ijk}T^k \ . \tag{A2}$$

Here $i,j,\ldots=1,2,\ldots,N$ where N is the dimension of the gauge group G and f^{ijk} are the structure constants which are assumed to be antisymmetric. The quadratic Casimir invariant is defined by

$$T^iT^i=C_2(G_R)I \ , \tag{A3}$$

where T^i is in a representation G_R of G of dimension d_R (i.e., T^i are $d_R\times d_R$ Hermitian matrices). The generators also satisfy the identity

$$\operatorname{tr}(T^iT^j)=N^{-1}C_2(G_R)d_R\delta^{ij} \ . \tag{A4}$$

If we let L^i denote the generators in the adjoint representation then the matrix elements of L^i are $(L^i)^j{}_k=-if^{ijk}$. From (A4) the structure constants are seen to satisfy

$$f^{ilm}f^{jlm}=C_2(G_{\mathrm{adj}})\delta^{ij} \ , \tag{A5}$$

$$f^{ijk}f^{ijk}=NC_2(G_{\mathrm{adj}}) \ , \tag{A6}$$

where G_{adj} denotes the adjoint representation.

Of special interest to us is the gauge group $G=\mathrm{SU}(n)$. In this case the group has dimension $N=n^2-1$, and the Casimir invariant of the adjoint representation is $C_2(G_{\mathrm{adj}})=n$, so that for $\mathrm{SU}(n)$ Eqs. (A5) and (A6) become

$$f^{ilm}f^{jlm}=n\delta^{ij} \ , \tag{A7}$$

$$f^{ijk}f^{ijk}=n(n^2-1) \ . \tag{A8}$$

The fundamental representation G_F of $\mathrm{SU}(n)$ has dimension $d_F=n$. The quadratic Casimir invariant for the fundamental representation is $C_2(G_F)=(n^2-1)/2n$. If F^i denote the generators in the fundamental representation, then from (A4)

$$\operatorname{tr}(F^iF^j)=\tfrac{1}{2}\delta^{ij} \ . \tag{A9}$$

From (A3),

$$F^iF^i=\left[\frac{n^2-1}{2n}\right]I \ . \tag{A10}$$

It is straightforward to use these results to derive a number of identities which are needed to evaluate the counterterms. These required expressions are

$$\operatorname{tr}(F^iF^jF^kF^k)=\frac{n^2-1}{4n}\delta^{ij} \ , \tag{A11}$$

$$\operatorname{tr}(F^iF^kF^jF^k)=-\frac{1}{4n}\delta^{ij} \ , \tag{A12}$$

$$\operatorname{tr}(F^iF^jF^k)\operatorname{tr}(F^iF^jF^l)=-\frac{1}{4n}\delta^{kl} \ , \tag{A13}$$

$$\operatorname{tr}(F^iF^jF^k)\operatorname{tr}(F^jF^iF^l)=\frac{n^2-2}{8n}\delta^{kl} \ . \tag{A14}$$

We now explain the terms occurring in our Lagrangian (3.1)–(3.3). $F^i_{\mu\nu}$ is the usual Yang-Mills field strength:

$$F^i_{\mu\nu}=\partial_\mu A^i_\nu-\partial_\nu A^i_\mu+gf^{ijk}A^j_\mu A^k_\nu \ . \tag{A15}$$

It proves convenient to also consider a matrix-valued gauge connection

$$A_\mu=A^i_\mu T^i \tag{A16}$$

and field strength

$$F_{\mu\nu}=F^i_{\mu\nu}T^i \ . \tag{A17}$$

Then

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-ig[A_\mu,A_\nu] \ . \tag{A18}$$

Defining

$$U(\theta)=e^{-i\theta_i(x)T^i} \ , \tag{A19}$$

where the $\theta_i(x)$ are the group parameters, A_μ and $F_{\mu\nu}$ behave like

$$A_\mu\to UA_\mu U^{-1}-\frac{i}{g}(\partial_\mu U)U^{-1} \ , \tag{A20}$$

$$F_{\mu\nu}\to UF_{\mu\nu}U^{-1} \tag{A21}$$

under a gauge transformation.

If $\psi(x)$ is any set of fields transforming like

$$\psi(x)\to U\psi(x) \tag{A22}$$

under a gauge transformation then the covariant derivative of ψ is defined by

$$D_\mu\psi(x)=\nabla_\mu\psi(x)-igA_\mu(x)\psi(x) \tag{A23}$$

and transforms as ψ does. Here ∇_μ is the spacetime-covariant derivative. This is the case for the Higgs field H in (3.2), which is a complex scalar field in the fundamental representation of $\mathrm{SU}(n)$. We have

$$(D_\mu H)^a=\partial_\mu H^a(x)-igA^i_\mu(x)(F^i)^a{}_bH^b(x) \tag{A24}$$

in component form, where $(F^i)^a{}_b$ denotes the matrix elements of the fundamental representation with $a,b,\ldots=1,\ldots,n$.

The other set of Higgs fields occurring in (3.2) are a set of real scalar fields denoted by Φ and transform under the adjoint representation:

$$\Phi\to U\Phi U^{-1} \ . \tag{A25}$$

These are the fields responsible for breaking the SU(5) symmetry. The covariant derivative of Φ is

$$D_\mu\Phi=\partial_\mu\Phi-ig[A_\mu,\Phi] \ . \tag{A26}$$

If we define (n^2-1) real scalar fields ϕ^i by

$$\Phi=\phi^iF^i \ , \tag{A27}$$

then

$$(D_\mu\Phi)^i=\partial_\mu\phi^i+gf^{ijk}A^j_\mu\phi^k \ . \tag{A28}$$

Because of the normalization of F^i in (A9) we have

$$\operatorname{tr}[D_\mu\Phi)(D^\mu\Phi)]=\tfrac{1}{2}(\partial_\mu\phi^i+gf^{ijk}A^j_\mu\phi^k)\times(\partial^\mu\phi^i+gf^{ilm}A^{l\mu}\phi^m) \ , \tag{A29}$$

which is the standard kinetic term.

Spinor notation is dealt with in Appendix B.

