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# Metaphysics of colliding self-gravitating plane waves

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We discuss certain global features of colliding plane-wave solutions to Einstein's equations. In particular, we show that the apparently local curvature singularities both in the Khan-Penrose solution and in the Bell-Szekeres solution are actually global. These global singularities are associated with the breakdown of nondegenerate planar symmetry in the characteristic initial data sets.

## I. INTRODUCTION

In general relativity, self-gravitating plane waves have a number of unusual properties. In this paper we shall be concerned with elucidating some of these properties, particularly global ones. It will be found that in both the Khan-Penrose[1] solution, and in the Bell-Szekeres[2] solution, global singularities develop in the future of the plane-wave intersection, with the result that these spacetimes are globally hyperbolic. This property is in contrast to certain single plane-wave solutions which have no singularities and are not globally hyperbolic. Our analysis, together with previous work by Tipler[3] and by Centrella and Matzner,[4] suggests that such behavior is generic for colliding plane waves.

In Sec. II, we shall give a brief review of the known global properties of self-gravitating plane waves, and develop some new techniques which will prove useful in studying the global properties of plane waves.

In Sec. III, we shall use these techniques to analyze the Khan-Penrose and the Bell-Szekeres colliding plane-wave solutions. We shall attempt to demystify a few rather strange properties of the Bell-Szekeres solution. In Sec. IV, we shall state and prove a singularity theorem concerning the existence of global singularities in the colliding plane-wave solutions. We shall use the notation of Hawking and Ellis.[5]

## II. COLLIDING PLANE-WAVE SOLUTIONS

Consider a spacetime which admits globally a pair of commuting spacelike Killing vectors, say $\partial/\partial x, \partial/\partial y$. Such a spacetime is said to be plane-symmetric. It has been shown (e.g., Refs. 1 and 6) that a plane-symmetric spacetime admits a coordinate system $\{u,v,x,y\}$ ($du,dv$ null), so that

$$ds^2 = -2e^{-M} du\, dv + g_{ab} dx^a dx^b + N_a du\, dx^a + P_a dv\, dx^a \,, \tag{2.1}$$

where $M = M(u,v)$, $N_a = N_a(u,v)$, $P_a = P_a(u,v), g_{ab} = g_{ab}(u,v)$, and $a,b = 2,3$. This metric, together with the assumption that $(u,v,x,y)$ are globally admissible, is the starting form for the "colliding plane-wave spacetimes" which are solutions of the Einstein equations that satisfy certain boundary conditions.

The boundary conditions and the resulting spacetimes are diagrammed in a standard way in Fig. 1. These spacetimes represent plane gravitational waves moving into an initially flat region. Thus region I is flat space, i.e.,

I: $(u<0, v<0)$

$$M = N_a = P_a = 0, \quad g_{ab} = \delta_{ab} \,. \tag{2.2}$$

It can be shown[6] that the Einstein-Maxwell equations combined with (2.2) imply $N_a = P_a = 0$ everywhere.

The 45° (null) lines $u=0, v=0$ are the wavefronts of gravitational radiation. Hence for $v<0, u>0$ one has the standard plane-wave metric

II: $(v<0, u>0)$

$$M = M(u), \quad V = V(u), \quad U = U(u) \,, \tag{2.3}$$

where

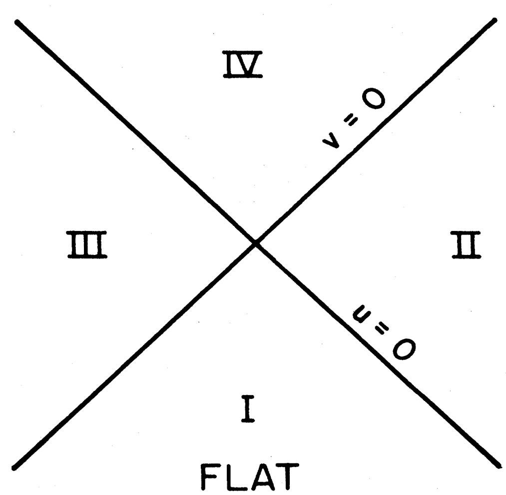


FIG. 1. The standard representation of a colliding plane-wave spacetime.

 

$$g_{ab} = \text{diag}\, e^{-U}(e^{V}, e^{-V})$$

(obviously this definition can be extended to the entire region $v<0$ so long as $U=V=M=0$, wherever $u<0$). The metric in region III is

$$M=M(v),\quad V=V(v),\quad U=U(v)\ ,$$
$$g_{ab} = \text{diag}\, e^{-U}(e^{V}, e^{-V})\ . \tag{2.4}$$

Continuity of the metric and first derivatives (the Lichnerowicz conditions) requires continuity of the functions $U, V, M$ (and of their derivatives) on the join surfaces; imposing the continuity of $U$ at the join surfaces defines the solution in region IV. In the case of colliding electromagnetic waves, Bell and Szekeres[2] show that these join conditions must be relaxed to allow certain derivatives across the join surface to be discontinuous; these are the O'Brien-Synge conditions,[7] which we will use in Sec. III.

There is no requirement that the waves in regions II and III be of finite duration. However, if they are, the analysis becomes considerably simpler. The waves before collision are then *sandwich waves,* and the parts of regions II and III behind the wave sandwich are *flat.* The simplest example of this type was one given by Khan and Penrose,[1] in which the waves have a δ-function impulsive profile.

The generic behavior of these solutions includes the evolution of a singularity in the future in region IV. For colliding gravitational waves, the Weyl tensor of one wave induces shear in the wavefront normals of the other as they pass through one another. Each wave emerges from the collision with nonzero shear, and since the square of the shear is a source for the convergence in the Sachs equations,[8] each eventually develops convergence as well. (Prior to the collision, the expansion and shear vanish in each wavefront.) One then anticipates the appearance of singularities at least on the extended initial wavefronts, as they are converged to a focus, with consequent infinite energy density. Examination of the solution via the Green's function shows that in fact in the vacuum case the singularity extends as shown in Fig. 2 across the future of region IV. In the case of electromagentic wave collisions, convergence—and not shear—is induced directly by the energy content of the wave; focusing again occurs. This is the case for the Bell-Szekeres solution in Sec. III below.

