Causal cosmological perturbations and implications for the Sachs-Wolfe effect

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(Received 16 December 1983)

Gravitational perturbations in a Robertson-Walker (RW) universe induce fluctuations $\delta T/T$ in the microwave background due to the Sachs-Wolfe effect. We find that in RW spacetimes there exist general-relativistic generalizations of energy-momentum conservation for perturbations; perturbations must satisfy certain integral constraints. When the constraint conditions are applied to causal perturbations, there is a decrease in the predicted magnitude of the Sachs-Wolfe effect, of the order $(1+z)^{-1}$. Here z is the red-shift when the universe becomes matter dominated. Exact solutions in position space of the perturbation variables are found by construction of a Green's function. In this form the solutions are manifestly causal.

I. INTRODUCTION

In Newtonian physics and special relativity, arbitrary perturbations of the stress energy are not possible; they must obey conservation of energy and momentum. By contrast, in general relativity energy goes into the gravitational field, so there is no Gauss's law for the balance of energy and momentum. However, we prove in a subsequent paper[1] that in Robertson-Walker (RW) spacetimes, arbitrary perturbations δT^{μ}_{ν} and $h_{\mu\nu}$ of the stress energy and metric are not possible. Rather, there exist integral constraints on allowed perturbations. These can be thought of as a general-relativistic generalization of energy-momentum conservation. In this paper we will impose the constraints on perturbations which are localized in space.

Gravitational perturbations lead to anisotropies $\delta T/T$ in the temperature of the microwave background. Observations show that $\delta T/T \lesssim 10^{-3}$.[2] One contribution to $\delta T/T$ is the change in a photon's four-momentum from the zeroth-order value, as it propagates on the perturbed null geodesic. Sachs and Wolfe[3] computed this effect for a flat, pressureless RW universe. Suppose a photon is emitted at $(t_E,\vec{x}_E)$ and received at $(t_0,\vec{0})$. They find $\delta T/T \simeq \frac{1}{10}A(\vec{x}_E)$, where A is the gravitational potential in a Poisson equation for the density perturbation $\delta\rho/\rho$. For example, for a plane-wave perturbation with comoving wave number k,

$$\frac{\delta T}{T}=\frac{1}{2}\left|\frac{\delta\rho_k}{\rho}\right|(t)\left[\frac{Ha}{k}\right]^2(t)\quad(\text{Ref. }3)\;.$$

[Here $a(t)$ is the scale factor of the universe and $H(t)=\dot{a}/a$ is Hubble's constant.] Of particular current interest are inflationary-universe scenarios which predict a Z'eldovich spectrum:[4,5] if t_H is the time when the scale k crosses its horizon $(Ha/k)(t_H)=1$, then the ampltiude $|\delta\rho_k/\rho|(t_H)$ is equal to ϵ_H independent of k. Therefore $\delta T/T=\frac{1}{2}\epsilon_H$.[4]

Now, instead of plane waves, consider a perturbation that is created by a causal process. Causality implies that the resulting perturbation is strictly zero outside the forward light cone of the initial disturbance. Figure 1 illustrates the Sachs-Wolfe effect[3] for such a localized density perturbation $\delta\rho$ that is created by a pressure fluctuation at t_1. The disturbance evolves in some complicated way while pressure is important, but stays within its horizon. After the universe becomes effectively pressureless at t_p, the spatial dependence of $\delta\rho$ remains constant, though gravitational waves still propagate along the forward light cone.

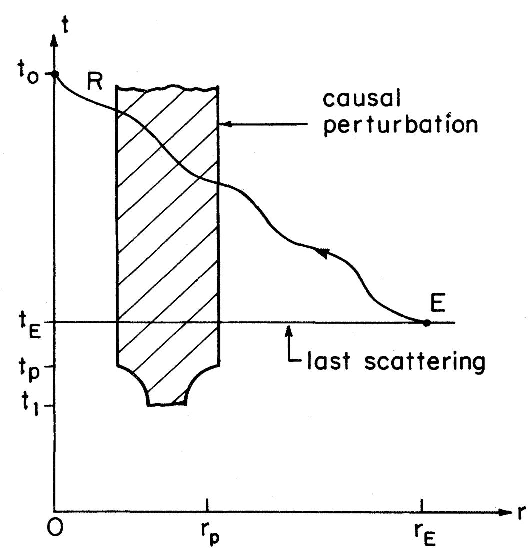

FIG. 1. A perturbation $\delta\rho$ is created by a pressure fluctuation at t_1. At t_p the universe becomes effectively pressureless. A photon is emitted at $(t_E,\vec{x}_E)$ and received at $(t_0,\vec{0})$. The null geodesic is perturbed from the background path, which causes a perturbation in the observed photon energy (see Sec. V). During the time when pressure is important, $\delta\rho$ propagates along the sound cone $dr/dt=c(t)/a(t)$, where $c(t)$ is the speed of sound (see Sec. VI).

When the emission point $\vec{x}_E$ is outside a perturbation, which is localized in space, we can write $A(\vec{x}_E)$ in a multipole expansion. The interesting point is that the constraints on RW perturbations imply that the monopole and dipole terms vanish, and consequently the magnitude of δT is smaller than if δT were computed without the constraints (wo). Explicitly, let $1/k$ be the coordinate length scale for $\delta\rho$, so the proper length scale is $L(t)=a(t)/k$. Then we will see that

$$\frac{\delta T}{T} \simeq \frac{1}{8} \left| \frac{\delta\rho}{\rho} \right| (t) \left[\frac{Ha}{k} \right]^2 (t) \left[\frac{H_0 a_0}{k} \right]^3$$
$$\simeq 4\times 10^{-15} \left| \frac{\delta\rho}{\rho} \right| (t_0) \left[\frac{M}{10^{15} M_\odot} \right]^{5/3} .$$

Further,

$$\frac{\delta T}{T} \simeq \frac{1}{4} \left[\frac{H_0 a_0}{k} \right]^2 \frac{\delta T_{\rm wo}}{T} \ll \frac{\delta T_{\rm wo}}{T} ,$$

since $(H_0 a_0/k) \ll 1$ whenever the multipole expansion is valid. If the universe has been RW since $t=0$, then causality implies $(H_0 a_0/k)^2 \lesssim (1+z_p)^{-1} \simeq 10^{-4}$. z_p is the red-shift when the universe becomes effectively pressureless.

The quantity that is measured in observations is the correlation in temperature between two source points $\vec{x}_1$ and $\vec{x}_2$, separated by an angle θ at the observer. Based on the galaxy correlation function, Peebles finds

$$\left\langle \frac{(T_1-T_2)^2}{T} \right\rangle^{1/2} \simeq 2\times 10^{-5} \left[\sin\frac{\theta}{2} \right]^{1/2}$$
$$\simeq \left[\sin\frac{\theta}{2} \right]^{1/2} \left[\frac{H_0 a_0}{k} \right]^{3/2} \left| \frac{\delta\rho}{\rho} \right|_0 \quad (\text{Ref.}\,6) .$$

In the calculation Peebles assumes that $\delta\rho$ is uncorrelated on some scale which is less than $|\vec{x}_1 - \vec{x}_2|$.

Now, let $\delta\rho$ be a sum of randomly scattered perturbations, each which is localized in space and has coordinate length scale $1/k$. Then $\delta\rho$ must satisfy the constraint conditions, and this changes the calculation of $\langle (T_1-T_2)^2 \rangle$. We find

$$\left\langle \frac{(T_1-T_2)^2}{T} \right\rangle^{1/2} \simeq \frac{1}{2} \left[\frac{H_0 a_0}{k} \right]^2 \left| \frac{\delta\rho}{\rho} \right|_0$$

and

$$\left| \left\langle \frac{2\delta T_1 \delta T_2}{T} \right\rangle \right|^{1/2} \simeq \frac{1}{16} \left[\frac{H_0 a_0}{k} \right]^{7/2} \left| \frac{\delta\rho}{\rho} \right|_0 \left[\sin\frac{\theta}{2} \right]^{-3/2} .$$

The second quantity is divergent in the case Peebles considers.

