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Thermodynamics of quark jets. I

Bindu Bambah
The Enrico Fermi Institute and the Department of Physics, University of Chicago, Chicago, Illinois 60637
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A model for quark jets based on the thermodynamics of a one-dimensional quark gas in a grand canonical ensemble is presented. The model makes predictions about such properties of jets as multiplicity and Koba-Nielsen-Olesen (KNO) scaling. A generalization to higher dimensions is outlined. KNO scaling in the statistical description of many-particle phenomena is also examined.

I. INTRODUCTION

Models for jet evolution fall into two categories—those based on parton fragmentation and QCD,[1] and those based on naive confinement models.[2] The confinement and the QCD picture apply to different stages of the jet evolution processes—QCD describes the evolution of fast partons and the naive confinement models address the question of the transformation of the on-shell partons into hadrons. The main problem in jet-evolution physics seems to be the following: how do the two different approaches join smoothly? Phenomenological models have tried to answer this question,[3] but the theoretical answer to this question seems to be equivalent to solving the problem of confinement.

In this paper we adopt a different approach to high-energy jet phenomena, hoping that by following two separate chains of thought we will find some point of intersection which should approximate the truth. Unlike jet parton models, the fundamental quantitities in this model are not quarks, partons, Regge poles, etc., but averaged thermodynamic quantities such as energy, entropy, etc. Such an approach to multiparticle production is not new; in fact it goes all the way back to Fermi and Landau,[4] since almost by definition, statistical mechanics is concerned with forecasting macroscopic behavior of physical systems given only partial information about their internal states. What is different about our model is that we start with an exact Hamiltonian[5] for the quarks in $1+1$ dimensions, which is derived from QCD in the Coulomb gauge. Thus we may get information about the multiplicity of hadrons in jets without any phenomenological inputs (adjustable parameters) except quark masses and string tension. The quarks are treated as classical particles, but their interactions through SU(N) gauge fields are treated exactly.

We also examine which statistical ensembles are reasonable in dealing with strongly interacting systems that show clustering. It is found, in particular, that the pressure-fixed grand canonical ensembles give correct Koba-Nielsen-Olesen (KNO) scaling. The details of these ensembles and the numerical work are shown in an accompanying paper.[6]

The basic physical situation that this model describes is the following: The e^+e^- ($p\bar{p}$) annihilate and produce a $q\bar{q}$ pair held together by a string of color flux, the interaction energy being given by Hamiltonian H [Eq. (2.1)]. The collision provides enough energy to create $q\bar{q}$ pairs. This results in the breaking of the initial string. Each new string represents a cluster of quarks in a singlet state. Our *a priori* postulate is that this system, which behaves like a one-dimensional gas, is not initially in equilibrium. However, the interactions are so strong that energy is quickly distributed among the various degrees of freedom according to statistical laws. This process takes place until entropy is maximized. Detailed arguments for the determination of the volume at which adiabatic expansion ensues are given in Sec. III. All these postulates will be justified *a posteriori*.

The next stage of the process is adiabatic, isentropic expansion. We assume that at each stage of the expansion, there is local equilibrium (in effect, like hydrodynamic expansion). As the system expands, breakup into clusters (hadrons) takes place. This behavior is contained in the equation of state derived from H, the exact Hamiltonian we derived earlier.[5(a)] We calculate $S(E)$ as a function of center-of-mass energy and relate it to the multiplicity of particles. The experimental sphericity[7,8] distribution indicates that the expansion is longer along the collision axis than in the transverse direction and if we do not consider vibrations of the string, the one-dimensional approach is justified.

In the next few sections we set up the thermodynamics of a one-dimensional quark gas, given an exact Hamiltonian, and obtain physical relevant quantities. We consider the volume-fixed ensemble and obtain a power-law behavior for the multiplicity.[9] The pressure-fixed ensemble is considered briefly to show KNO scaling, but is dealt with in more detail in an accompanying paper.[6]

II. THERMODYNAMICS OF A ONE-DIMENSIONAL GAS

We start with the Hamiltonian

$$H=-\sum_{k>l}^{1,n}\frac{\alpha}{4}\lambda_i^k\cdot\lambda_j^l\,|x_k-x_l| \tag{2.1}$$

which was derived in earlier papers.[5] The λ_i's are the SU(N) matrices appropriate to a quark and antiquark representation; $\alpha=g^2/2=$string tension. This Hamiltonian

gives the exact expectation values of entangled loop operators and their products in SU(N) gauge theories. The grand canonical partition function for this system can be calculated. For a volume-fixed ensemble we use the transfer-matrix method in the following way: to diagonalize H, one considers a well-ordered sector $x_1<x_2<\cdots<x_n$ and combines the quarks successively from left to right into a sequence of irreducible representations, the last one being an SU(N) singlet. Each such system defines an eigenstate of H with eigenvalues

$$E=\alpha\sum_{i=1}^{n-1}C_i\,|x_{i+1}-x_i|+\sum_{i=1}^{n}(m^2+k_i^2)^{1/2}\,;$$

C_i = Casimir operator of the ith representation ,

m = mass of the quark , (2.2)

k_i = momentum of the ith quark .

We use this Hamiltonian to study the thermodynamics of the quark gas in a volume-fixed ensemble. For this we employ the transfer-matrix formalism. The transfer matrix enables us to build an irreducible representation of $n+1$ particles from an n particle state, i.e.,

$$\psi_{n+1}=M_n\psi_n\ . \tag{2.3}$$

The Hilbert space for ψ is the space of all irreducible representations of SU(N). In a volume-fixed ensemble the transfer matrix can be defined as follows:

$$\psi_{n+1}(x)=\int_0^x zt\,e^{-\beta C(x-y)}\psi_n dy\,;\ z=2mK_1(m\beta)\ . \tag{2.4}$$

z is derived from the kinetic part of the Hamiltonian and is the single particle kinetic partition function. It depends on the flavor of the particles through the mass. The action of t on the vectors ψ_n of the Hilbert space is defined to be

$\langle\psi'\,|\,t\,|\,\psi\rangle=1$, if ψ' can be formed from ψ with the addition of a fundamental representation (quark or antiquark) of SU(N)

$=0$, otherwise .