The appearance of the singularity on the wavefronts $u=0$, $v=0$ appears precisely where these waves focus. Penrose has given a proof that for a single plane wave (not colliding) there exists no global Cauchy surface for the evolution. We will address the question of global hyperbolicity in colliding plane-wave spacetimes in Sec. IV.

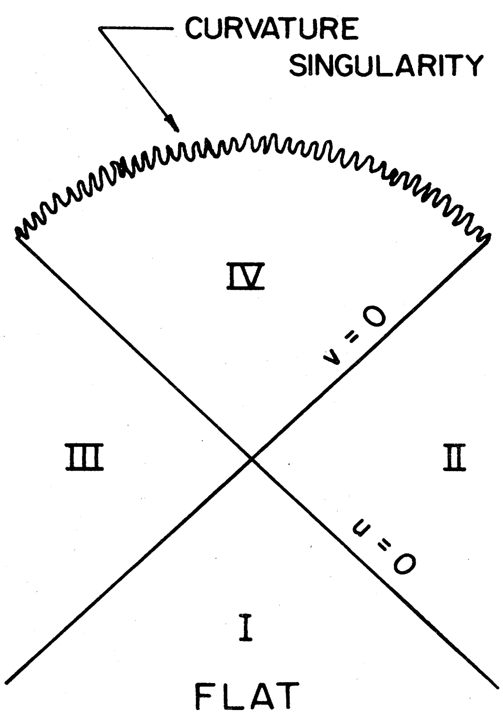


FIG. 2. Naive singularity structure of the Khan-Penrose solution.

## III. AN EXAMPLE: THE KHAN-PENROSE SOLUTION

Khan and Penrose[1] presented a solution in which two impulsive plane waves intersect. The Khan-Penrose metric is

$$ds^2 = \frac{-2t^3 du\, dv}{rw(pq+rw)^2} + t^2\left[\frac{r+q}{r-q}\right]\left[\frac{w+p}{w-p}\right]dx^2 + t^2\left[\frac{r-q}{r+q}\right]\left[\frac{w-p}{w+p}\right]dy^2\ ,$$

where

$$\begin{aligned} &p=u\theta(u),\quad q=v\theta(v)\ ,\\ &r=(1-p^2)^{1/2},\quad w(1-q^2)^{1/2}\ ,\\ &t=(1-p^2-q^2)^{1/2}=(r^2-q^2)^{1/2}=(w-p^2)^{1/2}\ . \end{aligned} \tag{3.1}$$

Figure 2 plots the naive singularity structure of this solution. Analysis of the Green's function for the solution shows[1,6] that curvature singularities occur at coordinates $u^2+v^2=1$. It is our intention to investigate the behavior of the regions $v=1, u\le 0$ and $u=1, v\le 0$. Because of the overall symmetry of this solution under the interchange $u\leftrightarrow v$, it will suffice to base our investigation on the behavior of the solution in the "strip" $v>0, u\le 0$. In this region, the metric is simply

$$ds^2 = -2du\, dv + (1+v)^2 dx^2 + (1-v)^2 dy^2\ . \tag{3.2}$$

(We emphasize that this form holds even *on* $u=0$, for $v>0$. This is a consequence of the demand for continuity of the metric.) As asserted in Sec. II, (3.2) is a metric for flat space. The transformation to standard null-Minkowskian coordinates $\tilde{x}^{\alpha}$ is

$$\begin{aligned} &x=\frac{\tilde{x}}{1+\tilde{v}},\quad y=\frac{\tilde{y}}{1-\tilde{v}}\ ,\\ &v=\tilde{v},\quad u=\tilde{u}+\frac{\frac{1}{2}\tilde{y}^2}{1-\tilde{v}}-\frac{\frac{1}{2}\tilde{x}^2}{1+\tilde{v}}\ , \end{aligned} \tag{3.3}$$

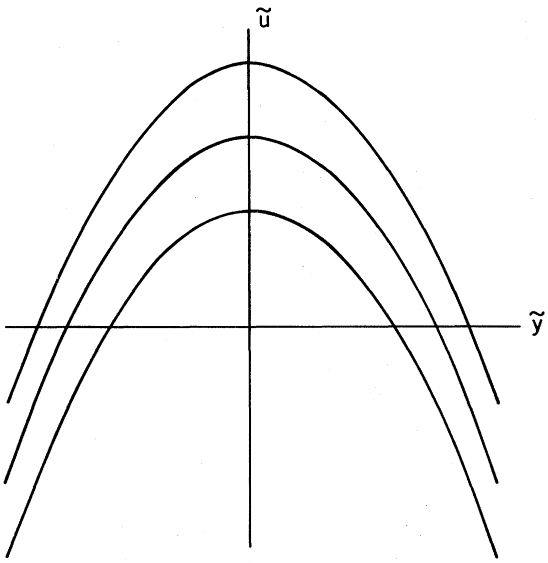


FIG. 3. The surfaces $u=u_1=$constant, $v=$constant projected into the $\tilde{u},\tilde{y}$ subspace of Minkowski space. As $v\rightarrow 1$, the surface $u=u_1$ has a similar shape, but the principal curvature of the surface is greater.

whence the metric is

$$ds^2=-2d\tilde{u}\,d\tilde{v}+d\tilde{x}^2+d\tilde{y}^2\ . \tag{3.4}$$

On the surface $v=1$, the coordinate transformation (3.3) is singular; nonetheless there appears no *a priori* reason preventing the continuation of the $\tilde{x}^{\alpha}$ to $\tilde{v}>1$, at least for $\tilde{u}<0$.

Let us investigate some of the properties of the surfaces $u=$constant. This is straightforward because we can work in the flat-space, null-Minkowski coordinates $\tilde{x}^{\alpha}$. Figure 3 gives a diagram of slices $u=$constant, $v=$constant, projected into the $\tilde{u},\tilde{y}$ subspace (i.e., of the lines $u$ constant, $v$ constant, $\tilde{x}$ constant). They are nested, similar downward-curving parabolas for constant $\tilde{v},\tilde{x}$ [cf. Eq. (3.3)]. For $\tilde{v}\rightarrow 1$ the curvature of these slices diverges. The surfaces $u=$constant, considered as embedded in the three-space $\tilde{x}=0$, consist of nested two-surfaces. They do not have any particular pathology, as surfaces, near the line $y=u=0,v=1$. However, the slices $v=$constant of these surfaces, which represent wavefronts of the propagating wave, do have an infinite principal curvature there (Fig. 4).