In the computation it has been assumed that $\delta\rho$ is uncorrelated on the scale $|\vec{x}_1 - \vec{x}_2|$, that is, $k|\vec{x}_1 - \vec{x}_2| \gg 1$, or equivalently, $\sin(\theta/2) \gg \frac{1}{4} H_0 a_0/k$. If the universe has been RW since $t=0$, this just says that we are comparing the temperature between two points that were causally disconnected at the emission time. Numerically this is $\theta \gtrsim 2°$, for $z_E \simeq 10^3$.

To apply the constraints we will need solutions for $h_{\mu\nu}$ and δT^μ_ν in which causality is manifest. The usual method (Refs. 3 and 7–9) is to write $h_{\mu\nu}$ as a sum of scalar, vector, and tensor modes, and then find solutions in an eigenfunction expansion. Sachs and Wolfe also transform each mode back to position space. However, when solutions are separated into modes, each mode has a long-range, action-at-a-distance part.

So we start in Sec. I by finding exact solutions in position space for $h_{\mu\nu}$ and δT^μ_ν, in a $k=0$, $p=0$ universe, by constructing the Green's function for the gravitational wave equation. Section II discusses the integral constraints on perturbations. In Sec. III, the late-time behavior of the perturbations is derived, as well as the solution for pure scalar modes. The simple example of a quadrupole source is worked out. The implication of the perturbation constraints for the Sachs-Wolfe effect is calculated in Sec. IV. Section V considers the case when pressure is nonzero.

II. EXACT SOLUTIONS FOR PERTURBATIONS OF A $k=0$, $p=0$, ROBERTSON-WALKER BACKGROUND BY CONSTRUCTION OF A GREEN'S FUNCTION

A. Background model and choice of coordinates

We shall consider perturbations from a $k=0$ Robertson-Walker (RW) universe with metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j$$
$$= g^{(0)}_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

and a perfect-fluid stress-energy tensor

$$T^\mu_{(0)\nu} = (\rho+p) u^\mu_{(0)} u_{(0)\nu} + p g^\mu_\nu . \tag{2}$$

Here ρ is the background density, p is the background pressure, and $\vec{u}_{(0)}$ is the unperturbed fluid four-velocity. In the coordinate system (1), $u^\mu_{(0)} = (1, \vec{0})$. Latin indices run from 1 to 3 and Greek indices from 0 to 3. Also, $\dot{f} \equiv df/dt$, $f' \equiv df/d\eta$, where $dt = a\, d\eta$.

The Einstein equations are

$$\left[\frac{\dot{a}}{a} \right]^2 = \tfrac{8}{3}\pi G\rho$$

and

$$\frac{d}{dt}(a^3\rho) = -3pa^2\dot{a} .$$

When $p=0$ the solutions are

$$a = a_0 \left[\frac{t}{t_0} \right]^{2/3} , \quad \rho = \rho_0 \left[\frac{a_0}{a} \right]^3$$

or, in terms of η,

$$a = a_0 \left[\frac{\eta}{\eta_0} \right]^2 ,$$

where

$$t_0=\tfrac{1}{3}a_0\eta_0 \ . \tag{3}$$

We can always choose synchronous coordinates in which the time direction is the tangent to the geodesic which is perpendicular to a family of spacelike hypersurfaces.[7–9] Then

$$ds^2=-dt^2+g_{ij}dx^idx^j \ .$$

Comoving coordinates are coordinates fixed to the fluid particles. So to have coordinates which are synchronous and comoving, the fluid paths must be geodesics which are perpendicular to the spacelike surfaces. The equations of motion for a fluid are $T^{\mu\nu}_{;\nu}=0$, or, for a perfect fluid,

$$\begin{aligned}&(\rho+p)\nabla_{\vec{u}}\vec{u}=-\nabla p-p_{;\alpha}u^{\alpha}\vec{u} \ ,\\ &\rho_{;\alpha}u^{\alpha}+(\rho+p)\nabla\cdot\vec{u}=0 \ .\end{aligned} \tag{4}$$

When $p=0$, these become

$$\begin{aligned}&\nabla_{\vec{u}}\vec{u}=0 \ ,\\ &(\rho u^{\nu}\sqrt{-g}\,)_{,\nu}=0 \ .\end{aligned} \tag{5}$$

Therefore in the pressureless case the fluid trajectories are geodesics. A necessary and sufficient condition for the fluid geodesics to be, in addition, perpendicular to a family of spacelike surfaces is that the fluid is irrotational:[7]

$$u_{i;j}-u_{j;i}=0 \ .$$

So in general we cannot have coordinates that are both synchronous and comoving.

We will use the synchronous gauge

$$g_{0\lambda}=0 \ .$$

Also, we will see that by choosing coordinates which are comoving with the irrotational part of the flow, the perturbation equations simplify.

There remain the following coordinate transformations that are compatible with the synchronous gauge (Refs. 7–9):

$$\begin{aligned}&\text{(i) For } x^{\mu}\to x^{\mu}-a^2(0,C^j(\vec{x})) \ ,\\ &g_{ij}\to g_{ij}+a^2(C_{i\,|\,j}+C_{j\,|\,i}) \ ,\\ &\text{(ii) For } x^{\mu}\to x^{\mu}-\left[C(\vec{x}),C(\vec{x})^{|j}a^2\int^t\frac{ds}{a^2}\right] \ ,\\ &g_{ij}\to g_{ij}+\left[2a^2\int^t\frac{ds}{a^2}\right]C(\vec{x})_{|\,i\,|\,j}+\frac{2\dot{a}}{a}g_{ij}C(\vec{x}) \ .\end{aligned} \tag{6}$$

Here $C(\vec{x})$ is an arbitrary function and $C^j(\vec{x})$ is an arbitrary three-vector both of which depend only on $\vec{x}$. $C_{i\,|\,j}$ is covariant differentiation with respect to the metric of the spacelike hypersurface g_{ij}.

B. Perturbation equations

Now consider perturbations from a flat RW background (1) and perfect-fluid stress energy (2),

$$\begin{aligned}&g_{\mu\nu}=g^{(0)}_{\mu\nu}+h_{\mu\nu}, \quad h_{0\lambda}=0\\ &T^{\mu}_{\nu}=T^{\mu}_{(0)\nu}+\delta T^{\mu}_{\nu} \ ,\\ &\delta T^0_0=-\delta\rho, \quad \delta T^i_j=\delta p g^i_j \ ,\\ &\delta T^0_i=(\rho+p)a^2v^i \ ,\\ &v^i\equiv\delta u^i \ .\end{aligned} \tag{7}$$

Substituting into the linearized Einstein equations $\delta R_{\mu\nu}=8\pi G\delta(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T)$ gives[8,9]

$$\begin{aligned}&\ddot{h}+2\frac{\dot{a}}{a}\dot{h}=-8\pi G(\delta\rho+3\delta p) \ ,\\ &\delta\dot{\rho}+3\frac{\dot{a}}{a}(\delta\rho+\delta p)=-(\rho+p)\left[\frac{\dot{h}}{2}+v^i_{,i}\right] \ ,\\ &\frac{1}{a^3}\frac{d}{dt}[a^5v^i(\rho+p)]=-\delta p_{,i} \ ,\\ &\frac{\partial}{\partial t}\left[h_{,i}-\frac{1}{a^2}h_{ik,k}\right]=16\pi Ga^2(\rho+p)v^i \ ,\\ &-\frac{1}{a^2}h_{ij,kk}+\ddot{h}_{ij}-\frac{\dot{a}}{a}\dot{h}_{ij}-2\frac{\dot{a}}{a}\frac{\ddot{a}}{a}h_{ij}\\ &\qquad=h_{,ij}-\dot{a}a\dot{h}\delta_{ij}+8\pi Ga^2\delta_{ij}(\delta\rho-\delta p)\\ &\qquad\quad-\frac{1}{a^2}(h_{ik,kj}+h_{jk,ki}) \ .\end{aligned} \tag{8}$$

A standard way of proceeding is to write h_{ij} as the sum of scalar, vector, and tensor modes, write the equations for each type of mode, and find solutions in Fourier space (Lifshitz and Khalatnikov[7] follow this procedure for $k=\pm1$, so they expand in spherical harmonics rather than plane waves.) Sachs and Wolfe fourier transform back to real space; one then has a solution of the form

$$h_{ij}=(Ag_{ij}+B_{,ij})+(E_{i,j}+E_{j,i})+C_{ij} \ , \tag{9}$$

where A,B are scalars, E_i a vector, C_{ij} a tensor, and

$$E_i{}^{;i}=C^i_i=C^j_{i,j}=0 \ .$$

Now, we are particularly interested here in the Sachs-Wolfe effect due to perturbations which are localized in space, so we will find solutions in position space not Fourier space. Further, we want the solutions in a form in which the causality of the Einstein equations is manifest. As will be shown, when solutions are written as the sum of modes the causality is hidden; the scalar modes have a long-range Coulombic part, for example. [This is like writing electromagnetism in the Coulomb gauge. The theory is causal although the scalar potential is written in an action-at-a-distance form (see, e.g, Jackson, *Classical Electrodynamics* 2nd ed. (Wiley, New York, 1975), problem 6.18)]. So we will not decompose $h_{\mu\nu}$ into a sum of modes.