APPENDIX B: SPINOR FIELDS IN CURVED SPACETIME

In order to deal with spinor fields in curved spacetime, one possible approach is to refer everything to a local orthonormal frame using the vierbein formalism.[38,83,84] Because the spinors occurring in the unbroken Georgi-Glashow model are all either right handed (RH) or left handed (LH) we shall treat them as two-component Weyl spinors. In addition we shall treat them as anticommuting which is not done in standard classical treatments of two-component spinors in curved spacetime.[85,86]

Let $\{\theta^{\hat{a}}(x)\}$ be orthonormal basis one-forms with

$$ds^2=\eta_{\hat{a}\hat{b}}\theta^{\hat{a}}(x)\otimes\theta^{\hat{b}}(x) . \tag{B1}$$

Here we use Latin indices with a caret symbol to refer to local orthonormal frame or vierbein indices. In terms of $g_{\mu\nu}$, $ds^2=g_{\mu\nu}(x)dx^{\mu}\otimes dx^{\nu}$ so that we may define

$$\theta^{\hat{a}}(x)=e^{\hat{a}}{}_{\mu}(x)dx^{\mu} , \tag{B2}$$

where

$$g_{\mu\nu}(x)=e^{\hat{a}}{}_{\mu}(x)e^{\hat{b}}{}_{\nu}(x)\eta_{\hat{a}\hat{b}} . \tag{B3}$$

The $\{e^{\hat{a}}{}_{\mu}(x)\}$ give the vierbein (or tetrad) field which transforms like a contravariant vector under local Lorentz transformations, and a covariant vector under a general coordinate transformation. The inverse of $e^{\hat{a}}{}_{\mu}$ we denote by $e_{\hat{a}}{}^{\mu}$ and take it to satisfy

$$\eta_{\hat{a}\hat{b}}=e_{\hat{a}}{}^{\mu}(x)e_{\hat{b}}{}^{\nu}(x)g_{\mu\nu}(x) . \tag{B4}$$

Any tensor with spacetime indices may be referred to the local orthonormal frame by contracting it with the appropriate number of vierbein fields (e.g., $T^{\hat{a}\hat{b}}=e^{\hat{a}}{}_{\mu}e^{\hat{b}}{}_{\nu}T^{\mu\nu}$).

Define connection one-forms $\omega^{\hat{a}}{}_{\hat{b}}$ with components

$$\omega^{\hat{a}}{}_{\hat{b}}=\omega^{\hat{a}}{}_{\hat{b}\mu}dx^{\mu} \tag{B5}$$

by

$$d\theta^{\hat{a}}+\omega^{\hat{a}}{}_{\hat{b}}\wedge\theta^{\hat{b}}=0 . \tag{B6}$$

The curvature two-form $\mathscr{R}^{\hat{a}}{}_{\hat{b}}$ is defined by

$$\mathscr{R}^{\hat{a}}{}_{\hat{b}}=d\omega^{\hat{a}}{}_{\hat{b}}+\omega^{\hat{a}}{}_{\hat{c}}\wedge\omega^{\hat{c}}{}_{\hat{b}} \tag{B7}$$

and has components

$$\mathscr{R}^{\hat{a}}{}_{\hat{b}}=-\tfrac{1}{2}R^{\hat{a}}{}_{\hat{b}\mu\nu}dx^{\mu}\wedge dx^{\nu} , \tag{B8}$$

where $R^{\hat{a}}{}_{\hat{b}\mu\nu}$ is the Riemann curvature tensor. [For more details see, for example, Ref. 82. Note that our conventions differ from this reference (see Appendix A).] From (B2) it is easy to show that

$$\omega^{\hat{a}}{}_{\hat{b}\mu}=e^{\hat{a}\nu}e_{\hat{b}\nu;\mu} , \tag{B9}$$

$$R^{\hat{a}}{}_{\hat{b}\mu\nu}=\omega^{\hat{a}}{}_{\hat{b}\mu,\nu}-\omega^{\hat{a}}{}_{\hat{b}\nu,\mu}+\omega^{\hat{a}}{}_{\hat{c}\nu}\omega^{\hat{c}}{}_{\hat{b}\mu}-\omega^{\hat{a}}{}_{\hat{c}\mu}\omega^{\hat{c}}{}_{\hat{b}\nu} , \tag{B10}$$

where $e_{\hat{b}\nu;\mu}=e_{\hat{b}\nu,\mu}-\Gamma^{\lambda}_{\mu\nu}e_{\hat{b}\lambda}$.

The Dirac γ matrices satisfy

$$\{\gamma^{\hat{a}},\gamma^{\hat{b}}\}=2\eta^{\hat{a}\hat{b}} \tag{B11}$$

and γ_5 is defined by

$$\gamma_5=-i\gamma^{\hat{0}}\gamma^{\hat{1}}\gamma^{\hat{2}}\gamma^{\hat{3}} . \tag{B12}$$

For two-component spinors, the Weyl representation for the γ matrices is the most useful:

$$\gamma^{\hat{0}}=\begin{bmatrix}0 & I\\ I & 0\end{bmatrix},\quad \gamma^{\hat{1}}=\begin{bmatrix}0 & \tau_x\\ -\tau_x & 0\end{bmatrix},$$
$$\gamma^{\hat{2}}=\begin{bmatrix}0 & \tau_y\\ -\tau_y & 0\end{bmatrix},\quad \gamma^{\hat{3}}=\begin{bmatrix}0 & \tau_z\\ -\tau_z & 0\end{bmatrix}, \tag{B13}$$

where

$$\tau_x=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$

$$\tau_y=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},$$

$$\tau_z=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

are the Pauli matrices. In this representation,

$$\gamma_5=\begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix} . \tag{B14}$$

Let Ψ be a four-component Dirac spinor and define

$$\Psi_L=\tfrac{1}{2}(1+\gamma_5)\Psi , \tag{B15}$$

$$\Psi_R=\tfrac{1}{2}(1-\gamma_5)\Psi . \tag{B16}$$

With the Weyl representation we may write

$$\Psi=\begin{bmatrix}\psi_L\\ \psi_R\end{bmatrix} , \tag{B17}$$

where ψ_L and ψ_R are two-component spinors which give the LH and RH parts of Ψ, respectively. The adjoint spinor is defined by $\overline{\Psi}=\Psi^{\dagger}\gamma^{\hat{0}}$ which in the Weyl representation has components

$$\overline{\Psi}=(\psi_R^{\dagger},\psi_L^{\dagger}) . \tag{B18}$$

We now adopt the two-component index notation

$$\psi_L=\begin{bmatrix}\psi_L^1\\ \psi_L^2\end{bmatrix} , \tag{B19}$$

$$\psi_R=\begin{bmatrix}\psi_{R\dot{1}}\\ \psi_{R\dot{2}}\end{bmatrix} , \tag{B20}$$