In fact, Eq. (3.3) may be manipulated to give

$$(\tilde{u}-u)(\tilde{v}-1)=\tfrac{1}{2}\tilde{y}^2-\tfrac{1}{2}\tilde{x}^2(\tilde{v}-1)/(\tilde{v}+1)\ . \tag{3.5}$$

Because the $\tilde{x}$ terms act near $v=1$ simply like a constant added to $\tilde{u}$, we consider only $\tilde{x}=0$. Equation (3.5) is thus an equation describing a null cone in $2+1$ Minkowski space $(\tilde{u},\tilde{v},\tilde{y})$ with apex at $\tilde{u}=u$, $\tilde{v}=1$. The $y$ homogeneity of the space is expressed in its action on the null rays lying in and generating this cone. The $y$ Killing vector simply moves one null ray into another through the same apex, in such a way that homogeneity-equivalent points lie along a $v=$constant curve. (It is no surprise that the $v=$constant curves, slices of a cone parallel to one side, come out to be parabolas.) The whole $2+1$ cone is swept out under this action. One of the generators of the cone is the ray $\tilde{v}=1$, $\tilde{x}=\tilde{y}=0$. The surface $u=0$ can be embedded in flat space, is homogeneous under the group, but it intersects the singularity at $u=0,v=1$. The image under the homogeneity of this $u=0,v=1$ singularity is the "line" $v=1$. More carefully stated, either the homogeneity breaks down or $v=1$ is a singularity. We note that because of the impulsive nature of the Khan-Penrose solution, the Riemann tensor is divergent ($\delta$ function) in the surface $u=0$. However, this divergence could be removed by smoothing the data. The amplitude, multiplying the $\delta$ function, diverges even after smoothing as the point $u=0,v=1$ is approached. This amplitude could be measured by its focusing effect, for instance. When $v=0$, the focusing effect on a bundle of parallel null rays crossing $u=0$ is impulsive but finite; the rays can travel a finite distance before focusing. As we consider rays crossing nearer $v=1$ the focusing becomes stronger, and at $u=0,v=1$ this focusing would diverge.[1]

We can show that the "surface" $v=\tilde{v}=1$, $u=$constant in region III is not merely a coordinate singularity, but in actuality is a singularity of spacetime in the sense that there does not exist a $C^1$ extension from region III to this surface. The proof is by contradiction. Suppose, on the contrary, that a $C^1$ extension existed. Then, by continuity the Killing equations could be extended from region III to

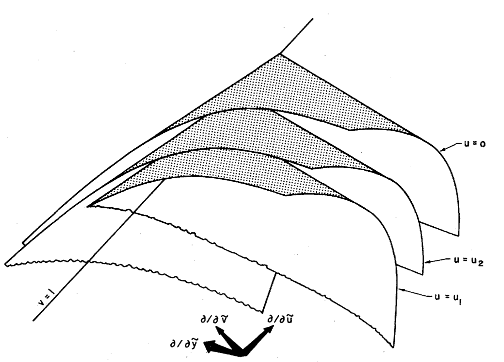


FIG. 4. A nested sequence of $u=$constant hypersurfaces, $u_3>u_2>u_1$. The surfaces coalesce at $v=1$.

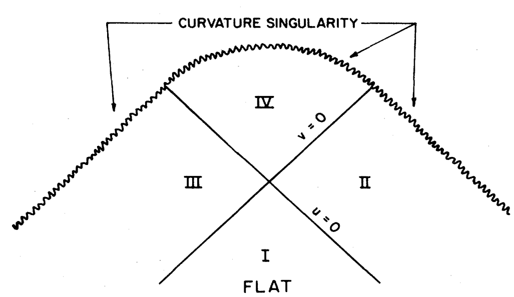


FIG. 5. The true singularity structure of the Khan-Penrose spacetime is shown in this two-dimensional slice through the solution.

the surface $v=1$; the vectors $\partial/\partial x$, $\partial/\partial y$ would in fact be Killing vectors (possibly degenerate) there also. But the above analysis shows that these symmetries would identify the $v=1$ surface with the singularity $u=0,v=1$. Thus the surface $v=1$ is actually a singularity. We call a singularity like that at $v=1$ (and the similar one at $u=1$) a "fold" singularity. The singularity structure in the Khan-Penrose solution is (as those authors pointed out) as given in Fig. 5.

This figure is a bit misleading in one respect, however. It suggests that the fold singularity whose existence was demonstrated above is distinct from the curvature singularity at $u=0,v=1$, whereas actually the latter is just one representative of the homogeneous set $v=1$. The above argument suggests this point, but does not prove it.

To investigate this question further, refer first to Fig. 4. In region III, the surface $u=0$ is a $u=$constant surface, identical, say, to $u=u_2$ in Fig. 4. But $u=0$ is the future boundary of region III; $u>0$ surfaces lie, by definition, in the nonflat region IV. *The flat region III is bounded by a pastward curving cap* $u=0$ (Fig. 4). This important point lets us understand the singularity at $v=1$. We consider null geodesics initially in region III with $dv/d\lambda>0$ ($\lambda$ is an affine parameter). The null ray $x=y=0$, $0>u=$constant crosses the boundary from region I to region III making a 45° angle in the diagram [Fig. 6(a)] and ending, apparently, at the $v=1$ singularity. Consider a null congruence of geodesics that are specified by giving their "initial" coordinates $x=0$, $y,u<0$ near $v=0$ in region III with their initial momenta chosen so that the congruence is nonexpanding (and nonrotating) and parallel to the ray $x=y=0,u=$constant. This latter ray can then be treated by taking limits within the congruence. (From the $x$-Killing property of the metric we see that we lose no generality by picking $x=0$.)