By defining the fugacity ξ such that $\psi_{n-1}=\xi\psi_n$ Eq. (2.4) becomes

$$\left[\frac{d}{dr}+\beta\alpha C-\xi zt\right]\psi_n=0\ . \tag{2.5}$$

The equation for the transfer matrix, i.e., (2.5) can be mapped onto a differential equation by associating with each ψ_n an antisymmetric function $g(y)$ such that

$$\psi=\sum C_n\psi_n \text{ is associated with } g(y)$$

$$=\sum\widetilde{C}_{n'}y_1^{\,n_1'}y_2^{\,n_2'}\cdots y_N^{\,n_N'}\ , \tag{2.6}$$

where n_i' label the irreducible representation of SU(N) which corresponds to a Young tableau with N rows. $n_1'>n_2'\cdots>n_N'$. The Casimir operator C for SU(N) is then given by the symmetric quadratic form

$$C=\tfrac{1}{2}\sum n_i'^{\,2}-\frac{1}{2N}\left[\sum n_i'\right]^2-\frac{N(N^2-1)}{24}\ . \tag{2.7}$$

C_n' is the natural antisymmetric extension of C_n. Further, we define

$$f(y)\prod_{i<j}(y_i-y_j)=g(y)\ . \tag{2.8}$$

The effect of t and C on ψ_n induces an effect on f and g given by the following: t induces a multiplication of f by

$$t\to y_1+y_2+\cdots y_N$$

for addition of a quark ,

$$\to\frac{1}{y_1}+\frac{1}{y_2}+\cdots\frac{1}{y_N}$$

for addition of an antiquark . (2.9)

The action of C on g is

$$n_i'\to y_i\frac{\partial}{\partial y_i}\ . \tag{2.10}$$

By these prescriptions Eq. (2.5) gets mapped to a differential equation for $g(y)$. We carry out a Fourier decomposition of g by associating $y_i=e^{ix_i}$, x_i real, and we get a differential equation reminiscent of the Schrödinger wave equation. The Casimir operator and the branching operator play the roles of the kinetic and potential energies of the system, respectively, in $N-1$ dimensions with the volume r corresponding to imaginary time. The condition for the positivity of the Casimir operator translates into an antisymmetry condition on the solutions of this equation. The Green's function

$$e^{-\beta Jr}\equiv\left\langle q\left|\left[\frac{d}{dr}+\beta\alpha C-\xi zt\right]^{-1}\right|q\right\rangle$$

$$=\sum_n|\langle n\,|\,q\rangle|^2e^{-E_nr} \tag{2.11}$$

may be calculated to give the grand partition function for the system where $|q\rangle$ is a one quark or one antiquark state. For large r only the smallest eigenvalue contributes

and (2.11) reduces to $|\langle n=1|q\rangle|^2 e^{-\lambda_1 r}=\mathrm{tr}G$. For small r, all eigenvalues contribute. Then $\partial\langle\beta J\rangle/\partial r=\langle\lambda\rangle$. We calculate $\langle q|G|q\rangle$ for SU(2) and SU(3) in the following limits: (a) The strong-coupling ($\xi z/\beta\alpha\gg 1$) limit (high-temperature limit) and (b) the semiclassical (WKB) limit ($2\xi z r>1$).

For SU(2), the equation for ψ_n corresponding to Eq. (2.5) is

$$\frac{d}{dr}\psi_n+\beta\alpha\left[\frac{n(n+2)}{4}\right]\psi_n-\xi z t\psi_n=0\ , \qquad C=\frac{n(n+2)}{4}\ , \tag{2.12}$$

where n labels the irreducible representation (IR) of SU(2) with isospin $I=n/2$. Since t induces transitions from n to $n\pm1$, if one associates a function $e^{2i(n+1)x}$ to each ψ_n we have

$$4\frac{d}{dr}\psi_n-\beta\alpha\left[\frac{\partial^2}{\partial x^2}+1\right]\psi_n-8\xi z\cos 2x\,\psi_n=0\ . \tag{2.13}$$

We obtain the Green's function by solving Eq. (2.11), which for SU(2) is

$$\frac{d}{dr}G_0-\frac{\beta\alpha}{4}\frac{\partial^2 G_0}{\partial x^2}+2\xi z(1-\cos 2x)G_0=\delta(r)\delta(x-x')\ , \tag{2.14}$$

where

$$G(x,x';r)=\left\langle x\left|\exp\left[\left[2\xi z+\frac{\beta\alpha}{4}\right]r\right]G_0\right|x'\right\rangle\ .$$

Since the coefficient function is non-negative, this equation can be viewed as a diffusion problem with absorption. For $2\xi z/\beta\alpha\gg 1$, we can approximate Eq. (2.14) by a time-dependent harmonic oscillator with a level spacing (frequency) $\omega=(\xi z\beta\alpha)^{1/2}$, the linear dimension r corresponds to imaginary time:

$$\frac{d}{dr}G_0-\frac{\beta\alpha}{4}\frac{\partial^2 G_0}{\partial x^2}+2\xi z x^2 G_0=\delta(r)\delta(x-x')\ . \tag{2.15}$$

Examining the case when we impose the boundary condition $n>0$ in consistency with the Casimir operators $C>0$.

We seek the Green's function which has the following properties:

$$G(x,x';r)=-G(-x,x';r)=-G(x,-x';r)\ . \tag{2.16}$$

This ensures $n>0$:

$$G(x,x';r)=\exp\left[\left[2\xi z+\frac{\beta\alpha}{4}\right]r\right]\left[\frac{4\xi z}{2\pi\beta\alpha\sinh 2t}\right]^{1/2}\left\{\exp\left[-\frac{4\xi z}{\beta\alpha\sinh 2t}[(x^2+x'^2)\cosh 2t-2xx']\right]-\exp\left[-\frac{4\xi z}{\alpha\beta\sinh 2t}[(x^2+x'^2)\cosh 2t+2xx']\right]\right\} \tag{2.17}$$

which is simply the Green's function for the time-dependent oscillator with the boundary conditions (2.16). Then

$$\langle q|G(x,x';r)|q\rangle=\frac{1}{\pi}\int_0^{2\pi}\int\sin 2x\sin 2x' G(x,x';r)\,dx\,dx'=C\exp\left[\left[2\xi z+\beta\frac{\alpha}{4}\right]\right]e^{-2\omega r}\left[\frac{\beta\alpha}{\xi z}\right]^{1/2}(\sinh\omega r)^{-3/2}\ . \tag{2.18}$$