so that ψ_L carries contravariant undotted indices, and ψ_R carries covariant dotted indices. We use capital Latin letters $A,B,\ldots$ to run over 1,2. The complex conjugates are defined by $(\psi_L^A)^* = \overline{\psi}_L^{\dot{A}}$ and $(\psi_{R\dot{A}})^* = \overline{\psi}_{RA}$. The antisymmetric $\epsilon^{AB}, \epsilon_{AB}$ are used to raise and lower indices by

$$\psi^A = \epsilon^{AB}\psi_B, \quad \psi_A = \psi^B \epsilon_{BA} , \tag{B21}$$

where $\epsilon^{12}=1$, $\epsilon_{12}=1$. $\epsilon_{\dot{A}\dot{B}} = (\epsilon_{AB})^*$, $\epsilon^{\dot{A}\dot{B}} = (\epsilon^{AB})^*$ are used to raise and lower dotted indices. [Note that $\epsilon^{AB} = (i\tau_y)^{AB}$.]

The charge conjugate of a Dirac spinor is defined by

$$\Psi^c = C\overline{\Psi}^T , \tag{B22}$$

where C is the charge-conjugation matrix which may be chosen to be $C = i\gamma^{\hat{2}}\gamma^{\hat{0}}$. [$C$ is defined by $C\gamma^{\hat{a}}C^{-1} = -(\gamma^{\hat{a}})^T$.] In terms of ϵ^{AB}, it follows that

$$C = \begin{bmatrix} \epsilon^{AB} & 0 \\ 0 & -\epsilon_{\dot{A}\dot{B}} \end{bmatrix} . \tag{B23}$$

Defining in the Weyl representation

$$\Psi^c = \begin{bmatrix} \psi_L^c \\ \psi_R^c \end{bmatrix}$$

we have

$$\psi_L^{cA} = \overline{\psi}_R^{A}, \quad \psi^c_{R\dot{A}} = \overline{\psi}_{L\dot{A}} . \tag{B24}$$

Because one-spinors are treated as anticommuting, that is,

$$\psi^A \chi^B = -\chi^B \psi^A , \tag{B25}$$

although we still have $\psi^A\chi_A = -\psi_A\chi^A$, it no longer follows that $\psi^A\psi_A = 0$ as in the usual approach.[85,86]

Every four-vector $V_{\hat{a}}$ is associated with a Hermitian two-spinor $V_{A\dot{B}}$ by

$$V_{A\dot{B}} = \sigma^{\hat{a}}{}_{A\dot{B}} V_{\hat{a}} , \tag{B26}$$

where $\sigma^{\hat{a}}{}_{A\dot{B}}$ is Hermitian and transforms as a contravariant vector on its Lorentz index. (The Hermiticity property is $\overline{\sigma}^{\hat{a}}{}_{\dot{A}B} = \sigma^{\hat{a}}{}_{\dot{A}B}$ where the order of dotted and undotted indices is unimportant.) Take

$$\sigma^{\hat{a}}{}_{A\dot{B}}\sigma^{\hat{b}A\dot{B}} = 2\eta^{\hat{a}\hat{b}} , \tag{B27}$$

which means that $V^{A\dot{B}}V_{A\dot{B}} = 2V^{\hat{a}}V_{\hat{a}}$. We also have

$$\sigma^{\hat{a}A\dot{C}}\sigma^{\hat{b}}{}_{\dot{C}B} + \sigma^{\hat{b}A\dot{C}}\sigma^{\hat{a}}{}_{\dot{C}B} = 2\eta^{\hat{a}\hat{b}}\delta^A_B . \tag{B28}$$

The connection between $\sigma^{\hat{a}}$ and $\gamma^{\hat{a}}$ is

$$\gamma^{\hat{a}} = \begin{bmatrix} 0 & \sigma^{\hat{a}A\dot{B}} \\ \sigma^{\hat{a}}{}_{\dot{A}B} & 0 \end{bmatrix} , \tag{B29}$$

where

$$\sigma^{\hat{a}}{}_{A\dot{B}} = (I, -\tau_x, \tau_y, -\tau_z)_{A\dot{B}} , \tag{B30}$$

$$\sigma^{\hat{a}}{}_{\dot{A}B} = (I, -\tau_x, -\tau_y, -\tau_z)_{\dot{A}B} . \tag{B31}$$

Define the covariant derivative of a spinor ψ^A by

$$\nabla_\mu \psi^A = \partial_\mu \psi^A + \Gamma_\mu{}^A{}_B \psi^B , \tag{B32}$$

where $\Gamma_\mu{}^A{}_B$ is the spin connection. ∇_μ is taken to be real and obey the usual rules for a covariant derivative. (See, for example, Refs. 85 and 86.) In addition it satisfies

$$\nabla_\mu \epsilon^{AB} = 0, \quad \nabla_\mu \epsilon_{AB} = 0 , \tag{B33}$$

$$\nabla_\mu \sigma^\nu{}_{A\dot{B}} = 0 , \tag{B34}$$

where $\sigma^\nu{}_{A\dot{B}} = e_{\hat{a}}{}^\nu \sigma^{\hat{a}}{}_{A\dot{B}}$. The first condition allows us to raise and lower spinor indices inside of a covariant derivative, and the second condition implies that $\nabla_\mu g_{\lambda\sigma} = 0$. It then follows from (B32) that

$$\nabla_\mu \psi_A = \partial_\mu \psi_A - \Gamma_{\mu A}{}^B \psi_B , \tag{B35}$$

$$\nabla_\mu \overline{\psi}^{\dot{A}} = \partial_\mu \overline{\psi}^{\dot{A}} + \overline{\Gamma}_\mu{}^{\dot{A}}{}_{\dot{B}} \overline{\psi}^{\dot{B}} , \tag{B36}$$

$$\nabla_\mu \overline{\psi}_{\dot{A}} = \partial_\mu \overline{\psi}_{\dot{A}} - \overline{\Gamma}_{\mu\dot{A}}{}^{\dot{B}} \overline{\psi}_{\dot{B}} . \tag{B37}$$

Coupled with the Liebnitz rule which ∇_μ is assumed to obey, Eqs. (B33) and (B35)–(B37) tell us how to take the covariant derivative of any spinor.