Finally, a further choice of gauge is possible in the $p=0$ case which simplifies the equations. A velocity perturbation can be generated by a scalar or vector perturbation

$$\begin{aligned}&\delta u^i=v^i+s^i \ ,\\ &v_i{}^{;i}=0 \ ,\\ &s_i=\phi_{;i} \ .\end{aligned}$$

Therefore, $\operatorname{curl}\vec{s}=0$, and we can choose coordinates that are synchronous and comoving with the scalar-generated (irrotational) part of the flow. In these coordinates $\delta u^i_{;i}=v^i_{;i}=0$. With this choice of gauge there are only essentially trivial degrees of coordinate freedom left, namely relabeling the coordinates within a hypersurface Eq. (6 i) and changing the time origin [C=constant in Eq. (6 ii)].

Integration along null geodesics for computing the Sachs-Wolfe effect is done most easily in terms of the conformally related metric

$$ds^2=a^2[-d\eta^2+(\delta_{ij}+\bar{h}_{ij})dx^i dx^j]$$
$$h_{ij}=a^2\bar{h}_{ij},\quad h=\bar{h}\ . \tag{10}$$

Finally, the linearized $k=0$ Einstein equations for $p=0$, $v^i_{,i}=0$, in the time variable η, are

$$h''+\frac{a'}{a}h'=-8\pi G\delta\rho a^2\ , \tag{11a}$$

$$\delta\rho'+3\frac{a'}{a}\delta\rho=-\rho\frac{h'}{2}\ , \tag{11b}$$

$$\frac{d}{d\eta}(a^5\rho v^i)=0\ , \tag{11c}$$

$$\frac{d}{d\eta}\bar{h}_{ik,k}=-16\pi G a^3\rho v^i+\bar{h}_{,i}\ , \tag{11d}$$

$$-\nabla^2\bar{h}_{ij}+\bar{h}''_{ij}+2\frac{a'}{a}\bar{h}'_{ij}=-\frac{a'}{a}\bar{h}'\delta_{ij}+\bar{h}_{,ij}+8\pi G a^2\delta\rho\delta_{ij}-(\bar{h}_{ik,kj}+\bar{h}_{jk,ki})\ . \tag{11e}$$

C. Solutions

The solutions for (11a)—(11d) are

$$h'(\vec{x},\eta)=b_1(\vec{x})\eta+b_2(\vec{x})\eta^{-4}\ ,$$
$$h(\vec{x},\eta)=\tfrac{1}{2}b_1(\vec{x})\eta^2-\tfrac{1}{3}b_2(\vec{x})\eta^{-3}+b_3(\vec{x})\ ,$$
$$\delta\rho(\vec{x},\eta)=\tfrac{1}{2}\rho(-\tfrac{1}{2}b_1\eta^2+\tfrac{1}{3}b_2\eta^{-3})\ , \tag{12}$$
$$v^i(\vec{x},\eta)=\frac{\alpha^i(\vec{x})}{a^5\rho},\quad \alpha^i_{,i}=0\ ,$$
$$\bar{h}_{ik,k}=-16\pi G\alpha^i(x)\int_{\eta_0}^{\eta}\frac{d\eta}{a^2}+h_{,i}+\beta_i(\vec{x})\ .$$

The functions b_1, b_2, b_3, α^i, and β_i are determined by the initial conditions. Note that coordinates can always be chosen such that $b_3=0$ by one of the remaining allowed transformations (6 i).[9]

In terms of proper time t,

$$\dot{h}(\vec{x},t)=C_1(\vec{x})t^{-1/3}+C_2(\vec{x})t^{-2}\ ,$$
$$\delta\rho(\vec{x},t)=\tfrac{1}{2}\rho[-\tfrac{3}{2}C_1(\vec{x})t^{2/3}+C_2(\vec{x})t^{-1}]\ . \tag{13}$$

The time dependence of $\delta\rho$ is the same as for perturbations in a Newtonian anlaysis.[8] This agreement is considered further in Appendix A.

Finally, we need to solve (11e),

$$\mathcal{O}\bar{h}_{ij}\equiv\left[-\nabla^2+\frac{\partial^2}{\partial\eta^2}+\frac{4}{\eta}\frac{\partial}{\partial\eta}\right]\bar{h}_{ij}=s_{ij}\ . \tag{14}$$

D. Construction of the Green's function

Let $G(\vec{x}\eta\,|\,\vec{x}'\eta')$ be defined by

$$\mathcal{O}G(\vec{x}\eta\,|\,\vec{x}'\eta')=\delta^{(3)}(\vec{x}-\vec{x}')\delta(\eta-\eta'),\quad \eta>\eta'$$
$$G(\vec{x}\eta\,|\,\vec{x}'\eta')=0,\quad \eta\le\eta'\ . \tag{15}$$

Then the solution to

$$\mathcal{O}u(\vec{x},\eta)=s(\vec{x},\eta)\ \text{for}\ \eta>\eta_0\ ,$$
$$u=0\ \text{for}\ \eta<\eta_0 \tag{16}$$

is

$$u(\vec{x},\eta)=\int d^3x'\int_{\eta_0}^{\eta}d\eta' s(\vec{x}',\eta')G(\vec{x}\eta\,|\,\vec{x}'\eta')+u_h\ ,$$

where u_h is any solution of the homogeneous equation.

By integrating (15) across the impulse time η', the problem to be solved for G can be restated as

$$\mathcal{O}G(\vec{x}\eta\,|\,\vec{x}'\eta')=0\ ,$$
$$G(\vec{x}\eta\,|\,\vec{x}'\eta')=0\ \text{for}\ \eta\le\eta'\ , \tag{17}$$
$$\partial_\eta G(\vec{x}\eta'\,|\,\vec{x}'\eta')=\delta^{(3)}(\vec{x}-\vec{x}')\ .$$

A solution to $\mathcal{O}G=0$ is[3]

$$G(\vec{x},\eta)=\frac{1}{\eta}\frac{\partial}{\partial\eta}\left[\frac{D(\vec{x},\eta)}{\eta}\right]\ , \tag{18}$$

where

$$(\partial_\eta{}^2-\nabla^2)D=0\ .$$

The initial conditions (17) on G will hold if D satisfies the initial conditions

$$D(\vec{x},\eta')=\frac{-\eta'^2}{4\pi|\vec{x}-\vec{x}'|}\ ,$$
$$\partial_\eta D(\vec{x},\eta')=\frac{-\eta'}{4\pi|\vec{x}-\vec{x}'|}\ . \tag{19}$$

The solution to $(\partial_\eta{}^2-\nabla^2)D=0$, $D(\vec{x},\eta')=f$, $\partial_\eta D(\vec{x},\eta')=g$ is[10]

$$D(\vec{x},\eta)=\frac{\partial}{\partial\eta}\{(\eta-\eta')\mu[f]\}+(\eta-\eta')\mu[g]\ , \tag{20}$$

where

$$\mu[g](\vec{x},\eta)=\frac{1}{4\pi}\int d\Omega\, g[\vec{x}+\vec{\Omega}(\eta-\eta')],\quad \vec{\Omega}\cdot\vec{\Omega}=1$$

is the spherical average of g.