We get the following thermodynamic functions:

$$\beta J=-\left[2\xi z-\frac{3\omega}{2}\right]r+\ln\left[\frac{\xi z}{\beta\alpha}\right]^{1/2}+\tfrac{3}{2}\ln[1-\exp(-2\omega r)]\ , \tag{2.19}$$

$$\beta P=-\frac{\partial J}{\partial r}=2\xi z-\frac{3\omega}{2}-\frac{3\omega}{(e^{2\omega r}-1)}\ , \tag{2.20}$$

$$\beta E=2\xi z r\ , \tag{2.21}$$

$$S=-\xi\frac{\partial J}{\partial T}=4\xi z r-\ln\frac{\xi z}{\beta\alpha}-\tfrac{3}{2}\ln(1-e^{-2\omega r})\ , \tag{2.22}$$

$$N=-\xi\frac{\partial J}{\partial\xi}=(2\xi z-\tfrac{3}{4}\omega)r-\tfrac{3}{2}\ln(1-e^{-2\omega r})\ , \tag{2.23}$$

$z\to T$ at high T. If S is expressed in terms of E and

$$S=2\sqrt{2E\xi r}-\ln\frac{E}{\alpha r}-\tfrac{3}{2}\ln(1-e^{-2\omega r})\ . \tag{2.24}$$

The crucial step in this calculation is the determination of the volume r at the stage of the expansion where adiabaticity sets in. By definition, in an adiabatic process the system is subject to external conditions which vary slowly in time. In this model the Hamiltonian $H(r)$ is a function of the volume r which plays the role of (imaginary) time. We suppose the Hamiltonian changes from a certain eigenvalue of H_0 at volume r_0 to an eigenvalue H_1 at a volume r_1. This change is due to string splitting and creation and annihilation of quarks. Let $R=r_1-r_0$ and $s=(r-r_0)/R$. Denote $H(s)$ the value taken by the Hamiltonian at $r=r_0+sR$. $H(s)$ is a continuous function of s and describes the evolution of the system from r_0 to r_1. It depends only on the parameter R measuring the evolution from H_0 to H_1. *The adiabatic theorem* tells us that in the

limit $R\to\infty$, in the case of slow adiabatic passage, the system initially in a stationary state of H_0 will at a later "time" r_1 have passed on to a stationary state of H_1, that derives from it by a continuous transformation. In our system the free energy J plays the role of H and the condition for adiabaticity set by the adiabatic theorem translates into the statement that the continuous change in the eigenvalue of the free energy with respect to T does not cause an abrupt transition in the system. Since ω corresponds to the level spacing we have

$$\left.\frac{\partial\langle\lambda\rangle}{\partial T}\right|_{r_0}\leq\omega r\,,\quad T\sim s\,, \tag{2.25}$$

which translates to

$$\omega r\geq 1\,;\ \text{since}\ \frac{\partial\beta J}{\partial r}=\langle\lambda\rangle \tag{2.26}$$

so the transition point is

$$\omega r_0\sim 1$$

so

$$r_0\sim\frac{1}{\omega}\sim\frac{1}{(\xi z\beta\alpha)^{1/2}}$$

$$\sim\text{hadronic size}\ (\sim\text{independent of}\ T)\,. \tag{2.27}$$

This implies that within a volume of the order of hadron size, the string splitting takes place until the system reaches equilibrium; thereupon adiabatic expansion ensues. The entropy remains constant and is given by

$$S_0\cong 4\xi z r_0-\tfrac{1}{2}\ln\frac{\xi z}{\beta\alpha}\sim 2\sqrt{2E\xi r_0}-\ln\frac{E}{2\alpha r_0}\,. \tag{2.28}$$

Since adiabaticity implies isentropic expansion with S_0 fixed, we have

$$E\sim\frac{{S_0}^2}{8\xi r}\,, \tag{2.29}$$

$$T\sim\frac{S_0}{4\xi r}\,. \tag{2.30}$$

For large r

$$N\sim\tfrac{1}{2}S_0+\ln\frac{\xi z}{\beta\alpha} \tag{2.31}$$

when cluster formation takes place, $T\sim\sqrt{\alpha}$,

$$N_c\sim\tfrac{1}{2}S_0\sim\sqrt{2}(E_0\sqrt{\xi/\alpha})^{1/2}-\tfrac{1}{2}\ln(E_0\sqrt{\xi/\alpha})\,. \tag{2.32}$$

Thus a power law for multiplicities is obtained.[9] In general, however, the harmonic approximation is not strictly valid. It is the high-temperature approximation. For corrections to this approximation we examine the WKB (semiclassical) limit.

III. SEMICLASSICAL (WKB) APPROXIMATION

For intermediate temperatures we can use the semiclassical approximation. Since the strong-coupling approximation is a first-order approximation for high temperature, another approximation might enable us to get a more comprehensive picture at intermediate temperatures. The WKB limit applies when

$$(2\beta\alpha\xi z)^{1/2}\left[\frac{2\xi z}{\beta\alpha}\right]^{1/2}r\gg 1\,,$$

$$2\xi z r\gg 1\,,$$

or for $\xi=1$, $2zr\gg 1$.

We start with the Hamiltonian

$$H=\frac{\beta\alpha}{4}[(n+1)^2-1]-2\xi z\cos x\,. \tag{3.1}$$

Since $(n+1)\to -i\,\partial/\partial x$, the Hamiltonian is

$$H=\frac{1}{2m}p^2-\frac{\beta\alpha}{4}-2\xi z\cos x\,,$$

correspondingly,

$$L=-\frac{m\dot{x}^2}{2}+2\xi z\cos x+\frac{\beta\alpha}{4}\,,\qquad \dot{x}=\frac{dx}{dr}\,. \tag{3.2}$$

The partition function

$$G(r)=\int e^{I(r)}D(\text{path})=\langle q\mid e^{-Hr}\mid q\rangle\,,$$
$$I(r)=I(\epsilon)=\int L\,dr\,, \tag{3.3}$$

which in the WKB limit we approximate by a classical path from $n=1$ to $n=1$. We also impose the antisymmetry condition on G. To do this, we evaluate $I(\epsilon)=\int L\,dr$ at the turning points $p_0=\epsilon$, antisymmetrize and set $\epsilon=1$. The antisymmetrization corresponds to

$$G=\frac{1}{[S'(\epsilon)]^{1/2}}e^{I(\epsilon)}-\frac{1}{[S'(-\epsilon)]^{1/2}}e^{I(-\epsilon)}$$

$$\sim e^{I(0)}\times f(\epsilon)\,. \tag{3.4}$$

In the WKB limit with fixed final and initial velocities we have the stationary points determined by

$$\ddot{x}-\omega^2\sin x=0\,,\quad \omega^2=\beta\alpha\xi z\,. \tag{3.5}$$

With $\dot{x}_0=0$, $\dot{x}_1=0$, there are three solutions:

$$\begin{aligned}&(1)\ x=\dot{x}=0\,,\\ &(2) x=\pi\,;\ \dot{x}=0\,,\\ &(3)\ \text{pendulum(oscillating) motion}\,.\end{aligned} \tag{3.6}$$

Each of these orbits contribute to $G(r)$. We do not, however, add up all three contributions. Solution (1) corresponds to a static solution which is trivially periodic for any period r. For $2\xi zr\gg 1$, the dominant contribution is from this solution and we have

$$G(r)=\frac{e^{2\xi zr}}{(\sinh\omega r)^{1/2}}\ , \tag{3.7}$$

which for $2\xi zr$ large gives the same results as Eq. (2.11) in the strong-coupling case. Nonetheless it should be noted that the derivation is qualitatively different. Although we drop higher powers of the fluctuations in path space about the classical path, the classical path itself is determined by

$$V(x)=-2\xi z\cos x-\frac{\beta\alpha}{4}$$

in its entirety.