Because region III is flat, the null rays constituting this congruence are easily written down, using the natural Minkowskian coordinates $\tilde{x},\tilde{y},\tilde{u},\tilde{v}$ of Eq. (3.3):

$$\begin{aligned}&\tilde{x}=0\ ,\\&\tilde{y}=\text{constant}=\tilde{y}_0\ ,\\&\tilde{u}=\text{constant}=\tilde{u}_0\ ,\\&\tilde{v}=\lambda\ .\end{aligned}\tag{3.6}$$

Because the homogeneity group sweeps out all directions, we can, by using the group to pick the particular ray $x=y=0$, define a congruence in any direction in the hyperplane $x=0$.

Using the homogeneity coordinates $x,y,u,v$, via Eq. (3.3), we have in region III, i.e., so long as $u\le 0$,

$$\begin{aligned}&x=0\ ,\\&y=\frac{y_0}{1-v}\ ,\\&u=\tilde{u}_0+\frac{1}{2}\frac{{y_0}^2}{1-v}\ ,\\&v=\lambda\ .\end{aligned}\tag{3.7}$$

Obviously, the ray leaves region III when

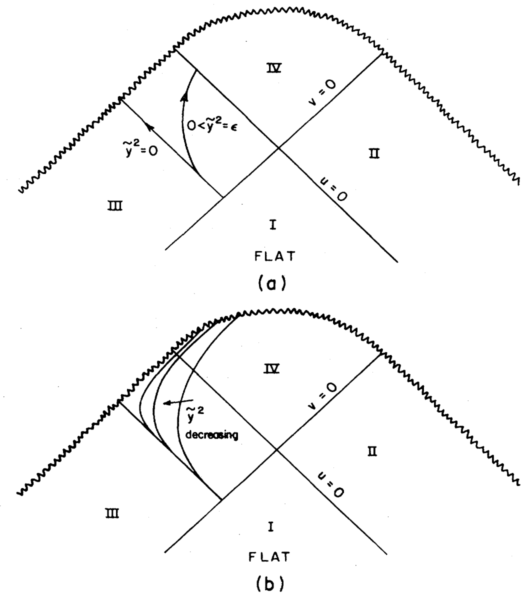


FIG. 6. (a) The description of null rays by projection into the plane $x=y=0$, under the homogeneity group, distorts the causal structure. We plot two null curves (geodesics) which are *parallel* in the Minkowski sense in the flat region III. The curve marked $\tilde{y}^2=0$ lies in the $x=y=0$ subspace. It remains in the flat region III until it hits the singularity at $v=1$. The curve marked $0<\tilde{y}^2=\epsilon$ has $x=0$, but nonzero and constant $\tilde{y}^2$ [see Eq. (3.6)]. When projected to the $u$-$v$ plane, it appears curved; it crosses $u=0$ into the curved region IV before reaching the singularity $v=1$. (b) Curves with nonzero $\tilde{y}^2$ can be continued in region IV and intersect the singularity ($u^2+v^2=1$) in a finite affine length. In a sequence of curves with $\tilde{y}^2\to 0$, the curves with smaller $\tilde{y}^2$ stay close to the $\tilde{y}=0$ curve longer, and turn across $u=0$ more abruptly, and cross $u=0$ at a point where the wave amplitude (measured by its focusing power) is stronger. The limiting ray ($\tilde{y}^2=0$) crosses into the curved part of the spacetime precisely where the gravitational wave amplitude is infinite and immediately runs into the curvature singularity. The $v=1$ and the $u^2+v^2=1$ singularities are thus seen to be in fact the same singularity. The apparent noncausality of diagrams like Fig. 6 is an artifact of the projection to the $u-v$ plane, which does not in general preserve causal relationships.

$$v=1-\left[\frac{\frac{1}{2}{\tilde{y}_0}^2}{-\tilde{u}_0}\right]<1\ .\tag{3.8}$$

We have plotted, in Fig. 6(a), the path of the portion of such a photon path that lies within region III, projected under the action of the homogeneity to the $x=y=0$ plane. This ray should be compared to the ray which has $x=y=0$ precisely. They are both null rays; *the homogeneity projection does not preserve conformal relations* for null rays that are not homogeneous to the $x=y=0$ ray.

Let us follow the geodesic after it crosses $u=0$. The components of its four-velocity are continuous at the crossing because the metric components Eq. (3.1) are continuous at $u=0,v>0$. Thus, just after entering region IV the tangent to the null ray has components

$$
\begin{aligned}
&dx/d\lambda=0\ ,\\
&dy/d\lambda=\tilde{y}_0/(1-v)^2\ ,\\
&du/d\lambda=\tfrac{1}{2}\tilde{y}_0{}^2/(1-v)^2>0\ ,\\
&dv/d\lambda=1\ ,
\end{aligned}
\tag{3.9}
$$

where $v$ has the value given by (3.8).

In region IV the metric has the form (we again ignore $x$)

$$ds^2=2g_{uv}(u,v)du\,dv+g_{yy}(u,v)dy^2\ .$$

Because the metric is independent of $y$ we have the immediate first integral $p_y=g_{yy}dy/\lambda=\text{constant}$. The null condition gives

$$-2g_{uv}\frac{du}{d\lambda}\frac{dv}{d\lambda}=\frac{p_y{}^2}{g_{yy}}\ . \tag{3.10}$$

Now $g_{uv}$, as given by Eq. (3.1), is nonpositive and bounded below in region IV. On the other hand, $(g_{yy})^{-1}>1$ and $(g_{yy})^{-1}\to\infty$ when $u^2+v^2\to 1$. Therefore the product $(du/d\lambda)/(dv/d\lambda)$ cannot change sign and cannot go to zero in region IV. Hence the geodesic reaches the singularity $u^2+v^2=1$ in a finite affine parameter. This behavior is schematically shown in Fig. 6(b).

It can be seen from Eq. (3.8) that goedesics with smaller $\tilde{y}_0$ cross the $u=0$ boundary with larger values of $v$, and in the limit $\tilde{y}_0\to 0$, the crossing occurs at $v=1$. This is also indicated in Fig. 6(b).