Therefore we need the integral

$$\int d\Omega \frac{1}{|\vec{w}-r\vec{\Omega}|} = \frac{4\pi}{|\vec{w}|} H(|\vec{w}|-r)+\frac{4\pi}{r}H(r-|\vec{w}|)\ . \tag{21}$$

$H(z)$ is the Heaviside function. Then the solutions for D and G are

$$D(\vec{x},\eta)=-\frac{\eta'}{4\pi}\left[\frac{1}{\eta}H(\Delta\eta-\Delta x)+\frac{1}{\Delta x}H(\Delta x-\Delta\eta)\right], \tag{22}$$

$$G(\vec{x}\eta\,|\,\vec{x}'\eta')=\frac{1}{\eta}\frac{\partial}{\partial\eta}\left[\frac{D}{\eta}\right]=\frac{\eta'}{4\pi}\left[\frac{1}{\eta^3}H(\Delta\eta-\Delta x)+\frac{\eta'}{\eta^2}\frac{\delta(\Delta\eta-\Delta x)}{\Delta x}\right], \tag{23}$$

where $\Delta\eta\equiv\eta-\eta'$ and $\Delta x\equiv|\vec{x}-\vec{x}'|$. Notice that $G\equiv 0$ outside the forward light cone of the source point $(\vec{x}',y')$. To check solutions (22) and (23), see Appendix B.

Finally, the solution to (16) is

$$u(\vec{x},\eta)=\frac{1}{4\pi}\int_{\Delta x\le\Delta\eta} d^3x'\left[\frac{1}{\eta^3}\int_{\eta_0}^{\eta-\Delta x}d\eta' s(\vec{x}',\eta')+\frac{1}{\eta^2}\frac{s(\vec{x}',\eta-\Delta x)}{\Delta x}(\eta-\Delta x)^2\right]+\frac{1}{\eta}\frac{\partial}{\partial\eta}\left[\frac{D}{\eta}\right], \tag{24}$$

where $(\partial\eta^2-\nabla^2)D=0$. The last term is the solution to the homogeneous equation. Now u is manifestly causal; the solution at $(\vec{x}_R,y_R)$ is determined by the source at points inside and on the backwards light cone of $(\vec{x}_R,\eta_R)$.

E. The solution for $\bar{h}_{ij}$

Using (12), (11e) becomes

$$\mathcal{O}\bar{h}_{ij}=f_{-3}(\vec{x})_{ij}\eta^{-3}+f_0(\vec{x})_{ij}+f_2(\vec{x})_{ij}\eta^2\ ,$$

where

$$f_2(\vec{x})_{ij}=-\tfrac{1}{2}b_1(\vec{x})_{,ij}\ ,$$

$$f_0(\vec{x})_{ij}=\frac{16\pi G}{3}(\alpha_{i,j}+\alpha_{j,i})\frac{\eta_0}{a_0^{\,2}}-(\beta_{i,j}+\beta_{j,i})-b_3(\vec{x})_{,ij}-5b_1(\vec{x})\delta_{ij}\ , \tag{25}$$

$$f_{-3}(\vec{x})_{ij}=-\frac{16\pi G}{3}(\alpha_{i,j}+\alpha_{j,i})\frac{\eta_0^{\,4}}{a_0^{\,2}}+\tfrac{1}{3}b_2(\vec{x})_{,ij}\ .$$

Suppressing indices, the solution to (25) is

$$u(\vec{x},\eta)=\frac{1}{4\pi}\int_{\Delta x\le\Delta\eta}d^3x'\left[\frac{1}{\eta^3}\{-f_{-3}(\vec{x}')[(\eta-\Delta x)^{-1}-\eta_0^{-1}]+\tfrac{1}{4}f_2(\vec{x}')[\eta-\Delta x)^4-\eta_0^4]+\tfrac{1}{2}f_0(\vec{x}')[(\eta-\Delta x)^2-\eta_0^2]\}+\frac{1}{\eta^2}\frac{1}{\Delta x}[f_{-3}(\vec{x}')(\eta-\Delta x)^{-1}+f_0(\vec{x}')(\eta-\Delta x)^2+f_2(\vec{x}')(\eta-\Delta x)^4\right]+\frac{1}{\eta}\frac{\partial}{\partial\eta}\left[\frac{D}{\eta}\right]. \tag{26}$$

The importance of writing h_{ij} in this form, instead of in Fourier space (e.g., Refs. 8 and 9) or as a sum of modes[3] will become clear when we want to apply integral constraints on perturbations.

III. INTEGRAL CONSTRAINTS ON PERTURBATIONS

In special relativity, time and space translational invariance imply the conservation laws

$$\partial_0\int_G d^3x\,\delta T^0_\alpha=-\int_{\partial G}da_k\delta T^k_\alpha\ .$$

Here G is a three-dimensional volume in flat space with boundary ∂G. When the surface terms vanish, and assuming the perturbation is zero at some initial time, these become

$$\int d^3x\,\delta\rho=0\ ,$$

$$\int d^3x\,\vec{x}\delta\rho=0\ .$$

By contrast, in general relativity there is no Gauss's law for the conservation of either T^μ_ν or δT^μ_ν. However, as is proved in a separate paper,[1] some spacetimes do have integral constraints on perturbations. Here we will simply state the results which have interesting implications for the Sachs-Wolfe effect.

Definition. Let G be a spacelike hypersurface with normal n and boundary ∂G. (Possibly $\partial G=0$, as in a closed RW universe.) An integral-constraint vector is a vector V such that

$$\int_G dv\, V^\mu \delta T^\alpha_\mu n_\alpha = \int_{\partial G} da_l B^l$$

for arbitrary perturbations $h_{\mu\nu}$ and $\dot{h}_{\mu\nu}$.

The boundary term B^l is a function of h_{ij} and V^μ, and is zero if h_{ij} vanishes on the boundary. The conditions for the existence of a constraint vector V in a given background geometry follow directly from the linearized Einstein equations, and depend only on the background and the slicing G chosen. V is independent of the type of stress energy assumed for δT^μ_ν and of the choice of gauge for $h_{\mu\nu}$. It is important to emphasize that most spacetimes do not have constraint vectors. However, we prove[1] that the RW $k=0$, $+1$, -1, and de Sitter spacetimes each have ten integral-constraint vectors.

Explicitly, let G be a subset of a $t=$constant surface in a $k=0$ universe, in the coordinate system of (1). Then the following are integral-constraint vectors:

$$V_{(0)}=\frac{\partial}{\partial t}-\frac{\dot a}{a}x^i\frac{\partial}{\partial x^i}\,, \tag{27}$$

$$V_{(k)}=x^k\frac{\partial}{\partial t}+\frac{\dot a}{a}\left(\tfrac{1}{2}\delta^{ki}r^2-x^ix^k\right)\frac{\partial}{\partial x^i},\quad k=1,2,3\,.$$

Also, the purely spatial Killing vectors

$$T_{(k)}=\frac{\partial}{\partial x^k}\,,$$

$$R_{(k)}=\epsilon^{kij}x^i\frac{\partial}{\partial x^j},\quad k=1,2,3$$

are integral-constraint vectors. We will call a perturbation local at time t_0 when there exists an R such that $h_{ij}(\vec{x},t_0)\equiv 0$ for $|\vec{x}|>R$. If a peturbation has local initial conditions, then we can always take ∂G big enough such that $h_{ij}=0$ on ∂G, as is clear from the solutions in Sec. I (see Fig. 2). Then the integral constraints become

$$\int_G dv\, V^\mu \delta T^0_\mu = 0\,.$$

For $V=V_{(0)}$, (27), this is

$$\int dv\left[\delta T^0_0-\frac{\dot a}{a}\delta T^0_k x^k\right]=0\,,$$

valid for any pressure and any δT^μ_ν. For the $p=0$ case considered in Sec. I, $\delta T^0_k=0$ when the flow is irrotational

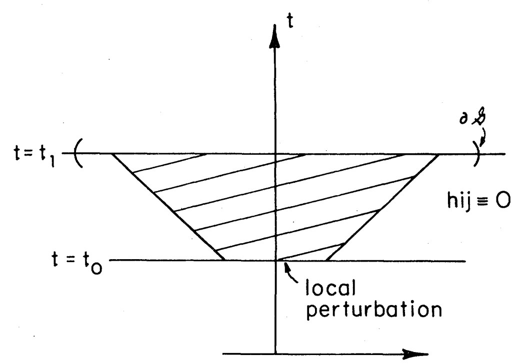

FIG. 2. A perturbation h_{ij} has local initial conditions at t_0. In a spatially infinite universe, at any later time t_1, we can always choose the region G large enough such that $h_{ij}\equiv 0$ on the boundary ∂G. Then the boundary term in the integral constraints is zero.