From (B33) it is easily seen that

$$\Gamma_\mu{}^A{}_B = \Gamma_{\mu B}{}^A . \tag{B38}$$

An explicit expression for $\Gamma_\mu{}^A{}_B$ may be found from the condition (B34). It is

$$\Gamma_\mu{}^A{}_B = \tfrac{1}{4}\sigma^{\hat{a}A\dot{C}}\sigma^{\hat{b}}{}_{B\dot{C}}\omega_{\hat{a}\hat{b}\mu} , \tag{B39}$$

where $\omega_{\hat{a}\hat{b}\mu}$ was defined in Eqs. (B5) and (B9). Define

$$\Gamma_\mu{}^A{}_B = \tfrac{1}{2}\omega_{\hat{a}\hat{b}\mu}(J^{\hat{a}\hat{b}})^A{}_B . \tag{B40}$$

Then

$$(J^{\hat{a}\hat{b}})^A{}_B = \tfrac{1}{4}(\sigma^{\hat{a}A\dot{C}}\sigma^{\hat{b}}{}_{\dot{C}B} - \sigma^{\hat{b}A\dot{C}}\sigma^{\hat{a}}{}_{\dot{C}B}) . \tag{B41}$$

Taking the complex conjugate of (B40) gives

$$\overline{\Gamma}_\mu{}^{\dot{A}}{}_{\dot{B}} = \tfrac{1}{2}\omega_{\hat{a}\hat{b}\mu}(\overline{J}^{\hat{a}\hat{b}})^{\dot{A}}{}_{\dot{B}} , \tag{B42}$$

where

$$\begin{aligned}(\overline{J}^{\hat{a}\hat{b}})^{\dot{A}}{}_{\dot{B}} &= [(J^{\hat{a}\hat{b}})^A{}_B]^* \\ &= \tfrac{1}{4}(\sigma^{\hat{a}\dot{A}C}\sigma^{\hat{b}}{}_{C\dot{B}} - \sigma^{\hat{b}\dot{A}C}\sigma^{\hat{a}}{}_{C\dot{B}}) .\end{aligned} \tag{B43}$$

Both $J^{\hat{a}\hat{b}}$ and $\overline{J}^{\hat{a}\hat{b}}$ satisfy the commutation relations for the Lorentz group. [See Eq. (B45) below.]

It is possible to show that

$$[\nabla_\mu, \nabla_\nu]\psi^A = -\tfrac{1}{2}R_{\mu\nu\hat{a}\hat{b}}(J^{\hat{a}\hat{b}})^A{}_B\psi^B \tag{B44}$$

using the basic definition (B32) along with (B40), the commutation relation

$$[J^{\hat{a}\hat{b}}, J^{\hat{c}\hat{d}}] = \eta^{\hat{a}\hat{c}}J^{\hat{d}\hat{b}} + \eta^{\hat{a}\hat{d}}J^{\hat{b}\hat{c}} + \eta^{\hat{b}\hat{c}}J^{\hat{a}\hat{d}} + \eta^{\hat{b}\hat{d}}J^{\hat{c}\hat{a}} \tag{B45}$$

and the definition of the curvature tensor in (B10). Because a spinor index may be lowered inside of a covariant derivative,

$$[\nabla_\mu,\nabla_\nu]\psi_A=\tfrac{1}{2}R_{\mu\nu\hat{a}\hat{b}}(J^{\hat{a}\hat{b}})_A{}^B\psi_B\ . \qquad (B46)$$

[Equation (B38) and definitions (B21) have been used to obtain this.] Taking the complex conjugate of (B44) and (B46) yields

$$[\nabla_\mu,\nabla_\nu]\bar{\psi}^{\dot{A}}=-\tfrac{1}{2}R_{\mu\nu\hat{a}\hat{b}}(\bar{J}^{\hat{a}\hat{b}})^{\dot{A}}{}_{\dot{B}}\bar{\psi}^{\dot{B}}\ , \qquad (B47)$$

$$[\nabla_\mu,\nabla_\nu]\bar{\psi}_{\dot{A}}=\tfrac{1}{2}R_{\mu\nu\hat{a}\hat{b}}(\bar{J}^{\hat{a}\hat{b}})_{\dot{A}}{}^{\dot{B}}\bar{\psi}_{\dot{B}}\ . \qquad (B48)$$

Contracting (B48) with $\sigma^\mu{}_{\dot{B}C}\sigma^{\nu C\dot{A}}$ and using (B28) and the identity

$$\sigma^\mu{}_{\dot{B}C}\sigma^{\nu C\dot{A}}(\bar{J}^{\hat{a}\hat{b}})_{\dot{A}}{}^{\dot{D}}R_{\mu\nu\hat{a}\hat{b}}=R\,\delta^{\dot{D}}_{\dot{B}} \qquad (B49)$$

leads after some calculation to

$$\sigma^\mu{}_{\dot{A}C}\sigma^{\nu C\dot{D}}\nabla_\mu\nabla_\nu\bar{\psi}_{\dot{D}}=(\Box+\tfrac{1}{4}R)\bar{\psi}_{\dot{A}}\ , \qquad (B50)$$

where $\Box=\nabla^\mu\nabla_\mu$.

The analogous results for a four-component Dirac spinor follow in a straightforward manner from the two-component ones above. From (B17), (B32), and (B37),

$$\nabla_\mu\Psi=\begin{bmatrix}\nabla_\mu\psi_L{}^A\\ \nabla_\mu\psi_{R\dot{A}}\end{bmatrix}=\begin{bmatrix}\partial_\mu\psi_L{}^A+\Gamma_\mu{}^A{}_B\psi_L{}^B\\ \partial_\mu\psi_{R\dot{A}}-\bar{\Gamma}_{\mu\dot{A}}{}^{\dot{B}}\psi_{R\dot{B}}\end{bmatrix}. \qquad (B51)$$

If we define the spin connection for Dirac spinors by

$$\nabla_\mu\Psi=\partial_\mu\Psi+\Gamma_\mu\Psi \qquad (B52)$$

we may immediately read off from (B51) that

$$\Gamma_\mu=\begin{bmatrix}\Gamma_\mu{}^A{}_B & 0\\ 0 & -\bar{\Gamma}_{\mu\dot{A}}{}^{\dot{B}}\end{bmatrix}. \qquad (B53)$$