We thus see that no extension is possible beyond $v=1$; the singularity marked along $v=1$ is actually an inaccurate diagrammatic representation of the singularity at $u^2+v^2=1$. Except for the special ray with $\tilde{y}=0$ exactly, the geodesics moving in region III leave region III (to region IV) and begin to feel (finite) curvature effects, which eventually diverge as they approach $u^2+v^2=1$. There is nothing noncausal in this behavior. It appears noncausal because the projection under the homogeneity does not respect the complete conformal structure. Those null geodesics with smaller $\tilde{y}^2$ defer their depature from region III until a larger $v$, and their departure appears more abrupt on this diagram. As they cross the boundary $u=0$, they feel a larger Riemann tensor. In the limit $\tilde{y}^2\to 0$ the departure appears discontinuous as the ray exits region III exactly at the curvature singularity of the $u=0$ wavefront.

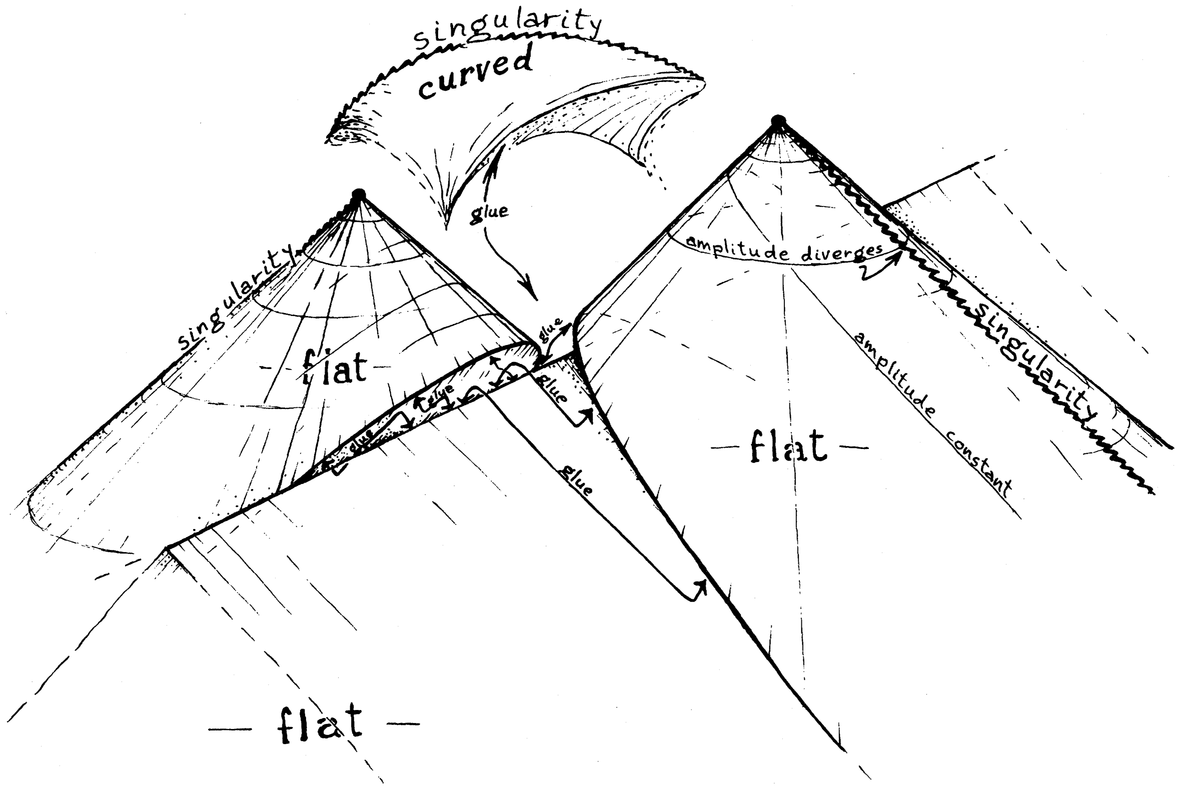


FIG. 7. Professor R. Penrose has drawn this figure giving a three-dimensional picture of the Khan-Penrose solution (three-dimensional version of Fig. 5). Because the principal curvatures of the $u=$constant and $v=$constant surfaces jump on crossing the wavefronts, we must "glue together"—i.e., identify points on opposite sides of the wave front that lie in surfaces with intrinsically different geometry. Notice that this applies too on crossing from region II or region III into the (curved) region IV. Note that near $v=1(u=1)$ we have a (mirror) copy of Fig. 4. The line marked "amplitude constant" is the locus of points where the amplitude of the waves and the principal curvature $K^y{}_y$ [cf. Eq. (3.7) and Sec. IV] are constant. As Fig. 4 and Eq. (3.3) show, these quantities diverge at the "fold" singularity.

Observe from Eq. (3.3) that the coordinate transformation connecting $x^\alpha$ and $\tilde{x}^\alpha$ is *not* the identity on the surface $v=\tilde{v}=0$. Immediately after the wave is encountered, the surfaces $u=$constant take on a curvature as evidenced by the last equation in (3.3). If, instead of relying on Fig. 5, a two-dimensional picture, one wants to continue Fig. 4, which is a three-dimensional picture of *part* of the solution to show the entire spacetime, trouble must be expected at the wavefronts. One will have the part of the solution near $v=1$ represented as in Fig. 5, and its mirror image near $u=1$. But at the wavefronts, one will have to connect up the intrinsically flat $u=$constant (or $v=$constant) surfaces on one side to intrinsically curved constant-coordinate surfaces on the other. The best one can expect is to indicate by arrows the points to be joined. Figure 7 is such a picture. (This is a previously unpublished diagram by Professor R. Penrose, and we thank him for his permission to use it.[9])

In the initially flat region the $u=$constant and $v=$constant surfaces are planes. At the wavefronts one has to identify ("glue together") points on that plane with points on the curved $u$ or $v$ surfaces after the wavefront. Also, there are jumps in curvature in going from region II into region IV, and in going from III to IV, since these also cross wavefronts (region IV is not flat and so cannot be *accurately* pictured as in Fig. 4). Hence the region-IV "cap" has to be glued onto the wavefront. Notice that, as advertised, we have a copy, and a mirror copy of Fig. 4 at $v=1$ and at $u=1$. Finally, the lines "amplitude constant" here corresponds to the loci where the principal curvature $K^y{}_y$ [Eq. (3.13) and Sec. IV] is constant. They are also homogeneous images of points on the $u=$constant, $x=y=0$ lines, i.e., they are the wavefronts of the waves. This principal curvature and the amplitude of the waves diverge as the "fold" singularity is approached, as we have already seen from Fig. 4.