(in the choice of gauge used). Also, we see in this section that at late times the growing (scalar) mode in $\delta\rho$ dominates the vector mode in δT^0_k. Therefore, either for irrotational flow, or at late times,

$$0\simeq\int dv\,\delta\rho\simeq\int dv\,\delta\rho_{\rm growing}\,. \tag{28a}$$

Similarly, using $V_{(1)}$, $V_{(2)}$, and $V_{(3)}$ gives

$$0\simeq\int dv\,\vec{x}\,\delta\rho\simeq\int dv\,\vec{x}\delta\rho_{\rm growing}\,. \tag{28b}$$

Therefore in a $k=0$, $p=0$ RW spacetime, we recover the special-relativity results that the monopole and dipole moments of $\delta\rho$ vanish.

IV. LATE-TIME BEHAVIOR OF SOLUTIONS AND EXAMPLE OF A QUADRUPOLE SOURCE

A. Late-time solution

Assume the source terms in the integrand of (26) are local, so $f_i(\vec{x})=0$ for $|\vec{x}|>R$. Then for $\eta-\eta_0\geq|\vec{x}|+R$ the integration in (26) can be extended to all $\vec{x}$. We use the integrals

$$-\frac{1}{4\pi}\int d^3x'\frac{f(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}=\nabla^{-2}f\,,$$

$$-\frac{1}{8\pi}\int d^3x' f(\vec{x}\,')\,|\vec{x}-\vec{x}\,'|=\nabla^{-4}f\,.$$

[To evaluate the second integral one subtracts a zero-point divergence:

$$16\pi^2\nabla^{-4}f=\int d^3z\, f(\vec{z})\int d^3y\left[\frac{1}{|\vec{x}-\vec{y}||\vec{y}-\vec{z}|}-\frac{1}{|\vec{x}-\vec{y}|^2}\right]+\int\frac{d^3y}{|\vec{x}-\vec{y}|^2}\int d^3z\, f(\vec{z})\,.$$

The second integral is zero for sources f whose integral over all space vanishes. In the case above this is true by the integral constraints (28a) and (28b). The first integral is done with the help of (21).] Substituting the definitions (25) of the source terms, one finds

$$h_{ij}(\vec{x},\eta)=\tfrac{1}{2}\eta^2\nabla^{-2}b_1(\vec{x})_{,ij}+5\delta_{ij}\nabla^{-2}b_1+\nabla^{-2}b_{3,ij}+5\Delta^{-4}b_{1,ij}+\nabla^{-2}(\beta_{i,j}+\beta_{j,i})-\frac{16\pi G\eta_0}{3a_0{}^2}\nabla^{-2}(\alpha_{i,j}+\alpha_{j,i})+O(1/\eta)\ . \tag{29}$$

The important point is that h_{ij} looks like a scalar perturbation at late times, independent of what mixture of modes is present in the initial data. Also notice that the next-order terms in η are η^0, and are a mixture of vector and scalar modes. The η^1 terms vanish upon integration by parts. Tensor modes enter the solution in the solution to the homogeneous equation.

B. Solutions for scalar modes

Next we want to compare the late-time solution (29) to the solution if one assumes h_{ij} is a scalar perturbation. Choose comoving coordinates, so $v^i=0$ in (11). As indicated in (9), and using the notation in Ref. 11, let

$$\bar{h}_{ij}=\delta_{ij}A+B_{,ij}\ ,$$
$$h=3A+\nabla^2 B\ .$$

Then (11d) becomes

$$\dot{A}_{,j}=0$$

and (11e) implies[11]

$$\ddot{A}+3\frac{\dot{a}}{a}\dot{A}=0\ ,$$
$$\ddot{B}+3\frac{\dot{a}}{a}\dot{B}=\frac{A}{a^2}$$

(Ref. 12).

The solutions for h_{ij} and $\delta\rho$, using (11a) and (11b) are

$$\bar{h}_{ij}=\tfrac{1}{10}\eta^2 A_{,ij}+A\delta_{ij}+\omega_{,ij}-b_{,ij}\eta^{-3}\ ,$$
$$\delta\rho=\tfrac{1}{2}\rho(-\tfrac{1}{10}\eta^2\nabla^2 A+\eta^{-3}\nabla^2 b)\ . \tag{30}$$

Here A, ω, and b are arbitrary functions of $\vec{x}$ only which are fixed by the initial conditions. One these are fixed then the six components h_{ij} are determined.

With some changes of notation, this agrees with the scalar modes part of the solution of Sachs and Wolfe.

Comparing to the previous solutions (12), we see that

$$b_1=\tfrac{1}{5}\nabla^2 A,\quad b_2=3\nabla^2 b,\quad b_3=3A+\nabla^2\omega\ . \tag{31}$$

For a purely scalar perturbation, it is impossible to have local initial conditions for h_{ij}. Suppose we pick three local functions L, M, and N for the initial data on $\delta\rho$, h, and h'. Then h_{ij} has a long-range part, since A, b, and ω are related to L, M, and N by the inverse Laplace operator. In particular, the integral constraints with zero boundary term cannot be applied to such perturbations. So, if we want to consider localized lumps (see, e.g., Fig. 1 in Sachs and Wolfe), we must include all modes.

From Sec. I, we known that if the initial conditions on perturbations are local, then the solution is strictly zero outside the forward light cone of the initial data. Therefore the integral constraints (see Sec. II) with zero boundary term must hold. However, calculationally it is difficult to use the exact solution (26) for h_{ij}, and we have seen that at late times the growing scalar modes dominate h_{ij}. So when we compute the Sachs-Wolfe effect which depends on $(\partial/\partial\eta)h_{ij}$, we will use the integral constraints, which are exact statements, and approximate h_{ij} by the growing scalar part only when doing the calculation. In this approximation the calculation can be done explicitly.[3]

As a check, suppose that only scalar modes are present in the initial data. Then the Green's-function solution (26) must reduce to the scalar solution (30). Indeed, with $\alpha^i=0$, (12) implies $\beta_i=-2A_{,i}$. Then with the change of notation (31) the late-time solution (29) agrees with the leading terms in (30).

C. Example of a pure quadrupole source

Here we calculate the solutions for the simplest local source that is allowed by the integral constraints, a quadrupole. Assume that only the growing mode is present and the flow is irrotational. Then the integral-constraint statements (28a), and (28b) are exact. So we want to solve (11) with the initial conditions

$$\delta\rho(\vec{x},\eta_0)=-\tfrac{1}{4}\rho\eta_0{}^2 b_1\ ,$$
$$h(\vec{x},\eta_0)=\tfrac{1}{2}\eta_0{}^2 b_1\ ,$$
$$h'(\vec{x},\eta_0)=\eta_0 b_1\ ,$$
$$b_1(\vec{x})=\lambda\delta(z)\delta(x)[\delta(y-l)+\delta(y+l)]$$
$$\qquad -\lambda\delta(z)\delta(y)[\delta(x-l)+\delta(x+l)]\ ,$$

where l is the coordinate quadrupole length and $\lambda=\epsilon l^3$. The condition $|\delta\rho/\rho|\ll 1$ is equivalent to $\epsilon\eta_0{}^2\ll 1$. For simplicity take $\beta_i=0$, and choose any local initial conditions on h_{ij} and h'_{ij} consistent with those on β_i, h, and h'. Then for $\eta-\eta_0>|\vec{x}|+l$, h_{ij} is given by (29) with

$$b_2=b_3=\beta_i=\alpha_i=0\ .$$

For the source assumed here, we interpret

$$\nabla^{-2}b_{1,ij}=-\frac{1}{4\pi}\int dv' b_1\partial_i\partial_j\left[\frac{1}{|\vec{x}-\vec{x}'|}\right]\ .$$