Using the two-component results in Eqs. (B47) and (B48) we have

$$[\nabla_\mu,\nabla_\nu]\Psi=\begin{bmatrix}-\tfrac{1}{2}R_{\mu\nu\hat{a}\hat{b}}(J^{\hat{a}\hat{b}})^A{}_B\psi_L{}^B\\ +\tfrac{1}{2}R_{\mu\nu\hat{a}\hat{b}}(\bar{J}^{\hat{a}\hat{b}})_{\dot{A}}{}^{\dot{B}}\psi_{R\dot{B}}\end{bmatrix}. \qquad (B54)$$

We may define

$$[\nabla_\mu,\nabla_\nu]\Psi=-\tfrac{1}{2}R_{\mu\nu\hat{a}\hat{b}}\mathcal{J}^{\hat{a}\hat{b}}\Psi\ , \qquad (B55)$$

where from (B54) it is apparent that

$$\mathcal{J}^{\hat{a}\hat{b}}=\begin{bmatrix}(J^{\hat{a}\hat{b}})^A{}_B & 0\\ 0 & -(\bar{J}^{\hat{a}\hat{b}})_{\dot{A}}{}^{\dot{B}}\end{bmatrix}. \qquad (B56)$$

Note that with this definition the Dirac spin connection in (B53) becomes

$$\Gamma_\mu=\tfrac{1}{2}\omega_{\hat{a}\hat{b}\mu}\mathcal{J}^{\hat{a}\hat{b}}\ . \qquad (B57)$$

From Eqs. (B41), (B43), and (B29) it follows from (B56) that

$$\mathcal{J}^{\hat{a}\hat{b}}=\tfrac{1}{4}[\gamma^{\hat{a}},\gamma^{\hat{b}}]\ . \qquad (B58)$$

Contraction of (B55) with $\gamma^\mu\gamma^\nu$ and use of the identity

$$R_{\hat{a}\hat{b}\hat{c}\hat{d}}\gamma^{\hat{a}}\gamma^{\hat{b}}\gamma^{\hat{c}}\gamma^{\hat{d}}=-2R \qquad (B59)$$

leads to

$$\gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu\Psi=(\Box+\tfrac{1}{4}R)\Psi\ , \qquad (B60)$$

which is analogous to the two-component result in (B50).

To complete this appendix we describe the fermion fields occurring in the Lagrangian (3.2) and (3.3). The fermions in the Georgi-Glashow part (3.2) of the Lagrangian are placed in a reducible $n^*\oplus\tfrac{1}{2}n(n-1)$ representation of SU(n). The $\tfrac{1}{2}n(n-1)$ representation is formed from the antisymmetric tensor product of two n representations. $\psi_L^{ab}=-\psi_L^{ba}$ denotes the LH Weyl spinor which transforms under this representation. Because of our index conventions in (B19) it has a contravariant undotted spinor index. (Recall that $a,b=1,\ldots,n$ here.) The covariant derivative of this spinor is

$$(D_\mu\psi_L)^{ab}=\nabla_\mu\psi_L^{ab}-igA^i_\mu(F^i)^a{}_c\psi_L^{cb}-igA^i_\mu(F^i)^b{}_c\psi_L^{ac}\ , \qquad (B61)$$

where ∇_μ includes the spin connection and is defined in (B32). (We have suppressed the two-component spinor indices.)

The spinor χ_{La} is taken to be a LH Weyl spinor which transforms under the n^* representation of SU(n). (Equally well one may take the charge-conjugate RH Weyl spinor which transforms under the n representation; see Ref. 49.) Its covariant derivative is

$$(D_\mu\chi_L)_a=\nabla_\mu\chi_{La}+ig\chi_{Lb}(F^i)^b{}_aA^i_\mu\ , \qquad (B62)$$

where again spinor indices have been suppressed. (χ_{La} carries contravariant undotted indices.)

The spinor fields occurring in Eq. (3.3) are those introduced by Chang, Das, and Perez-Mercader[11] in order to obtain an asymptotically free theory. Θ^a is a set of n four-component Dirac spinors transforming under the n representation. The covariant derivative of Θ^a is

$$(D_\mu\Theta)^a=\nabla_\mu\Theta^a-igA^i_\mu(F^i)^a{}_b\Theta^b\ , \qquad (B63)$$

where $\nabla_\mu\Theta^a$ was defined in (B52), (B57) and (B58). The spinors denoted by B_i are (n^2-1) four-component Dirac spinors. If $B=B_iF^i$, then B is taken to transform under the adjoint representation of SU(n). Its covariant derivative is like that for the Higgs field Φ in (A26) and (A28):

$$D_\mu B=\nabla_\mu B-ig[A_\mu,B]\ , \qquad (B64)$$

$$(D_\mu B)^i=\nabla_\mu B^i+gf^{ijk}A^j_\mu B^k\ . \qquad (B65)$$

For the relationship with the notation used by Chang, Das, and Perez-Mercader[11] see Appendix D. The various factors of i occurring in the Lagrangian (3.1) ensure that the action is real when the spinor fields are treated as anticommuting.

APPENDIX C: NORMAL-COORDINATE EXPANSIONS AND POLE PARTS OF GREEN'S FUNCTIONS

In this section we derive a number of results concerning the Green's functions which are needed in order to calculate the counterterms in the main body of this paper. The method used is the momentum-space representation of the normal-coordinate expansion of the Green's function which was introduced by Bunch and Parker.[17]

Let $G^i{}_j(x,x')$ be a Green's function which satisfies

$$[\delta^i_k \nabla^\mu \nabla_\mu + Q^i{}_k(x)] G^k{}_j(x,x') = \delta^i_j \delta(x,x') \ . \tag{C1}$$

Here we use labels $i,j,\ldots$ to indicate any appropriate indices carried by the fields of interest (e.g., spinor or vector). $Q^i{}_k(x)$ is any function with indices of the indicated type. $\delta(x,x')$ is the biscalar Dirac distribution defined on a test function $f(x)$ by

$$\int dv_x \delta(x,x') f(x) = f(x') \ . \tag{C2}$$

[Feynman boundary conditions are understood in (C1).] The covariant derivative ∇_μ is taken to act upon the x dependence of $G^k{}_j(x,x')$ in (C1) and is defined by

$$\nabla_\mu G^i{}_j(x,x') = \partial_\mu G^i{}_j(x,x') + \Gamma_\mu{}^i{}_k(x) G^k{}_j(x,x') \ , \tag{C3}$$

where $\Gamma_\mu{}^i{}_k(x)$ is the appropriate connection for the given spin. The results for scalars, spinors, and vectors in the $\omega=1$ gauge all involve Green's functions which satisfy (C1).