We should point out that if the $\delta$-function Riemann-tensor profiles of the wavefront were spread slightly, there would be a continuous distortion of the $u=$constant surface on passing through the wave, in place of the discontinuous distortion here.

In order to proceed in general, we wish to extract a geometrical characterization of the surface bending of the Khan-Penrose solution. Intuitively, this a problem in the embedding of a three-surface in an enveloping pseudo-Riemannian space. There is a straightforward way to measure the principal curvatures of the two-surface, as embedded in the $(u,x,y)$ null three-space. Define a vector[10]

$$^0\underline{e}_{(y)}=(g^{yy})^{1/2}\partial_y \,. \tag{3.11}$$

This is a (metrically) unit vector pointing in the $y$ direction. We also use the coordinate basis vectors—written *sans* superscript 0:

$$\{\underline{e}_{(\alpha)}\}=\{\underline{e}_{(u)},\underline{e}_{(v)},\underline{e}_{(x)},\underline{e}_{(y)}\}=\left\{\frac{\partial}{\partial u},\frac{\partial}{\partial v},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right\}.$$

Compute

$$\begin{aligned}\nabla_{^0\underline{e}_{(y)}}{}^0\underline{e}_{(y)}&=g^{yy}\Gamma^\alpha_{yy}\underline{e}_{(\alpha)}\\&=g^{yy}\Gamma^u_{yy}\underline{e}_{(u)}+g^{yy}\Gamma^v_{yy}\underline{e}_{(v)}\\&=g^{yy}g^{uv}\Gamma_{vyy}\underline{e}_{(u)}+g^{yy}g^{uv}\Gamma_{uyy}\underline{e}_{(v)}\,.\end{aligned} \tag{3.12}$$

The connections appearing here are the coordinate-basis Christoffel symbols. On the left of Eq. (3.12) we have the classical definition of the curvature of the orbit of $\partial_y$.[11] Because the $x$-$y$ part of the metric is diagonal, the normalization (3.11) is simple, otherwise see Ref. 10; we also use in (3.12) the plane-wave property that the $u$-$v$ block of the metric is separate from the $x$-$y$ block—no $g_{ux}$ for instance, and we use the fact that there is homogeneity in the $x$ and the $y$ directions. Thus $\nabla_{^0\underline{e}_{(x)}}{}^0\underline{e}_{(x)}$ and $\nabla_{^0\underline{e}_{(y)}}{}^0\underline{e}_{(y)}$ are the principal curvatures of the $x,y$ subspace of the $(u,x,y)$ null three-space.[11]

Apply this to the Khan-Penrose solution in the region we have just analyzed, $v>0, u\le 0$. Then $\Gamma^v_{yy}=0$, and

$$\begin{aligned}\nabla_{^0\underline{e}_{(y)}}{}^0\underline{e}_{(y)}&=g^{yy}(\Gamma^u_{yy}\underline{e}_{(u)})\\&=g^{yy}\tfrac{1}{2}g_{yy,v}\underline{e}_{(u)}\\&=-\frac{1}{1-v}\frac{\partial}{\partial u}\,.\end{aligned} \tag{3.13}$$

As one compares a unit $y$-direction spacelike vector at a point in the $x$-$y$ surface to a parallel-transported copy of a nearby such vector, one finds that they deviate by a protrusion into the null direction $u$. This can be simply seen, graphically, by referring to Fig. 4. And, clearly, one has a divergence of this protrusion as $v\to 0$. This infinite curvature along the whole "line" $-\infty<y<\infty$ for $x=0$, $u\le 0$ when $v=1$ provides the "fold singularities" already discussed in the Khan-Penrose solution. (As may be expected, one finds $\nabla_{^0\underline{e}_{(x)}}{}^0\underline{e}_{(x)}=-[1/(1+v)]\partial/\partial v$, which is finite at $v=1$.)

We may apply a similar analysis to the Bell-Szekeres[2] solution. This is the solution which describes the collision of two self-gravitating electromagnetic waves. The principal feature in such a collision is that one expects only convergence, not shear, to be induced as the waves collide. The solution is displayed in Fig. 8. The metric is

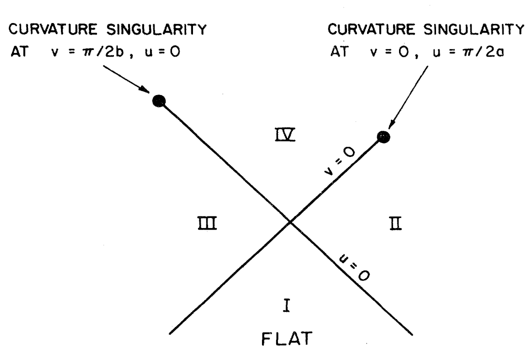


FIG. 8. Naive singularity structure of the Bell-Szekeres spacetime.

$$\text{region I, flat: } ds^2 = -2du\,dv + dx^2 + dy^2\,,$$
$$\text{region II: } ds^2 = -2du\,dv + \cos^2 au\,(dx^2+dy^2)\,,$$
$$\text{region III: } ds^2 = -2du\,dv + \cos^2 bv\,(dx^2+dy^2)\,,$$
$$\text{region IV: } ds^2 = -2du\,dv + \cos^2(au-bv)dx^2 + \cos^2(au+bv)dy^2\,.$$

No loss of generality is suffered by taking $a>0, b>0$. Notice that the region-III metric is valid on the segment $u=0, v>0$. In the Bell-Szekeres solution, one finds by explicit calculation that there are curvature singularities at $u=0, v=\pi/2b$ and at $v=0, u=\pi/2a$. A naive (local) analysis finds no others. We will again study the geometry of the $u=$constant hypersurfaces $(u\le 0, v>0)$ to determine if a fold singularity occurs in this solution also. The Bell-Szekeres solution obeys O'Brien-Synge boundary conditions;[7] e.g., across $u=0$ the derivative $g_{\alpha\beta,u}$ has a jump, while $g_{\alpha\beta,v}$, which is related to the geometry *in* the surface $u=0$, is continuous. In region III, compute $\nabla_{{}^0\underline{e}_{(y)}}{}^0\underline{e}_{(y)}$

$$\nabla_{{}^0\underline{e}_{(y)}}{}^0\underline{e}_{(y)} = g^{yy}\nabla_{\partial/\partial y}\frac{\partial}{\partial y} = g^{yy}\Gamma^u_{yy}\frac{\partial}{\partial u}\,, \tag{3.14}$$

where we again use the explicit properties of the metric to simplify the right-hand side.