An expansion in l/r yields

$$h_{ij}=-\eta^2\frac{1}{8\pi}\frac{\lambda l^2}{r^5}\begin{bmatrix}-6 & 105xy(y^2-x^2)/r^4 & 30xz/r^2\\ & 6 & -30yz/r^2\\ & & 20(y^2-x^2)/r^2\end{bmatrix}$$

$$+\frac{15}{8\pi}\frac{\lambda l^2}{r^5}\begin{bmatrix}\frac{1}{9}-(y^2-x^2)/r^2 & \frac{5}{2}(y^2-x^2)xy/r^4 & xz/r^2\\ & \frac{2}{3}-(y^2-x^2)/r^2 & -yz/r^2\\ & & -7(y^2-x^2)/r^2\end{bmatrix}+O\left[\lambda\frac{l^3}{r^6}\right]+O(1/\eta)\ .$$

V. INTEGRAL CONSTRAINTS AND THE SACHS-WOLFE EFFECT

The observed isotropy of the microwave background is one of the strongest reasons for using a RW model as a zeroth-order approximation to the Universe. Perturbations lead to anisotropies $\delta T/T$ in the background temperature. There is a contribution to δT from adiabatic perturbations and Doppler-type scattering due to peculiar velocities on the last scattering surface.[13–16] A second contribution to δT is the Sachs-Wolfe effect: there is a perturbation to the photon four-momentum as it propagates on the perturbed null geodesic.[3,6,17,18] Here we will compute δT due to the perturbation of the geodesic. Sachs and Wolfe calculated the temperature anistropy for $k=0$, assuming only growing scalar modes were important. We will apply the integral constraints to the $k=0$ case, and see that there is a decrease in the predicted effect. In a separate paper,[1] we will consider $k=1$ as well, so here we will summarize the relevant formalism with greater generality than Ref. 3 (see also Ref. 17).

As in Ref. 3, integrate the null geodesics in the conformally related metric $\bar{g}_{\mu\nu}$,

$$ds^2=a^2d\bar{s}^2=a^2(\bar{g}_{\mu\nu}+\bar{h}_{\mu\nu})dx^\mu dx^\nu\ .$$

A photon is emitted at $E=(\eta_E,\vec{x}_E)$ and received at $R=(\eta_0,\vec{0})$. The photon four-velocity is $k^\mu=(1/a^2)\bar{k}^\mu$ and the four-velocity of the observer is $u^\mu=(1/a)\bar{u}^\mu$.[3] The geodesic equation is

$$\frac{d}{d\omega}[(\bar{g}_{\alpha\beta}+\bar{h}_{\alpha\beta})\bar{k}^\beta]=\tfrac{1}{2}(\bar{g}_{\mu\nu,\alpha}+\bar{h}_{\mu\nu,\alpha})\bar{k}^\mu\bar{k}^\nu \tag{32}$$

and the zeroth-order null geodesics are

$$x^\mu_{(0)}(\omega)=(\eta_0-\omega,\vec{n}\omega)\ ,$$
$$\bar{k}^\mu_{(0)}=(-1,\vec{n}),\quad \vec{n}\cdot\vec{n}=1\ . \tag{33}$$

The first-order geodesic equation is

$$\frac{d}{d\omega}(\bar{g}_{\alpha\beta}\delta\bar{k}^\beta)-g_{\mu\nu,\alpha}\delta\bar{k}^\mu\bar{k}^\nu_{(0)}=-\bar{h}_{\alpha\beta,\nu}\bar{k}^\nu_{(0)}\bar{k}^\beta_{(0)}-2\bar{h}_{\alpha\beta}\frac{d}{d\omega}\bar{k}^\beta_{(0)}+\bar{h}_{\mu\nu,\alpha}\bar{k}^\mu_{(0)}\bar{k}^\nu_{(0)}\ .$$

In the synchronous gauge, substituting (33) gives

$$\delta\bar{k}^0(s)=-\frac{1}{2}\int_0^s\left[\frac{\partial}{\partial\eta}\bar{h}_{ij}n^in^j\right]_{(0)}d\omega\ . \tag{34}$$

The subscript (0) means that the integrand is evaluated on the unperturbed path (33). Now,

$$\frac{T_E}{T_R}=\frac{(k\cdot u)_E}{(k\cdot u)_R}=\frac{a_R}{a_E}[1-\delta\bar{k}^0(\Delta\eta)+n^ju_j(E)-n^ju_j(R)]\ ,$$

where $\Delta\eta=\eta_0-\eta_E$. Substituting the solutions (30) or (26) gives

$$\frac{\delta T}{T}(n)=\frac{\delta T}{T}(\text{grav})+\frac{\delta T}{T}(\text{Dop 1})+\frac{\delta T}{T}(\text{Dop 2})\ ,$$
$$\frac{\delta T}{T}(\text{grav})=\tfrac{1}{10}(A[0]-A[\Delta\eta])\quad(\text{Ref.3})\ ,$$
$$\frac{\delta T}{T}(\text{Dop 1})=\tfrac{1}{10}(\eta_0n^iA_{,i}[0]-\eta_En^iA_{,i}[\Delta\eta])\quad(\text{Ref.3})\ , \tag{35}$$
$$\frac{\delta T}{T}(\text{Dop 2})=\frac{n^ju_j}{a}(\eta_0)-\frac{n^ju_j}{a}(\eta_E)\ .$$

In evaluating the path integral (34), h_{ij} has been approximated by the growing scalar modes, as was discussed in Sec. III. This will be a good approximation when $\eta_E\gg\eta_p$, so that in the range of integration h_{ij} is a late-time perturbation. In the gauge of Sec. I, the scalar velocity field is zero, but in some other coordinate system grad A would be related to the velocity of the observer; therefore (as in Ref. 3) the grad$A$ term has been called a Doppler effect.

The scenario is summarized in Fig. 1. (η_0 and t_0 are understood to label the same spatial hypersurface.) A local perturbation is centered at $\vec{x}_p$. The initial data are given at η_p, and we assume the $p=0$ solutions apply for $\eta>\eta_p$.

A is the gravitational potential in the Poisson equation (30) for $\delta\rho$. If we write the solution for A in a multipole expansion, then the integral constraints (28a) and (28b) imply that the monopole and dipole terms vanish. Therefore,

$$A(\vec{x})\sim\frac{2}{k^3|\vec{x}-\vec{x}_p|^3}(Ha)^2\left|\frac{\delta\rho}{\rho}\right|\ , \tag{36a}$$

whereas without the constraints

$$A(\vec{x})_{\rm wo} \sim \frac{2}{k\,|\vec{x}-\vec{x}_p|}(Ha)^2 \left|\frac{\delta\rho}{\rho}\right| . \tag{36b}$$

If $t_E \ll t_0$, then $|\vec{x}_E-\vec{x}_p| \simeq 2/(H_0 a_0)$, and the anisotropic part of $\delta T_{\rm grav}$ is

$$\begin{aligned}\frac{\delta T}{T}(\vec{n}_E) &= \tfrac{1}{10}A(\vec{x}_E)\\ &\simeq \frac{1}{40}\left|\frac{\delta\rho}{\rho}\right|(t)\left[\frac{Ha}{k}\right]^2(t)\left[\frac{H_0a_0}{k}\right]^3\\ &\simeq 10^{-15}\left|\frac{\delta\rho}{\rho}\right|_0\left[\frac{M}{10^{15}M_\odot}\right]^{5/3} .\end{aligned} \tag{37}$$

The time-dependent quantities can be evaluated at any convenient time t. M is the time-independent mass contained in a sphere of coordinate radius $1/k$, $M=\frac{4}{3}\pi\rho(a/k)^3=(H_0{}^2/2G)(a_0/k)^3$.

The important point is to compare δT to the value one would predict without the constraints

$$\frac{\delta T}{T} \simeq \frac{1}{4}\left[\frac{H_0a_0}{k}\right]^2\frac{\delta T_{\rm wo}}{T} \ll \frac{\delta T_{\rm wo}}{T} , \tag{38}$$

since $(H_0a_0/k) \ll 1$ whenever the multipole expansion is valid. The effect of the "energy-momentum" constraints is to reduce the magnitude of δT.

We note that if the universe has been RW since $t=0$, then causality implies $a(t_p)/k < t_p$ and therefore $(H_0a_0/k)^2 < (1+z_p)^{-1} \simeq 10^{-4}$.