The idea now is to study $G^i{}_j(x,x')$ for x in a neighborhood of x'. The method which we use is to introduce a Riemann normal-coordinate system with origin at x'; that is, define $x^\mu = x'^\mu + y^\mu$. The following results are required:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \tfrac{1}{3} R_{\mu\rho\sigma\nu} y^\rho y^\sigma + \cdots \ , \tag{C4}$$

$$g^{\mu\nu}(x) = \eta^{\mu\nu} + \tfrac{1}{3} R^\mu{}_{\rho\sigma}{}^\nu y^\rho y^\sigma + \cdots \ , \tag{C5}$$

$$\Gamma^\lambda_{\mu\nu}(x) = -\tfrac{1}{3}(R^\lambda{}_{\mu\rho\nu} + R^\lambda{}_{\nu\rho\mu}) y^\rho + \cdots \ , \tag{C6}$$

$$[-g(x)]^{1/2} = 1 + \tfrac{1}{6} R_{\rho\sigma} y^\rho y^\sigma + \cdots \ , \tag{C7}$$

$$\Gamma_\mu{}^i{}_j(x) = \tfrac{1}{4} R_{\mu\rho\hat{a}\hat{b}} (J^{\hat{a}\hat{b}})^i{}_j y^\rho + \cdots \ , \tag{C8}$$

$$Q^i{}_j(x) = Q^i{}_j + \cdots \ . \tag{C9}$$

Here all quantities occurring on the right-hand sides of (C4)–(C8) are evaluated at the origin of the normal-coordinate system (viz., x') where only the lowest-order terms are retained. In (C8) $R_{\mu\rho\hat{a}\hat{b}}$ is the Riemann curvature tensor with two vierbein indices, and $(J^{\hat{a}\hat{b}})^i{}_j$ is the matrix element of the Lorentz generator $J^{\hat{a}\hat{b}} = -J^{\hat{b}\hat{a}}$ for the representation appropriate to the field under consideration (e.g., spinor or vector). Our notation is such that

$$[J^{\hat{a}\hat{b}}, J^{\hat{c}\hat{d}}] = \eta^{\hat{a}\hat{c}} J^{\hat{d}\hat{b}} + \eta^{\hat{a}\hat{d}} J^{\hat{b}\hat{c}} + \eta^{\hat{b}\hat{c}} J^{\hat{a}\hat{d}} + \eta^{\hat{b}\hat{d}} J^{\hat{c}\hat{a}} \ . \tag{C10}$$

A straightforward expansion of (C1) leads to

$$[\eta^{\mu\nu}\partial_\mu\partial_\nu G^i{}_j + \tfrac{1}{3} R^\mu{}_{\rho\sigma}{}^\nu y^\rho y^\sigma \partial_\mu\partial_\nu G^i{}_j + \tfrac{1}{2} R^\mu{}_{\nu\hat{a}\hat{b}} (J^{\hat{a}\hat{b}})^i{}_l y^\nu \partial_\mu G^l{}_j + \tfrac{2}{3} R^\mu{}_\nu y^\nu \partial_\mu G^i{}_j + Q^i{}_l G^l{}_j + \cdots] = \delta^i_j \delta(y) \ , \tag{C11}$$

where $G^i{}_j$ denotes $G^i{}_j(x'+y,x')$ and is regarded as a function of y. [Derivatives in (C11) are with respect to y.] Following Bunch and Parker[17] we solve (C11) by substituting a momentum-space expansion for $G^i{}_j$:

$$G^i{}_j(x,x') = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot y} (G_2{}^i{}_j + G_3{}^i{}_j + G_4{}^i{}_j + \cdots) \ , \tag{C12}$$

where $k\cdot y = \eta_{\mu\nu} k^\mu y^\nu$. The quantities $G_n{}^i{}_j$ are taken to fall off like k^{-n} for large k ($n \geq 2$). By substituting (C12) into (C11) and equating terms on both sides with the same large-momentum behavior it is not difficult to show that

$$G^i{}_j(x,x') = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot y} \left[\frac{-\delta^i_j}{k^2+i0} + \frac{\tfrac{1}{3} R \delta^i_j - Q^i{}_j}{(k^2+i0)^2} - [\tfrac{2}{3} \delta^i_j R_{\mu\nu}] \frac{k^\mu k^\nu}{(k^2+i0)^3} + O(k^{-5}) \right] \ . \tag{C13}$$

(Feynman boundary conditions are denoted by k^2+i0 which is to be interpreted as k^2 plus a small positive imaginary part.)

A similar procedure may be used to obtain the normal-coordinate expansion for the vector propagator if the gauge-fixing parameter $\omega \neq 1$. It satisfies the following differential equation:

$$\left[\delta^\mu_\lambda \nabla^\rho \nabla_\rho + R^\mu{}_\lambda - \left[1 - \frac{1}{\omega} \right] \nabla^\mu \nabla_\lambda \right] G^\lambda{}_\nu(x,x') = \delta^\mu_\nu \delta(x,x') \ . \tag{C14}$$

The analysis of this equation proceeds as described above, but is considerably more complicated due to the presence of the $\nabla^\mu \nabla_\lambda$ term in the differential operator. An extremely lengthy calculation leads to

$$\begin{aligned} G^\mu{}_\nu(x,x') = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot y} \{ & -\delta^\mu_\nu (k^2+i0)^{-1} - (\omega-1) k^\mu k_\nu (k^2+i0)^{-2} + \tfrac{1}{3}[R\delta^\mu_\nu + (\omega-3) R^\mu{}_\nu](k^2+i0)^{-2} \\ & + \tfrac{2}{3}[-\delta^\mu_\nu R_{\rho\sigma} k^\rho k^\sigma + (\omega-1) R k^\mu k_\nu + 2\omega R^\mu{}_{\rho\sigma\nu} k^\rho k^\sigma](k^2+i0)^{-3} \\ & - 3(\omega-1) R_{\rho\sigma} k^\rho k^\sigma k^\mu k_\nu (k^2+i0)^{-4} + O(k^{-5}) \} \ . \end{aligned} \tag{C15}$$

If the curvature is taken to be zero, then (C15) reduces to the well-known flat-spacetime result. In the case $\omega=1$ it may be observed that (C15) gives a result in agreement with (C13) when specialized to the vector case. [Note that the spin connection in (C3) is just the Christoffel symbol.]