In the region $2\xi zr\gg 1$ it is not necessary to add the contribution of the nontrivial orbit. The nontrivial orbit (3) contribution is given by

$$G_3(r)=e^{S(x_c(r))}\Delta_1(r)\ . \tag{3.8}$$

$x_c(r)$ corresponds to the periodic orbit. $\Delta_1(r)$ is the "quantum correction" factor, given by

$$\Delta_1(t)=\int \mathscr{D}y(r)\exp\left\{-\int_0^r\left[\left[\frac{1}{\beta\alpha}\dot{y}(r)^2+2\xi z\cos y(r)y(r)^2\right]\right]dr\right\}, \tag{3.9}$$

$y(r)$=perturbation around the classical path .

r is the period of the orbit given by

$$\begin{aligned}\dot{x}&=\left[\frac{\beta\alpha}{4}\right]^{1/2}(E-2\xi z\cos x)^{1/2}\ ,\\ r&=\int_0^{\phi_0}\frac{2dx}{\sqrt{\beta\alpha}(E-2z\xi\cos x)}\\ &=\frac{1}{\sqrt{\beta\alpha\xi z}}2\sqrt{2}K\left[2\left[1+\frac{E}{2\xi z}\right]^{1/2}\right],\end{aligned} \tag{3.10}$$

where K=elliptic integral of the second kind

$$E=-2\xi z\cos\phi_0\ . \tag{3.11}$$

We now have

$$G_3(r)=\frac{e^{S(x_c(r))}}{\dot{x}(0)}\left|\frac{dE}{dr}\right|^{1/2}\ . \tag{3.12}$$

For the particular case $\partial G/\partial r\mid_{r=r_0}=0$ we have

$$r_0=\frac{1}{\sqrt{\beta\alpha\xi z}}2\sqrt{2}K(2)=\frac{3.708}{\sqrt{\beta\alpha\xi z}}\ ;\ E=0 \tag{3.13}$$

and

$$G(r)=e^{\beta\alpha r/4}\frac{I_0(2\xi zr)}{\left[\omega r+\left[\frac{\beta\alpha}{2\xi z}\right]^{1/2}\right]^{3/2}}\ , \tag{3.14}$$

$$J(r)=-T\left\{\frac{\beta\alpha r}{4}+\ln I_0(2\xi zr)-\tfrac{3}{2}\ln\left[\omega r+\left[\frac{\beta\alpha}{2\xi z}\right]^{1/2}\right]\right\}. \tag{3.15}$$

Since the WKB limit is valid when

$$(2\beta\alpha\xi z)^{1/2}\left[\frac{2\xi z}{\beta\alpha}\right]^{1/2}r\gg 1$$

or

$$2\xi zr\gg 1\ ,$$

this also gives

$$N\sim CE^{1/2}-\tfrac{3}{2}\ln E\ . \tag{3.16}$$

So the power law for multiplicities is reproduced by the WKB approximation.

IV. RESULTS FOR SU(3)

The equation for $\psi_{n,m}$ is

$$\frac{d}{dr}\psi_{n,m}+\frac{\beta\alpha}{3}[(n+1)^2+(m+1)^2+(m+1)(n+1)-3]\psi_{n,m}-\xi zt\psi_{n,m}=0\cdots\ , \tag{4.1}$$

where t induces transitions from (m,n) to $(m+1,n)$, $(m,n+1)$, and $(m\pm1,n\pm1)$. As in the case for SU(2) we associate the function $\psi_{m,n}=\eta e^{i(m+1)u-i(n+1)v}$ to each IR (m,n) so that the condition (2.5) is compatible with

$$\frac{d}{dr}\eta-\frac{\beta\alpha}{3}\left[\frac{\partial^2}{\partial u^2}-\frac{\partial}{\partial u}\frac{\partial}{\partial v}+\frac{\partial^2}{\partial v^2}-3\right]\eta-2\xi z[\cos u+\cos v+\cos(u+v)]\eta=0\ . \tag{4.2}$$

Under the condition $C_{m,n}>0$; $m,n>0$, we have the boundary condition that η is antisymmetric under the symmetry group S_3 whose elements are $u,v,-u-v$, corresponding to Weyl reflection. By making the change of variables $u,v\to x,y$; $x=u+v, y=u-v$ we have

$$\frac{d}{dr}\eta-\frac{\beta\alpha}{3}\left[\frac{\partial^2}{\partial x^2}+\frac{3\partial^2}{\partial y^2}-3\right]\eta-2\xi z\left[\cos\left[\frac{x+y}{2}\right]+\cos\frac{(x-y)}{2}+\cos x\right]\eta=0\ . \tag{4.3}$$

The equation for the Green's function is

$$\frac{d}{dr}G-\frac{\beta\alpha}{3}\left[\frac{\partial^2}{\partial x^2}+3\frac{\partial^2}{\partial y^2}-3\right]G-2\xi z\left[\cos\frac{(x+y)}{2}+\cos\frac{(x-y)}{2}+\cos x\right]G=0\ . \tag{4.4}$$

We calculate G in limiting cases.