The measurable quantity is correlations in temperature between two points $\vec{x}_1$ and $\vec{x}_2$ separated by an angle θ. Let ξ be the correlation function

$$\begin{aligned}\xi(z) &= \left\langle \frac{\delta\rho}{\rho}(\vec{u}+\vec{z})\frac{\delta\rho}{\rho}(\vec{u})\right\rangle\\ &= \frac{1}{V}\int d^3u\,\frac{\delta\rho}{\rho}(\vec{u}+\vec{z})\frac{\delta\rho}{\rho}(\vec{u}) ,\end{aligned}$$

where V is the coordinate volume.

Consider the case when $\delta\rho$ is the sum of uncorrelated separte lumps, each which is localized and has length scale $1/k$. Then $\xi(z)=0$ for $z \gg 1/k$. This is equivalent to noting that points separated by an angle θ with $\sin(\theta/2) \gg \frac{1}{4}(H_0a_0/k)$ are uncorrelated. Now,

$$\langle (T_1-T_2)^2\rangle = \langle \delta T_1{}^2\rangle + \langle \delta T_2{}^2\rangle + \langle 2\delta T_1\delta T_2\rangle ,$$

if each of these integrals exists. In fact, the integral $\langle \delta T_1 \delta T_2\rangle$ has a divergence which can be subtracted out when the monopole moment of the source vanishes, which is true for the scenario being considered. With repeated use of the constraints (28a) and (28b), and expanding in $(k\,|\vec{x}_1-\vec{x}_2|)^{-1}=\frac{1}{4}(sin\theta/2)^{-1}(H_0a_0/k)$, we find

$$\begin{aligned}\left|\left\langle\frac{2\delta T_1\delta T_2}{T}\right\rangle\right|^{1/2} &\simeq \frac{1}{16}\left[\frac{H_0a_0}{k}\right]^{7/2}\left|\frac{\delta\rho}{\rho}\right|_0\left[\sin\frac{\theta}{2}\right]^{-3/2}\\ &\simeq 10^{-9}\left[\frac{M}{10^{15}M_\odot}\right]^{7/6}\left|\frac{\delta\rho}{\rho}\right|_0\left[\sin\frac{\theta}{2}\right]^{-3/2} .\end{aligned} \tag{39}$$

On the other hand, $\langle (T_1\text{-}T_2)^2\rangle$ is dominated by $\langle \delta T_1{}^2\rangle + \langle \delta T_2{}^2\rangle$. This is the contribution to the variance which is independent of θ:

$$\begin{aligned}\left\langle\frac{(T_1-T_2)^2}{T}\right\rangle^{1/2} &\simeq \frac{1}{2}\left[\frac{H_0a_0}{k}\right]^2\left|\frac{\delta\rho}{\rho}\right|_0\\ &\simeq 5\times 10^{-6}\left[\frac{M}{10^{15}M_\odot}\right]^{2/3}\left|\frac{\delta\rho}{\rho}\right|_0 ,\end{aligned} \tag{40}$$

again valid for $\sin(\theta/2) \gg \frac{1}{4}(H_0a_0/k)$.

Peebles computes $\langle (T_1-T_2)^2\rangle$ in terms of the galaxy correlation function. In this calculation the constraints are not imposed, and $\langle \delta T_1\delta T_2\rangle$ is divergent. In fact, Peebles' leading term depends on the monopole moment of the source, and was zero in our computation (40):[6]

$$\begin{aligned}\left\langle\frac{(T_1-T_2)^2}{T}\right\rangle^{1/2}_{\rm wo} &\simeq 2\times 10^{-5}\left[\sin\frac{\theta}{2}\right]^{1/2}\\ &\quad\times\left[\frac{H_0}{100\ {\rm km\,sec^{-1}\,Mpc^{-1}}}\right]^{3/2}\\ &\quad\times\left[\frac{a_0{}^3\int dr\,r^2\xi(r)}{960\ {\rm Mpc}^3}\right]^{1/2}\\ &\simeq \left[\sin\frac{\theta}{2}\right]^{1/2}\left[\frac{H_0a_0}{k}\right]^{3/2}\left|\frac{\delta\rho}{\rho}\right|_0 .\end{aligned} \tag{41}$$

Again, the effect of the constraints is to decrease the rms temperature fluctuation, and to change the dependence on θ.

As an example, evaluate (40) and (41) at the horizon crossing time for the scale $1/k$. (This would be relevant, for example, if there existed causal fluctuations during a period of inflation.) Then

$$\left\langle\frac{(T_1-T_2)^2}{T}\right\rangle^{1/2} = \frac{1}{2}\left|\frac{\delta\rho}{\rho}\right|(t_H)$$

compared to

$$\left\langle\frac{(T_1-T_2)^2}{T}\right\rangle^{1/2}_{\rm wo} = (1+z_H)^{1/4}\left[\sin\frac{\theta}{2}\right]^{1/2}\left|\frac{\delta\rho}{\rho}\right|(t_H) .$$

VI. PERTURBATIONS WITH NONZERO PRESSURE

So far we have assumed that perturbations are local on some initial-value surface where $p=0$. However, the formation and early evolution of such a peturbation will occur when pressure is important. Here we will show that local initial conditions do result from a pressure fluctuation at earlier times.

Consider a scenario where all perturbation quantities are zero for $t<t_1$, and then there is a pressure impulse at $t=t_1$,

$$\delta p = f(\delta\rho) + F(\vec{x})\delta(t-t_1),\quad t \ge t_1 . \tag{42}$$

f is the equation of state for the perturbation and F is a function which is localized in space.

A δ function in the pressure and hence in the acceleration implies a step in the velocity and in $\delta\rho$, since the overall volume expansion means $p\,dv$ work is being done. Precisely, put (42) in (8) and integrate across t_1 to derive the initial conditions

$$\begin{aligned} &h(\vec{x},t_1)=0\ ,\\ &\dot{h}(\vec{x},t_1)=-24\pi GF\ ,\\ &\delta\rho(\vec{x},t_1)=-3\frac{\dot{a}}{a}F\ ,\\ &a^2(\rho+p)v^i(\vec{x},t_1)=-F_{,i}\ . \end{aligned} \tag{43}$$

Next, linearize the equation of state and assume that the speed of sound $c(t)$ is independent of $\vec{x}$,

$$\delta p=c^2(t)\delta\rho\ .$$

Let

$$y=\dot{h},\ \ \delta=\frac{\delta\rho}{\rho+p},\ \ w^i=\frac{a^2}{c^2}v^i$$

and define the six-vector of unknowns

$$z=(h,y,\delta,w^1,w^2,w^3)\ .$$

Then (8) becomes a sixth-order linear hyperbolic symmetric system

$$L(z)\equiv A^{\mu}z_{,\mu}+Bz=0\ . \tag{44}$$

The A^{μ} are 6×6 symmetric matrices,

$$A^{(0)}=\mathrm{diag}\left[1,\frac{1}{\rho+p},\frac{a^2}{c^2},1,1,1\right]\ ,$$

$$A^{(1)34}=A^{(1)43}=1\ ,$$

$$A^{(2)35}=A^{(2)53}=1\ ,$$

$$A^{(3)36}=A^{(3)63}=1\ ,$$

$$B^{12}=-1,\ \ B^{22}=2\frac{\dot{a}}{a}\frac{1}{\rho+p},\ \ B^{32}=\frac{1}{2}\frac{a^2}{c^2}\ ,$$

$$B^{33}=\frac{a^2}{c^2}\left[\frac{\dot{\rho}+\dot{p}}{\rho+p}+3\frac{\dot{a}}{a}(1+c^2)\right],\ \ B^{23}=8\pi G(1+3c^2)s\ ,$$

$$B^{44}=B^{55}=B^{66}=-2\frac{c}{a}\frac{d}{dt}\left[\frac{a}{c}\right]+\frac{\frac{d}{dt}[a^5(\rho+p)]}{a^5(\rho+p)}\ ,$$

$$A^{(j)ik}=B^{ik}=0\ \text{otherwise}\ .$$

Then the solution z to the system (44) with initial conditions (43) exists, is unique, and is nonzero only within the forward sound cone of the initial disturbance $F(\vec{x})$.[19] (see Fig. 1). The solution at any point P depends only on $L(z)$ inside and on the backward ray cone from P, and the initial data inside the cone.