Another useful result which we use below is that in normal coordinates

$$[\nabla^\mu\nabla_\mu+\tfrac{1}{3}R(x)]\delta(x,x')=\eta^{\mu\nu}\frac{\partial^2}{\partial y^\mu \partial y^\nu}\delta(y)\ . \tag{C16}$$

In order to evaluate the Feynman integrals which arise, we need results for a number of momentum integrals. These may all be evaluated using standard techniques and the formulas of dimensional regularization.[41] We tabulate a number of useful results below [$\epsilon=(4\pi)^2(d-4)$ and P.P. denotes the pole part of the indicated quantity]:

$$\text{P.P.}\int\frac{d^dk}{(2\pi)^d}\begin{Bmatrix}1\\ k_\mu\\ k_\mu k_\nu\end{Bmatrix}(k^2-m^2+i0)^{-1}[(k-p)^2-m^2+i0]^{-1}=\begin{Bmatrix}-2i\epsilon^{-1}\\ -i\epsilon^{-1}p_\mu\\ -i\epsilon^{-1}\eta_{\mu\nu}(m^2-\tfrac{1}{6}p^2)-\tfrac{2}{3}i\epsilon^{-1}p_\mu p_\nu\end{Bmatrix}, \tag{C17}$$

$$\text{P.P.}\int\frac{d^dk}{(2\pi)^d}\begin{Bmatrix}1\\ k_\mu\\ k_\mu k_\nu\end{Bmatrix}(k^2-m^2+i0)^{-2}[(k-p)^2-m^2+i0]^{-1}=\begin{Bmatrix}0\\ 0\\ -\dfrac{i}{2}\epsilon^{-1}\eta_{\mu\nu}\end{Bmatrix}, \tag{C18}$$

$$\text{P.P.}\int\frac{d^dk}{(2\pi)^d}k_\rho k_\sigma k_\mu k_\nu(k^2-m^2+i0)^{-4}=-\frac{i}{12}\epsilon^{-1}(\eta_{\rho\sigma}\eta_{\mu\nu}+\eta_{\rho\mu}\eta_{\sigma\nu}+\eta_{\rho\nu}\eta_{\sigma\mu})\ . \tag{C19}$$

Massless tadpoles

$$\int\frac{d^dk}{(2\pi)^d}(k^2+i0)^{-1}$$

are regularized to zero as usual in dimensional regularization.

Using these results, from (C13) it is easily seen that in the coincidence limit

$$\text{P.P.}[G^i{}_j(x',x')]=2i\epsilon^{-1}[Q^i{}_j-\tfrac{1}{6}\delta^i_jR]\ . \tag{C20}$$

The quantity in parentheses on the RHS may be recognized as the E_1 coefficient in the heat-kernel or Schwinger-DeWitt expansion. (See, for example, Ref. 57 whose notation for the coefficients we use.) Similarly, from (C15) we find for the vector propagator in the $\omega\neq1$ gauge:

$$\text{P.P.}[G^\mu{}_\nu(x',x')]=+\frac{i}{2}(\omega+3)\epsilon^{-1}(R^\mu{}_\nu-\tfrac{1}{6}\delta^\mu_\nu R)\ . \tag{C21}$$

This calculation is equivalent to a determination of the E_1 coefficient for differential operators of the form given in (C14) and we believe provides a new result. The case $\omega=1$ gives a result in agreement with (C20) when specialized to the vector case.

The scalar field propagator for a massless field satisfies

$$[\nabla^\mu\nabla_\mu+\xi R(x)]\Delta(x,x')=-\delta(x,x')\ , \tag{C22}$$

so that $Q(x)=\xi R(x)$ and $\Gamma_\mu(x)=0$. [Note the overall negative sign relative to the definition in (C1).] Thus, from (C13), the normal-coordinate expansion reads

$$\Delta(x,x')=\int\frac{d^dk}{(2\pi)^d}e^{ik\cdot y}[(k^2+i0)^{-1}+(\xi-\tfrac{1}{3})R(k^2+i0)^{-2}+\tfrac{2}{3}R_{\mu\nu}k^\mu k^\nu(k^2+i0)^{-3}+O(k^{-5})]\ . \tag{C23}$$

The pole part of the coincidence limit is found from (C20) to be

$$\text{P.P.}[\Delta(x',x')]=-2i\epsilon^{-1}(\xi-\tfrac{1}{6})R\ . \tag{C24}$$

Differentiation of (C23) with respect to x'^ν leads to

$$\partial'_\nu\Delta(x,x')=\int\frac{d^dk}{(2\pi)^d}e^{ik\cdot y}[-ik_\nu(k^2+i0)^{-1}-i(\xi-\tfrac{1}{3})Rk_\nu(k^2+i0)^{-2}-\tfrac{2}{3}iR_{\rho\sigma}k^\rho k^\sigma k_\nu(k^2+i0)^{-3}+O(k^{-4})]\ . \tag{C25}$$

Up to terms of order k^{-3} we have also that

$$\partial_\nu\Delta(x,x')=-\partial'_\nu\Delta(x,x')\ . \tag{C26}$$

Differentiation of (C25) with respect to x^μ gives

$$\partial_\mu \partial'_\nu \Delta(x,x') = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot y}[k_\mu k_\nu (k^2+i0)^{-1} + (\xi - \tfrac{1}{3})R k_\mu k_\nu (k^2+i0)^{-2} + \tfrac{2}{3} R_{\rho\sigma} k^\rho k^\sigma k_\mu k_\nu (k^2+i0)^{-3} + O(k^{-3})] \,. \tag{C27}$$

The propagator for a LH Weyl spinor was given in Eqs. (3.11) and (3.12). It is observed that $Q_{\dot A}{}^{\dot B} = \frac{1}{4} R \delta_{\dot A}{}^{\dot B}$ so that (C13) leads to

$$G_{\dot A}{}^{\dot B}(x,x') = \delta_{\dot A}{}^{\dot B} \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot y}[-(k^2+i0)^{-1} + \tfrac{1}{12} R (k^2+i0)^{-2} - \tfrac{2}{3} R_{\mu\nu} k^\mu k^\nu (k^2+i0)^{-3} + O(k^{-5})] \,. \tag{C28}$$