First, we consider the strong-coupling limit $\xi z\gg 1$. The Green's function with the appropriate symmetry under S_3 elements $[(x+y)/2,(x-y)/2,-x]$

$$G(r)=\frac{e^{(6\xi z+\beta\alpha)r}}{2(\sinh t)^{5/2}}\left[\frac{1}{2}-\frac{2}{3\coth^2 t+1}\right],\qquad t=(2\xi z\beta\alpha)^{1/2}r\ . \tag{4.5}$$

The antisymmetry under the elements of the Weyl symmetry group ensure that forbidden transitions are canceled; i.e., $\psi_{n,m}=0$ for values of n and m such that $C_{m,n}<0$:

$$J=6\xi zrT-\tfrac{7}{2}\ln\sinh t-\ln(3\coth^2 t+1)\ . \tag{4.6}$$

The general expressions for energy and entropy are

$$S=(6\xi z+\beta\alpha)r-\tfrac{7}{2}\ln\sinh(2\xi z\beta\alpha)^{1/2}r-\ln[3\coth^2(\xi z\beta\alpha)^{1/2}r+1]\ ,\qquad E\sim 6\xi zTr\ . \tag{4.7}$$

The condition $\omega r=1$ gives

$$S=(6\xi z+\beta\alpha)r\ , \tag{4.8}$$

$$E\sim\frac{S^2}{12\xi r}\ , \tag{4.9}$$

$$N\sim\sqrt{2}(E_0\sqrt{\xi/\alpha})^{1/2}-\tfrac{7}{2}\ln(E_0\sqrt{\xi/\alpha})\ . \tag{4.10}$$

The results are similar to those of SU(2). However detailed analysis of the clustering process shows that both baryons and mesons[6] are formed. On the average the number of mesons is much larger than the number of baryons, hence in the gross analysis of the multiplicities the effect does not show up.

In the WKB limit for SU(3),

$$G(r)=\frac{e^{(6\xi zr+\beta\alpha r)}}{(6\xi zr)\left[\omega r+\left[\frac{\beta\alpha}{\xi z}\right]\right]^{7/2}}\ , \tag{4.11}$$

$$J=-T(\beta\alpha r)+6\xi zr-\ln 6\xi zr-\tfrac{7}{2}\ln\left[\frac{\beta\alpha}{\xi z}\right]-\tfrac{7}{2}\ln\left[1+\frac{1}{2\xi zr}\right]\ . \tag{4.12}$$

The condition $2\xi zr\gg 1$ gives for

$$S\sim(6\xi z+\beta\alpha)r \quad \text{as in the strong-coupling case}\ , \tag{4.13}$$

$$N\sim\sqrt{2}(E_0\sqrt{\xi/\alpha})^{1/2}-\tfrac{7}{2}\ln(E_0\sqrt{\xi/2})\ . \tag{4.14}$$

The leading-order term is unaffected by the group structure.

The possibility of including gluons may be realized by treating them as classical particles belonging to higher-dimensional (adjoint) representations of SU(N). Since we have seen that the overall multiplicity of the hadrons is unaffected by group structure, we will explicitly examine the effect of allowing particles assigned to higher-dimensional representations of SU(2).

V. GENERALIZATION TO n FLAVORS AND HIGHER REPRESENTATIONS

If we do not restrict ourselves to a single flavor, but allow the quarks to have one of n flavors, the expression (2.4) changes to

$$\psi_n(r)=\int_0^r e^{-\beta\alpha C(x-y)}\left[\sum_{i=1}^{n_f}\xi_i z_i t_i\right]\psi_{n-1}(y)\ , \tag{5.1}$$

t_i is the branching operator for each flavor i of the quark. Assume the quarks of various flavors have the masses $m_1,m_2,\ldots,m_n$. The system behaves like a mixture of gases, where z_i represents the kinetic-energy contribution due to each of the flavors and

$$\langle \mathrm{IR}(1)\,|\,t\,|\,\mathrm{IR}(2)\rangle=1\ ,\ \text{if IR(2) can be formed from IR(1) by addition of a quark, antiquark, or a gluon } (I=1) \text{ or } I=\tfrac{3}{2}\ ,$$
$$=0\ ,\ \text{otherwise}\ . \tag{5.2}$$

$d=2I+1$ is the dimensionality of the representation. Suppose each of the flavors have the same fugacity but ξ_d represents the fugacity of the various representations d. For example, restricting ourselves to $I=\frac{1}{2}$, $I=1$, and $I=\frac{3}{2}$ we have

$$\frac{d}{dr}\psi_n-\frac{\beta\alpha}{4}\left[\frac{\partial^2}{\partial x^2}+1\right]\psi_n-\left\{2\sum_{i=1}^{n_f}z_i[\xi_{1/2}\cos\theta+\xi_1(\cos(2\theta)+1)+\xi_{3/2}(\cos3\theta+\cos\theta)]\right\}\psi_n=0\ . \tag{5.3}$$

In the strong-coupling limit we can solve for the thermodynamic functions to get

$$J=-\left[2\sum_{i=1}^{n_f}z_i\xi_{1/2}+4\sum_{i=1}^{n_f}z_i\xi_1+4\sum_{i=1}^{n_f}z_i\xi_{3/2}-\tfrac{3}{2}\omega\right]r+\frac{1}{2}\ln\left\{\frac{\left[\sum_{i=1}^{n_f}z_i(\xi_{1/2}+2\xi_1+2\xi_{3/2})\right]}{\beta\alpha}\right\}+\tfrac{3}{2}\ln(1-e^{-2\omega r})\ , \tag{5.4}$$

$$\beta E=\left[2\sum_{i=1}^{n_f}z_i\xi_{1/2}+4\sum_{i=1}^{n_f}z_i\xi_1+4\sum_{i=1}^{n_f}z_i\xi_{3/2}\right]r\ , \tag{5.5}$$

$$S=2\left[2\sum_{i=1}^{n_f}z_i\xi_{1/2}+4\sum_{i=1}^{n_f}z_i\xi_1+4\sum_{i=1}^{n_f}z_i\xi_{3/2}\right]r-\frac{1}{2}\ln\left[\sum_{i=1}^{n_f}\frac{z_i(\xi_{1/2}+2\xi_1+2\xi_{3/2})}{\beta\alpha}\right]-\tfrac{3}{2}\ln(1-e^{-\omega r})\ , \tag{5.6}$$

$$\omega=\left[\sum_{i=1}^{n_f}z_i(\xi_{1/2}+2\xi_1+2\xi_{3/2})\beta\alpha\right]^{1/2}\ , \tag{5.7}$$

$$N\approx\left[2\sum_{i=1}^{n_f}z_i\xi_{1/2}+4\sum_{i=1}^{n_f}z_i\xi_1+4\sum_{i=1}^{n_f}z_i\xi_{3/2}\right]r\ . \tag{5.8}$$

For large T all the flavors are degenerate because $z_i\to T$ independent of m_i. Thus

$$\beta E=n_fTr(\xi_{1/2}+2\xi_1+2\xi_{3/2})\ , \tag{5.9}$$

$$S=2[2E_n(\xi_{1/2}+2\xi_1+2\xi_{3/2})rn_f]^{1/2}-\ln\frac{E}{2\alpha r}-\frac{3}{2}\ln(1-e-2\omega r)\ . \tag{5.10}$$