The sound rays, or bicharacteristics, are the directions along which surfaces of discontinuity, or wave fronts, propagate. The equation which determines the characteristic surfaces $\phi(t,\vec{x})$ for the system (44) is

$$\det A^{(\mu)}\phi_{,\mu}=\phi_{,0}{}^4\left[\phi_{,0}{}^2-\frac{c^2}{a^2}\phi_{,r}{}^2\right]=0\ .$$

There are six roots which decribes six modes of propagation. Two sound-wave modes propagate along the bicharacteristics

$$\frac{dr}{dt}=+\frac{c}{a}(t)\ .$$

The fourth-order root of stationary modes corresponds to two vorticity modes and the two independent coordinate degrees of freedom (6) that are compatible with synchronous gauge.

To recover the $p=0$ case, put $p=c^2=0$. The sound-wave bicharacteristics degenerate to the double root $\phi_{,0}{}^2=0$, which represent the growing and dying modes (12) of $\delta\rho$ found before.

In addition to Eqs. (44) for the unknowns z, there is the equation in (8) for h_{ij}. If z is known, then the right-hand side of this equation is known. So, by considering the homogeneous equation we check that the gravitational waves propagate on the null geodesics

$$\frac{dr}{dt}=\pm\frac{1}{a}\ .$$

In Secs. I—IV, we have examined perturbations on a $p=0$ background, assuming local initial conditions for $\delta\rho$ and h_{ij}. This section shows that such initial conditions will be established by impulsively striking a background with nonzero pressure which was previoulsy smooth.

VII. CONCLUSION

Perturbations in a RW universe cause fluctuations in the temperature of the microwave background. If $\delta\rho$ is the sum of randomly scattered perturbations each which has zero monopole and dipole moment, then the magnitude of $\delta T/T$ is decreased by a factor of order $(H_0a_0/k)^2\ll 1$ compared to the magnitude of $\delta T/T$ when $\delta\rho$ does not obey these conditions. The rms temperature fluctuation is also changed,

$$\frac{\langle (T_1-T_2)^2\rangle^{1/2}}{\langle (T_1-T_2)^2\rangle^{1/2}_{\mathrm{wo}}}\simeq\left[\sin\frac{\theta}{2}\right]^{-1/2}\left[\frac{H_0a_0}{k}\right]^{1/2}\ll 1\ .$$

This means that a perturbation with a given length scale can have a larger amplitude $|\delta\rho/\rho|$ and be compatible with the observational limits on δT.

A disturbance which is created by a causal process is localized in space. Local perturbations in a RW $k=0$ universe must in fact obey certain integral constraints, which in the pressureless case reduce to the statements that the monopole and dipole moments are zero. Therefore the anisotropy in the microwave background is quite different if $\delta\rho$ is a bunch of causal, uncorrelated disturbances or if $\delta\rho$ is a pure plane wave.

Finally, we note that in a Ostriker-Cowie[20]-type model for galaxy formation, there is no anisotopic Sachs-Wolfe effect from scalar perturbations. In this model perturbations are created at late times $z\le 100$. This implies that

$A(\Delta\eta)=0$ in (35) and hence the contribution to $\delta T_1-\delta T_2$ from scalar modes is zero.

ACKNOWLEDGMENTS

Many thanks for the helpful and enjoyable discussions with D. Eardley and R. Brandenberger. This work was supported in part by NSF Grant No. PHY 80-07351 at Harvard University and NSF Grants Nos. PHY 81-07384 and PHY 77-27084 at the University of California at Santa Barbara.

APPENDIX A: Newtonian Peturbations

The solutions for $\delta\equiv\delta\rho/\rho$ in a Newtonian analysis[8] are identical to the solutions for δ in the gauge of Sec. I (synchronous and $v^i_{,i}=0$). Call these $S1$ coordinates. In some other choice of gauge δ will have a different time dependence,[21] but having chosen the $S1$ coordinates, the $t^{2/3}$ and t^{-1} modes represent the physical degrees of freedom.

The Newtonian equations for $p=0$, in comoving coordinates x^i, for $p=0$, are[8]

$$\ddot{\delta}+2\frac{\dot{a}}{a}\dot{\delta}=4\pi G\rho\delta ,\qquad \dot{\delta}+u^i_{,i}=0 , \tag{A1}$$

where

$$u^i=\frac{dx^i}{dt} .$$

The linearized relativistic equations (8) can be rewritten as

$$\ddot{\delta}+2\frac{\dot{a}}{a}\dot{\delta}=4\pi G\rho\delta ,\qquad \dot{\delta}+\frac{\dot{h}}{2}+u^i_{,i}=0 . \tag{A2}$$

Once we pick $S1$ coordinates at t_0, then we stay in this coordinate system. On the other hand, if we tried to make (A1) and (A2) identical by choosing $\dot{h}=0$, this would necessitate a new change of coordinates on each time slice.

In the Newtonian picture, a time rate of change of δ in a region R implies a flux of particles across the boundary of R, and hence a div u term. In the relativistic model, the change of δ inside R implies a change in the proper volume, and so in h. By comparing these we see that the physical identification between the Newtonian model, and the relativistic one in $S1$ gauge, is that the mean-free path Δl of the Newtonian particle in time step Δt equals the change in the radius of R in Δt:

Newtonian (the radius of R is l):

$$-\int_R \dot{\delta}\, dv=\int_{\partial R} d\sigma_i u^i=4\pi l^2\frac{\Delta l}{\Delta t} .$$

Relativistic:

$$h=\frac{3\Delta l}{l} ,$$

$$-\int_R \dot{\delta} dv=\int_R \frac{3}{l}\frac{\Delta l}{\Delta t}=4\pi l^2\frac{\Delta l}{\Delta t} .$$

APPENDIX B: CHECKING THE GREEN'S FUNCTION

Rewrite the solution (22) for D as

$$D(\vec{x},\eta)=\frac{-\eta'}{4\pi}\left[1+\frac{H(\Delta x-\Delta\eta)}{\Delta x}\eta' + H(\Delta x-\Delta\eta)\frac{(\Delta\eta-\Delta x)}{\Delta x}\right] .$$

Then D is the sum of three terms each of which satisfies the wave equation. This is checked easily, using

$$\frac{d}{d\lambda}H(\lambda)=\delta(\lambda)$$

and

$$\lambda\delta(\lambda)=0 .$$

Each of the three solutions yields a term in G, which independently satisfy $\mathcal{O}G=0$.

To verify that the initial conditions are satisfied, one must show, for example

$$\frac{\partial}{\partial\eta}G(\vec{x}\eta' \mid \vec{x}'\eta')=\delta^{(3)}(\vec{x}-\vec{x}') .$$

That is, for a test function $\phi(\vec{x})$

$$\lim_{\eta\to\eta'}\int d^3x'\phi(\vec{x}')\frac{\partial}{\partial\eta}G=\phi(\vec{x}) .$$

From (23),

$$\frac{\partial}{\partial\eta}G(\vec{x}\eta \mid \vec{x}'\eta')=\frac{\eta'}{4\pi}\left[-\frac{3}{\eta^4}H(\Delta\eta-\Delta x) + \frac{1}{\eta^3}\delta(\Delta\eta-\Delta x)\left[\frac{\Delta x-2\eta'}{\Delta x}\right] + \frac{\delta'(\Delta\eta-\Delta x)}{\Delta x}\frac{\eta'^2}{\eta^2}\right] .$$

The third term in this expression gives

$$\begin{aligned}\lim_{\Delta\eta\to 0}\frac{1}{4\pi}\frac{\eta'^2}{\eta^2}\int d\Omega\, u^2du\, \phi(\vec{u}+\vec{x})\frac{\delta'(\Delta\eta-u)}{u} &=\lim\frac{1}{4\pi}\frac{\eta'^2}{\eta^2}\int d\Omega\, du\, \delta(\Delta\eta-u)\frac{\partial}{\partial u}(u\phi)\\ &=\lim\frac{1}{4\pi}\frac{\eta'^2}{\eta^2}\int d\Omega\, \phi(\vec{u}+\vec{x})\big|_{u=\Delta\eta}\\ &=\phi(\vec{x}) .\end{aligned}$$

The second term is treated similarly, and yields zero. The first term goes to zero as η approaches η'.

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