Because [see Eq. (B37)]

$$\nabla_\nu G_{\dot A}{}^{\dot B}(x,x') = \partial_\nu G_{\dot A}{}^{\dot B}(x,x') - \bar{\Gamma}_{\nu \dot A}{}^{\dot C}(x) G_{\dot C}{}^{\dot B}(x,x') \tag{C29}$$

in normal coordinates we have, using (C8),

$$\nabla_\nu G_{\dot A}{}^{\dot B}(x,x') = \partial_\nu G_{\dot A}{}^{\dot B}(x,x') - \tfrac{1}{4} R_{\hat a \hat b \nu \mu} (\bar{J}^{\hat a \hat b})_{\dot A}{}^{\dot C} y^\mu G_{\dot C}{}^{\dot B}(x,x') + \cdots \,. \tag{C30}$$

It is straightforward to use (C30), (C28), and (3.11) and show that

$$S^{A\dot B}(x,x') = \sigma^{\nu A \dot C} \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot y} \Big[-i\delta_{\dot C}{}^{\dot B} k_\nu (k^2+i0)^{-1} + \frac{i}{12} \delta_{\dot C}{}^{\dot B} R k_\nu (k^2+i0)^{-2} - \tfrac{2}{3} i \delta_{\dot C}{}^{\dot B} R_{\rho\sigma} k^\rho k^\sigma k_\nu (k^2+i0)^{-3} - \frac{i}{2} (\bar{J}^{\hat a \hat b})_{\dot C}{}^{\dot B} R_{\nu\mu\hat a \hat b} k^\mu (k^2+i0)^{-2} + O(k^{-4}) \Big] \,. \tag{C31}$$

A similar calculation for Dirac spinors leads to (spinor indices suppressed)

$$S(x,x') = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot y} \Big[-i\gamma^{\hat a} k_{\hat a} (k^2+i0)^{-1} + \frac{i}{12} R \gamma^{\hat a} k_{\hat a} (k^2+i0)^{-2} - \tfrac{2}{3} i \gamma^{\hat a} R_{\rho\sigma} k^\rho k^\sigma k_{\hat a} (k^2+i0)^{-3} + \frac{i}{2} \gamma^{\hat a} \mathcal{J}^{\hat c \hat d} R_{\hat c \hat d \hat a \mu} k^\mu (k^2+i0)^{-2} + O(k^{-4}) \Big] \,. \tag{C32}$$

The momentum-space expansion of the Dirac-spinor Feynman propagator was first given by Bunch and Parker.[17]

Finally we require the pole parts of a number of expressions involving products of Green's functions. These may be found by using the momentum-space expansions above and performing a number of Feynman integrals using results in (C17)–(C19). Once the pole parts have been determined in the normal-coordinate system we may immediately write the unique answer in general coordinates. Details of these lengthy calculations will not be given here. We merely note the following final results:

$$\text{P.P.}[G^{\mu\nu}(x,x')\Delta(x,x')] = \frac{i}{2}(3+\omega)\epsilon^{-1} g^{\mu\nu} \delta(x,x') \,, \tag{C33}$$

$$\text{P.P.}[G^{\mu\nu}(x,x')\partial'_\nu \Delta(x,x')] = \frac{i}{2}(\omega-3)\epsilon^{-1} \partial^\mu \delta(x,x') \,, \tag{C34}$$

$$\text{P.P.}[G^{\mu\nu}(x,x')\partial_\nu \Delta(x,x')] = -\frac{i}{2}(\omega-3)\epsilon^{-1} \partial^\mu \delta(x,x') \,, \tag{C35}$$

$$\text{P.P.}[G^{\mu\nu}(x,x')\partial_\mu \partial'_\nu \Delta(x,x')] = i\epsilon^{-1}[\tfrac{3}{2}(\omega-1)\nabla^\mu \nabla_\mu + (2\omega\xi + \tfrac{1}{6}\omega - \tfrac{1}{2})R]\delta(x,x') \,, \tag{C36}$$

$$\text{P.P.}[S^{A\dot B}(x,x') S_{A\dot B}(x,x')] = -2i\epsilon^{-1}(\nabla^\mu \nabla_\mu + \tfrac{1}{6}R)\delta(x,x') \,, \tag{C37}$$

$$\text{P.P.}\{\text{Tr}[S(x,x')S(x',x)]\} = 4i\epsilon^{-1}(\nabla^\mu \nabla_\mu + \tfrac{1}{6}R)\delta(x,x') \,. \tag{C38}$$

APPENDIX D: EIGENVALUE CONDITIONS

Here we quote the results of Chang, Das, and Perez-Mercader[11] on the eigenvalue conditions which lead to an asymptotically free theory. Because our notation differs from that used in Ref. 11 we summarize the comparison. In our notation $(a,b,\lambda,2\alpha,2\beta,|\gamma|,4|\Gamma|,\mu_\phi{}^2,\mu_H{}^2)$ in Eqs. (3.2) are $(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\sqrt{2}|h|,|h'|,\mu^2,\nu^2)$ in the notation of Chang, Das, and Perez-Mercader.[11] For Eq. (3.3), $[\sqrt{2}B_i(F^i)^a{}_b,\Theta^a,\sqrt{2}\phi^i(F^i)^a{}_b,k_2,k_4,k_5,k_6]$ in our notation are $(B^\alpha_\beta,\chi^\alpha,\phi^\alpha_\beta,k_2,k_4,k_5,k_6)$ in the notation of Chang, Das, and Perez-Mercader.[11] (Factors of i in the Yukawa terms are necessary to ensure reality of the action because we treat the spinor fields as anticommuting.) It is then just a simple matter of using the results given in Ref. 11 and translating to our notation.

The results for the two models given in Ref. 11 are listed below:

Three-generation model	Seven-generation model
$a=0.029\,244g^2$	$a=0.017\,20g^2$
$b=0.457\,611g^2$	$b=0.662\,75g^2$
$\lambda=1.196\,053g^2$	$\lambda=2.872\,32g^2$
$\alpha=-0.006\,187g^2$	$\alpha=-0.030\,07g^2$
$\beta=0.454\,585g^2$	$\beta=1.204\,07g^2$
$\gamma=-0.359\,784g$	$\gamma=-0.094\,158g$
$\Gamma=-0.186\,693g$	$\Gamma=0$
$k_2=-0.635\,486g$	$k_2=-0.903\,92g$
$k_4=-0.942\,053g$	$k_4=-1.338\,4g$
$k_5=-0.809\,565g$	$k_5=-0.965\,76g$
$k_6=0.706\,639g$	$k_6=0.692\,65g$

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