The stage of adiabatic expansion sets in at

$$r\sim\frac{1}{\omega}\sim\frac{1}{[\alpha n_f(\xi_{1/2}+2\xi_1+2\xi_{3/2})]^{1/2}} \tag{5.11}$$

so that

$$N_c\sim\tfrac{1}{2}S_0\sim\sqrt{2}\left\{\frac{E_0}{\alpha^{1/2}}[(\xi_{1/2}+2\xi_1+2\xi_{3/2})n_f]^{1/2}\right\}^{1/2}\ , \tag{5.12}$$

so that in the average multiplicity allowing n_f flavors and higher-order representations simply means modifying the fugacity ξ to $(\xi_{1/2}+2\xi_1+2\xi_{3/2})n_f$ in the strong-coupling limit. (This fugacity is an interesting feature of quark-gas ensembles and will be commented upon in Ref. 6.) Detailed comparison of this formula with experiment will be done in Sec. VII.

VI. KNO SCALING IN THERMODYNAMIC ENSEMBLES

In the last few sections we have applied the transfer-matrix method to a volume-fixed ensemble. This enabled us to determine the thermodynamic functions of the volume r. Alternatively, we could have applied this method to a pressure-fixed grand canonical ensemble. This is considered in detail in the accompanying paper.[6] For our present purposes we need the volume-fixed partition function given by

$$Z_V\sim Ae^{+a\xi zr};\ a \text{ is a constant}\ . \tag{6.1}$$

Fixing the pressure-fixed partition function is equivalent to solving $(\alpha C+p-\lambda^{-1}t)\psi=0$. In the thermodynamic limit $(Z_p)^{(n)}\sim(zT\lambda_0)^n$.[5] So the grand canonical pressure partition function is given by

$$\sum_{n=1}^{\infty}\xi^nZ_p^{(n)}\sim\frac{1}{(-\xi za+\beta p)}\ , \tag{6.2}$$

$\lambda_0=$lowest eigenvalue of (2.5)

in the thermodynamic limit.

The quantity of interest in multiplicity measurement in jets is the deviation of the multiplicity from the mean value, i.e., the mean-square fluctuations of additive quantities, i.e., volume and entropy (thus, multiplicity).

If we consider the mean-square fluctuation in the number of particles in the volume- and pressure-fixed ensembles, we get different results. In the volume-fixed ensembles, the fluctuations in the average number of particles form a Gaussian distribution

$$\bar{N}\cong-\xi\frac{\partial-\ln e^{+a\xi zr}}{\partial\xi}\bigg|_{\text{vol fixed}}=a\xi zr=-\xi\frac{\partial J}{\partial\xi}\ ,$$
$$(\Delta\bar{N})^2\,|_v=\xi\frac{\partial\bar{N}}{\partial\xi}\bigg|_v=a\xi zr\sim\bar{N}\ . \tag{6.3}$$

Thus

$$(\Delta\bar{N})\,|_v=(\bar{N}^2-\bar{N}^2)^{1/2}\sim\sqrt{\bar{N}}\ ,$$

which is the standard statistical law of fluctuations for Gaussian distributions. In the pressure-fixed ensemble, however,

$$\bar{N}\cong-\xi\frac{\partial-\ln\dfrac{1}{(-a\xi z+\beta p)}}{\partial\xi}\bigg|_p=\frac{\xi za}{(-a\xi z+\beta p)}\ , \tag{6.4}$$

$$(\Delta\bar{N})^2=\left.\frac{\partial\bar{N}}{\partial\xi}\right|_p=\frac{a\xi z}{-a\xi z+\beta p)}+\frac{\xi^2z^2a^2}{(-a\xi z+\beta p)^2}$$

$$\cong+\bar{N}+\bar{N}^2\ . \tag{6.5}$$

So, for $\bar{N}$ large

$$(\Delta\bar{N})^2\,|_p\sim\bar{N}^2 \tag{6.6}$$

and

$$\frac{\Delta\bar{N}}{\bar{N}}\sim1\ . \tag{6.7}$$

Thus the fluctuations are much larger than those of a Gaussian distribution. This type of scaling, called KNO scaling, has been extensively studied in hadron-hadron inelastic processes and e^+e^- annihilation and has become a useful phenomenological tool.[10,11] *It arises naturally in pressure-fixed ensembles.* It must be noted, however, that there is no discrepancy in dealing with the volume-fixed ensemble to calculate average multiplicity, since the volume does not enter the final result. The dispersion in the final multiplicity is entirely due to the uncertainty inherent in determining r_0, the volume at which adiabatic expansion ensues.

Although the pressure-fixed ensemble gives KNO scaling, the pp and $p\bar{p}$ data[9] shows that $(\Delta n/\langle n\rangle)^2\sim0.29\pm0.1$, so that within experimental error the pressure-fixed ensemble gives KNO scaling with a different constant. However, better agreement may be obtained if we do not restrict ourselves to $1+1$ dimension, but incorporate transverse momentum effects. In our model this can be done in the following way: in addition to string breaking in one dimension, we allow each pair of quarks at momentum P_T to start its own longitudinal jet.

The system consists of k coupled grand canonical ensembles; each one-dimensional quark system behaves as a grand canonical ensemble with fugacity ξ and each branch contributes to the grand canonical partition function. We assume that the branching process stops at finite k. The exact value of $D^2/\langle n\rangle^2$ strongly depends on the value of k and gives $(\Delta n/\langle n\rangle)^2\sim0.29\pm0.01$ at $k\sim3$–4. However there is no strict theoretical argument for finite k except the following heuristic one[11]: typical of any branching process, the energy at each step gets diminished by a constant factor c. The process continues until after k branchings the energy is of the order of magnitude of hadronic mass. This determines a phenomenological limit for the branching process,[12]

$$M\sim\frac{E}{c^k}\ ,$$

$E=$c.m. energy, $M=$Mass of pion ,

$$k\sim\frac{1}{c}\ln\frac{E}{M}\ ,$$

with this assumption

$$Z\sim\left[\frac{1}{-a\xi z+\beta p}\right]^k\ ,$$

TABLE I. Comparison of the moments of the multiplicity distribution in hadron-hadron collision with the branching model of k branches (see Ref. 13).