Hence

$$\nabla_{{}^0\underline{e}_{(y)}}{}^0\underline{e}_{(y)} = \frac{1}{\cos^2 bv}\left(\tfrac{1}{2}\cos^2 bv\right)_{,v}\frac{\partial}{\partial u}$$
$$= -b\tan bv\frac{\partial}{\partial v}\,. \tag{3.15}$$

(The $\nabla_{{}^0\underline{e}_{(x)}}{}^0\underline{e}_{(x)}$ curvature in this case is identical.) We see that again, the surfaces $u=$constant harbor infinitely strong principal curvatures, here at $bv=\pi/2$. Again, this would be only a coordinate singularity, except for the fact that there is a curvature singularity at $u=0, v=\pi/2b$. The sign of the curvature in this case is the same as we found in studying the Khan-Penrose solution. If we could draw a picture like Fig. 4 for this Bell-Szekeres solution,[12] we would see the surface containing $u=0, v=\pi/2b$ fold downward as in Fig. 4. When one realizes that the surfaces $u>0$, for $v>0$, also fold *down* and warp down to

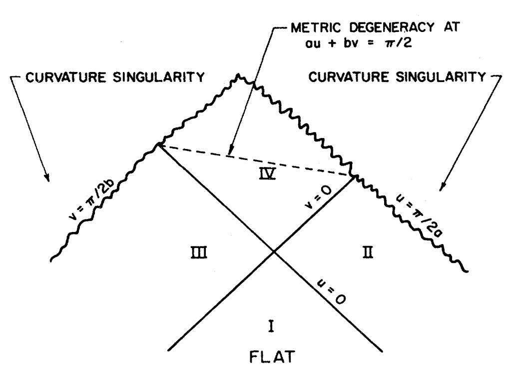


FIG. 9. True singularity structure of the Bell-Szekeres spacetime.

the curvature singular point, one realizes that the entire line $v=\pi/2b$ is, in fact, singular. Repeating the argument for $u=\pi/2a$, the actual structure of the Bell-Szekeres solution is shown in Fig. 9, and is topologically equivalent to the singularity structure in the Khan-Penrose solution. The apparent metric degeneracy at $au+bv=\pi/2$ has been shown by Bell and Szekeres[2] to be removable by a coordinate transformation. It is, however, the location of *zeros* of the length of the spacelike Killing vector $\partial/\partial y$. We are unable by our techniques to discover whether this homogeneity failure leads to a true singularity, as occurs for other apparent coordinate singularities. We do note, however, that the singularities at $v=\pi/2b$ and that at $u=\pi/2a$ are separated by a nonzero spatial distance in this hypersurface.

## IV. COLLIDING-WAVE SPACETIMES IN GENERAL

Because the general colliding-plane-wave metric has no cross terms between the $u,v$ and the $x,y$ blocks, we have the important symmetry of the connection

$$\Gamma_{uxx} = -\Gamma_{xux}$$

with similar equations when $u$ is substituted by $v$ and/or $x$ is substituted by $y$. This allows us to connect our principal curvature calculation to a calculation involving the optical scalars of the $u$ and $v$ congruences. The vector $\partial/\partial v$ is proportional to the tangent of a ray lying in the surface $u=$constant. Hence it describes how the photon in the null trajectory evolves as it travels to larger $v$. We thus propose to consider

$$\nabla_{{}^0\underline{e}_{(x)}}\left[\frac{\partial}{\partial v}\right] = (g^{xx})^{1/2}\Gamma^\alpha_{xv}\underline{e}_{(\alpha)} \tag{4.1}$$

and

$$\nabla_{{}^0\underline{e}_{(y)}}\left[\frac{\partial}{\partial v}\right] = (g^{yy})^{1/2}\Gamma^\alpha_{yv}\underline{e}_{(\alpha)}\,. \tag{4.2}$$

(As before, we need consider these quantities only in a region where metric variables only depend on $v$.) Because of the (at least block) diagonality of the colliding-wave metrics, we have here

$$K^x{}_x\,{}^0\underline{e}_{(x)} \equiv \nabla_{{}^0\underline{e}_{(x)}}\left[\frac{\partial}{\partial v}\right] = [g^{xx}\Gamma_{xxv}]\,{}^0\underline{e}_{(x)} \tag{4.3}$$

and

$$K^y{}_y\,{}^0\underline{e}_{(y)} \equiv \nabla_{{}^0\underline{e}_{(y)}}\left[\frac{\partial}{\partial v}\right] = [g^{yy}\Gamma_{yyv}]\,{}^0\underline{e}_{(y)}\,. \tag{4.4}$$

[In general one may have $y$-metric cross terms, whence (4.3) and (4.4) contain on the right a linear combination of ${}^0\underline{e}_{(x)}$ and ${}^0\underline{e}_{(y)}$. This generalization is immediate.[10] Because we deal with diagonal examples, we continue to assume diagonality here.]

Equations (4.3) and (4.4) are the components of a *null* extrinsic curvature as defined by Hawking and Ellis;[5] they are also expressible as the optical scalars $\rho,\sigma$ (or $\mu,\lambda$), as defined by Newman and Penrose[13] (see below). The important point to notice is that

$$\nabla_{^0\underline{e}_{(y)}}\left[\frac{\partial}{\partial v}\right]=(g^{yy}g^{uv}\Gamma_{vyy})(-g_{uv})^0\underline{e}_{(y)}\ ; \qquad (4.5)$$

the coefficient of $(-g_{uv}\ ^0\underline{e}_{(y)})$ is exactly the magnitude of the $y$-principal curvature of our warping two-surface, when we consider a region with metric dependent on $v$ only. Also notice that $g_{uv}=-1$ can always be chosen consistently in a region where the metric components only depend on $v$, simply by rescaling $v$ (only); the choice that makes $g_{uv}=-1$ is the choice that makes $v$ affine. We assume that this has been done subsequently. This also eliminates any ambiguity arising from comparing the coefficients of $\nabla_{^0\underline{e}_{(y)}}\ {^0\underline{e}_{(y)}}$ rather than the vector itself.