	Experimental data (Ref. 17)						Theoretical predictions		
Name of experiment	CERN ISR (pp)	CERN ISR (pp)	UA5 (pp)	UA5 (CERN SPS) ($p\bar{p}$)	CERN ISR (pp)	UA1 (CERN SPS) $p\bar{p}$	$k=3$	$k=4$
Energy $\sqrt{s}$ (GeV)	30	44	53	53	63	540	$3E_0$	$4E_0$
$\langle n\rangle$	9.4±0.2	11.1±0.3	11.55 ±0.17	11.47 ±0.16	12.4 ±0.3	26.8 ±2.1	$1.62E^{1/2}$	$1.88E^{1/2}$
γ_2			0.291±0.015	0.304±0.014	0.297±0.010	0.308±0.010	$0.33+O\left[\frac{1}{\langle n\rangle}\right]$	$0.25+O\left[\frac{1}{\langle n\rangle}\right]$
γ_3			0.110±0.016		0.122±0.006	0.127±0.006	0.216	0.131
γ_4				0.051±0.006	0.027±0.008		0.21	0.09

TABLE II. Comparison of moments of multiplicity distributions for e^+e^- collisions with a branching model of k branches (see Ref. 13).

Name of experiment	Experimental data (Ref. 15) PLUTO	PLUTO (Ref. 21)	JADE	JADE	JADE (Ref. 21)	Theoretical predictions $k=6$	$k=7$	$k=8$
$\frac{E}{\sqrt{s}}$ (GeV)	9.4	29.9	12	30	35	$6E_0$	$7E_0$	$8E_0$
$\langle n\rangle$	7.6±0.1	11.8±0.4	7.8 ±0.1	12.0 ±0.5	12.4 ±0.5	$2.3E^{1/2}$	$2.48E^{1/2}$	$2.67E^{1/2}$
γ_2			~0.135±0.010	0.135±0.010	0.135±0.010	$0.166+O\left[\frac{1}{n}\right]$	$0.142+O\left[\frac{1}{\langle n\rangle}\right]$	$0.125+O\left[\frac{1}{\langle n\rangle}\right]$
γ_3						0.054	0.040	0.030

$$\ln Z = -k\ln(-\xi z a+\beta p)\,,$$

$$N=-\xi\frac{\partial}{\partial\xi}-\ln Z=\frac{k\xi z a}{(-a\xi z+\beta p)}\,,$$

$$\xi\frac{\partial^2}{\partial\xi^2}\xi\ln Z=\frac{k\xi^2 z^2 a^2}{(-a\xi z+\beta p)^2}+\frac{k\xi z a}{(-a\xi z+\beta p)}\,.$$

So[13]

$$\frac{\langle n^2\rangle-\langle n\rangle^2}{\langle n\rangle^2}\sim\frac{1}{k}\,,$$

in the strict thermodynamic limit[13] $\langle N\rangle\to\infty$.

A detailed analysis of the multiplicity distribution can be done in terms of the higher moments[14]

$$\gamma_3=\langle(n-\langle n\rangle)^3\rangle/\langle n\rangle^3\sim\frac{2}{k^2}+O\left[\frac{1}{k\langle\bar n\rangle}\right]+\left[\frac{1}{\langle\bar n\rangle^2}\right],$$

$$\gamma_4=\langle(n-\langle n\rangle)^4\rangle-3\langle(n-\langle n\rangle)^2\rangle^2/\langle n\rangle^4\,,$$

$$\cong O\left[\frac{6}{k^3}\right].$$

The experimental data[15] is tabulated in Table I, and the comparison with our predictions for values of k equal to 3 and 4 is shown.

In the case of e^+e^- reactions,[16] the multiplicity distribution is narrower: $(D/\langle n\rangle)^2\sim0.136\pm0.001$. Thus in our "branching" model, k is equal to 6, 7, and 8; the comparison with data is shown in Table II.

It is interesting to note that the number of "branches" (fireballs) in e^+e^- collisions is higher than in hadron-hadron collisions.[17]

A possible interpretation of the difference in the multiplicity distributions of e^+e^- and hadron-hadron collisions is the following heuristic one. Each of the colliding protons (antiprotons) in a pp ($p\bar p$) collision has a substructure composed of three quarks (partons). The incident collision energy is thus smeared over the quark (parton) distribution and the available energy per quark is reduced. Since the branching process stops at $k\sim C\ln E/m$, a reduction in the available energy will terminate the process after a fewer number of steps. Furthermore, there is an uncertainty in the energy distribution which contributes to the dispersion in the multiplicity distributions. In other words

$$\frac{\Delta\bar n^2}{\langle n\rangle^2}=\frac{\Delta\bar n^2}{\langle n\rangle^2}\bigg|_{E\text{ fixed}}+\frac{\Delta\bar n^2}{\langle n\rangle^2}\bigg|_{\text{due to }\Delta E^2}.$$

Thus, in general, the dispersion is more in the case of $p\bar p$ and pp collisions than in e^+e^- collisions. More quantitative estimates can only be made if the details of the parton distribution are known.

In the next section we analyze the implications of this model.

VII. ANALYSIS OF RESULTS, AND COMMENTS

We have presented a model which predicts some of the characteristics of the final-state hadrons in e^+e^-, pp, and $p\bar p$ collisions. In particular, our model predicts the energy dependence of the multiplicity distribution to be of the form $N=cE^{1/2}$, where c is determined exactly by the number of flavors of quarks n_f, and the string tension α.

If the system is considered in a pressure-fixed grand canonical ensemble, it shows KNO scaling. Although generic KNO scaling is present, better agreement with the experimentally observed situation is possible if we depart from the $1+1$ dimensional approximation (where the Hamiltonian is exact) and allow "branching" in transverse directions. The branching process is finite and stops at three to four branches for pp and $p\bar p$ collisions and at six to eight branches for e^+e^- collisions. Although the postulation of a finite number of branches destroys the exact aspect of the model, we shall show that the two principal features of hadron production, i.e., energy dependence of the mean-charged multiplicity and KNO scaling are consistent with each other.

Figure 1 shows how the mean charged multiplicity varies with energy for e^+e^- collisions. The phenomenological fit is $N=(2.3\pm0.1)E^{1/2}$. (E in GeV.) Figure 2 shows a similar plot for pp collisions. The phenomenological fit is $N=(1.668\pm0.012)E^{1/2}$.