The O'Brien-Synge[7] junction conditions require that $g_{uv,v}$, which is the derivative determining the affineness of $\partial/\partial v$, be continuous at the join $u=0$; hence affineness is guaranteed on $u=0$ also.

With this choice, (4.3) and (4.4) are a null generalization of the formula for the extrinsic curvature for a spacelike three-surface. For that spacelike case, one computes $\nabla_\alpha n$, where $n$ is a metrically normalized normal to the three-surface.[14] One finds in that case that this tensor has no projections out of the three-surface. For the null case considered here, if $\partial/\partial v$ is affine, then the metrically raised contravariant form of $du$ is $\partial/\partial v$; and one may take the affine condition as the null analog of normalization. In that case one may proceed as in (4.5) by considering $\nabla_\alpha du$, where $du$ is the (affinely) normalized normal to the null three-surface. One then finds also that this tensor has no projections out of the null three-surface, and in fact none out of the $x,y$ two-surface, and (4.3) and (4.4) are a complete characterization of the embedding of the $u,x,y$ surface, entirely analogous to the spacelike three-surface embedding problem.

The connection between the optical scalars and the null extrinsic curvature allows us to prove a theorem which suggests that the collapse of a null surface into a singularity is a generic feature of colliding plane-wave spacetimes. By "physical singularity" we shall mean a singularity through which there is no $C^1$ extension.

*Theorem.* Suppose on the null hypersurface $u=$constant there is a physical singularity at $v=1$. If the convergence $\rho\to+\infty$ or $\rho\to-\infty$ at this singularity, then either ($u<$constant, $v=1,x,y$) or ($u>$constant, $v=1,x,y$) also is a physical singularity, provided the spacetime can be extended that far.

The proof is immediate from the fact that $\rho$ is equal to the trace of the components of the null extrinsic curvature of the $u=$constant hypersurface. Since $\rho$ diverges at the singularity, it follows that at least one of the $K^x{}_x$ or $K^y{}_y$ also diverges at the singularity. Our geometrical analysis then implies that either the half-hyperplane ($u<$constant, $v=1,x,y$) or the half-hyperplane ($u>$constant, $v=1,x,y$) is actually a singularity.

The singularities which collapse the null half-hyperplanes in the Khan-Penrose and in the Bell-Szekeres solutions make these spacetimes globally hyperbolic. This is in contrast to the nonglobally hyperbolic nature of the single gravitational plane-wave geometry. In the single plane-wave case, the analog of the singular null half-hyperplane is a surface—call it $S$—on which the planar symmetry breaks down. Since there are no singularities anywhere in the single plane-wave geometry, this surface also is nonsingular, and so one can extend across it. Let $p$ be a point in $I^-(Q)$, where $Q$ is the gravitational plane wave. Then $S\subset I^+(Q)$, and all the null geodesic generators of $\dot{I}^+(p)$—with one exception—intersect $Q$ and are focused by $Q$ onto a region on $S$ or in $I^+(S)$. The exceptional generator of $I^+(p)$ is the null geodesic which propagates parallel to $Q$. Since it never encounters the curvature in $Q$, it is never focused. This means that there is always a null generator of $\dot{I}^-(q)$, for all $q$ in the region where the generators of $\dot{I}^+(p)$ are focused, which never intersects $\dot{I}^+(p)$ (see Penrose[15] for details). Thus $\dot{I}^+(p)\cup\dot{I}^-(q)$ is not compact, which implies that the spacetime is not globally hyperbolic. One has a breakdown in global hyperbolicity only if $S$ or the region beyond it is included in the spacetime.

The singularities in the colliding plane-wave spacetimes we have studied collapse the surface $S$ onto a singularity, and thus prevent a region to the future of $S$ from existing. This means that the maximally extended colliding plane-wave spacetimes we have studied are globally hyperbolic. Our results suggest that global hyperbolicity is generic for colliding plane-wave spacetimes.

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[1]K. Khan and R. Penrose, Nature 229, 185 (1971).

[2]P. Bell and P. Szekeres, Gen. Relativ. Gravit. 5, 275 (1974).

[3]F. J. Tipler, Phys. Rev. D 22, 2929 (1980).

[4]J. M. Centrella and R. A. Matzner, Astrophys. J. 230, 311 (1979).

[5]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, England, 1973).

[6]P. Szekeres, J. Math. Phys. 13, 286 (1972).

[7]S. O'Brien and J. L. Synge, *Jump Conditions at Discontinuities in General Relativity,* Communications of the Dublin Institute of Advanced Studies A, No. 9 (1952).

[8]R. K. Sachs, Proc. R. Soc. London A270, 103 (1962).

[9]R. Penrose, private communication.

[10]We assume the diagonality of the $x$-$y$ metric sub-block. If this diagonality does not hold, we simply take ${}^0\underline{e}_{(x)}\,{}^0\underline{e}_{(x)}+{}^0\underline{e}_{(y)}\,{}^0\underline{e}_{(y)}$ with ${}^0\underline{e}_{(a)}$ the symmetric square root of $g^{ab}\underline{e}_{(a)}\otimes\underline{e}_{(b)}$, $a,b=x,y$. Then we will find spatial as well as null components of the principal curvature; we proceed by considering only the null part.

[11]L. P. Eisenhart, *An Introduction to Differential Geometry* (Princeton University Press, Princeton, 1947).

[12]We cannot draw such a picture because in contrast to the situation in the Khan-Penrose solution above, region III is not flat in the Bell-Szekeres solution.

[13]E. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).

[14]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[15]R. Penrose, Rev. Mod. Phys. 37, 215 (1965).