In our model $N=cE^{1/2}$, where

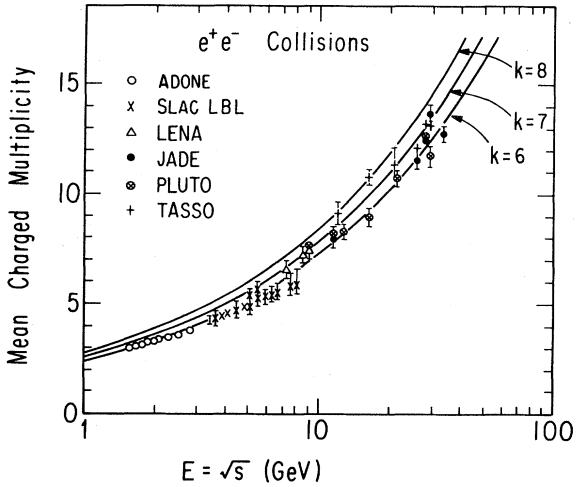

FIG. 1. Mean charged multiplicity $\bar{N}$ vs center-of-mass energy $E=\sqrt{s}$ for e^+e^- reactions. Solid curves represent multiplicity predictions for $k=6$, 7, and 8 branches. Symbols represent experimental data.

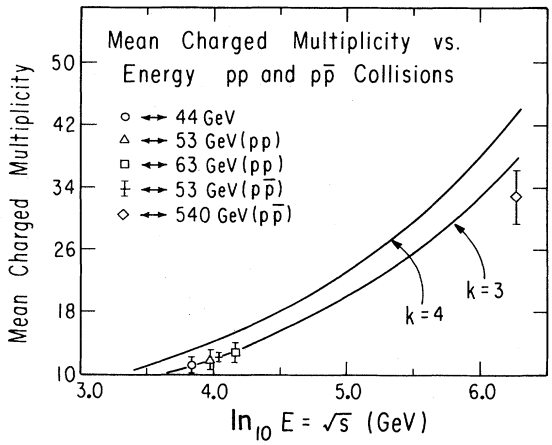

FIG. 2. Mean charged multiplicity $\bar{N}$ vs center-of-mass energy (logarithmic scale) for pp and $p\bar{p}$ collisions. Solid curves represent multiplicity predictions for $k=3$ and 4 branches. Symbols represent experimental data.

$$C=\tfrac{1}{2}\times\tfrac{2}{3}\sqrt{2}(\xi\,\mathrm{eff})^{1/4}\left[\frac{n_f}{\alpha}\right]^{1/4}$$

$$\sim 0.94 \text{ for } \alpha\sim 0.2 \text{ GeV}, \quad n_f=5 ,$$

$$\xi_{1/2}=\xi_1=1 .$$

E is the total c.m. energy. The factor $\frac{1}{2}\times\frac{2}{3}$ accounts for the fact that only charged hadrons are observed. For e^+e^- reactions, KNO scaling is consistent with six to eight "branches." If the total c.m. energy is shared by each of these branches $E=(7\pm1)E_0$, so that

$$N=(2.48\pm0.94)\sqrt{E_0} .$$

For pp and $p\bar{p}$ collisions, $E=(3.5\pm0.5)\bar{E}_0$ so that

$$N=(1.75\pm0.47)\sqrt{E_0} \text{ (Ref. 18)} .$$

In a self-consistent way we have shown that the results predicted by this model are in good agreement with observed e^+e^- data.[17]

In conclusion, we would like to remark that thermodynamic models in general show KNO scaling[14,18]; however exact details of the observed data can be deduced by knowing the microscopic details of the system such as the parton distributions, etc. The exact results of this model are valid for $1+1$ dimensions, which is a mathematical laboratory where confinement of quarks takes place naturally. Gross features of the hadron multiplicity in e^+e^-, pp and $p\bar{p}$ collisions can be predicted. If we sacrifice the exactness of this model in a self-consistent way, better agreement with the data can be shown. A crucial test of this model will come when multiplicity data at higher energy is observed; as at those energies the multiplicity with a power-law dependence will be significantly different from the energy dependence of the multiplicity shown by perturbative QCD.[20]

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[12]This determines a *speculative* limit for the branching process

$$M=\frac{E}{C^k};\ k=\frac{1}{C}\ln\frac{E}{M}\ .$$

It must be stressed that this is only a speculative argument for the finiteness of k. QCD models for KNO scaling also postulate the existence of a finite number of clusters. The experimental evidence for the variation with energy for k is the constancy of the moment γ_2. Only a complete analysis (see Ref. 17) of the leading particle contributions will resolve this issue.

[13]The moment $\gamma_2\sim 1/k$ only in the thermodynamic limit $\bar{N}\to\infty$. In actual practice, however, $(\overline{\Delta N})=(1/k)\bar{N}^2+\bar{N}$; so that $\gamma_2\sim 1/k+1/N$. Experimental data for e^+e^- collisions yields $N\sim 10$—20; so that since $\bar{N}\sim cE^{1/2}$, γ_2 is a decreasing function of energy. In $p\bar{p}$ data there may be some indication of this fact [Ref. 15(b)]. In e^+e^- data, to our knowledge (Ref. 15) there seems little indication of it.

[14]The definition of KNO scaling we have used is that the moments of the distribution are independent of energy. We have shown asymptotically that KNO scaling holds. However, for the energies observed so far, the moments show a decrease with increasing energy (Ref. 13). The distribution function $\psi(n/\bar{n})$ may be calculated from the partition function and the moments:

$$\psi\left[\frac{n}{\bar{n}}\right]=\frac{k^k}{(k-1)!}\left[\frac{n}{\bar{n}}\right]^{k-1}e^{-kn/\bar{n}}\ \text{(asymptotically)}$$

for values of k between 3 and 4 this shows agreement with data.

[15]J. G. Branson, MIT Technical Report No. 113, 1983 (unpublished).

[16]This is assuming the final state hadrons are mostly pions. This is consistent with our Monte Carlo analysis (Ref. 5) and experimental data, which show that the baryon-to-meson ratio is ~ 0.1.

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[18]The leading-particle effect leads to a universality of scaling behavior of e^+e^- and pp collisions. In our model it corresponds to replacing $E_{\text{c.m.}}$ by the available energy $E_{\text{c.m.}}/2$ for the nonleading final hadrons (Ref. 18). Hence

$$\begin{aligned}N_{\text{had}}&=\frac{2.48\pm 0.94\sqrt{E}}{2}\\&=1.76\pm 0.67\sqrt{E}\\&=N_c\ ,\end{aligned}$$

so our analysis does not contradict Ref. 19.

[19]After this paper was written, the work of P. Carruthers and C. C. Shih [Phys. Lett. 127B, 242 (1983)] came to our attention. This has interesting results relevant to statistical models of KNO scaling.

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