Spontaneous breaking of chiral symmetry for confining potentials

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Using the Bogoliubov-Valatin variational method, we show that the chiral-invariant vacuum is unstable for a color, fourth-component vector powerlike potential r^α $(0<\alpha<3)$ independently of the strength of the coupling constant. The fermion self-energy is negative and dominates over the positive potential energy, destabilizing the vacuum by $\bar{\psi}\psi$ pair condensation. This self-energy is finite but infrared singular, reflecting the behavior of the potential at large distances. We give an analytical proof of the fact that the energy of the unbroken vacuum is not minimum. The proof extends to logarithmic potentials as $\alpha \rightarrow 0$, but breaks down for $\alpha \geq 3$ (number of spatial dimensions) due to severe infrared singularities. If the confining potential possesses a spin-spin piece, there are critical values of its strength, depending on the power α, beyond which the stability of the chiral-invariant vacuum is restored. In the case of the harmonic oscillator $\alpha=2$, the gap equation reduces to a nonlinear second-order differential equation. We find (besides the usual chiral degeneracy) an infinite number of solutions breaking chiral symmetry, higher in energy as the number of their nodes increases. We compute the expectation value of $\bar{\psi}\psi$ and the mass gap for the new vacuum, the lowest solution in energy. The infrared singularity of the massless fermion self-energy is removed for the stable broken solution.

I. INTRODUCTION

Quantum chromodynamics must undergo dynamical chiral-symmetry breaking[1] (CSB) to implement the successful scheme of current algebra. Many ideas have been expressed as to the origin of this CSB. A certain trend associates it with confinement, another necessary property of QCD. Strong-coupling lattice treatments precisely seem to demonstrate at the same time CSB and confinement.[2] But which is precisely the logical connection between the two properties?

We must first emphasize that other pieces of the quark-quark potential, as the one generated by short-distance one-gluon exchange, may also lead to CSB. If the gluon coupling is bigger than some critical value $\alpha_s^{\rm crit}$, there is indeed instability of the chiral-invariant vacuum. One finds[3,4] $\alpha_s^{\rm crit}$ of the order of 1, and therefore short-distance contributions to CSB are possible. That confinement alone implies CSB has been suggested by lattice-gauge-theory calculations. It has been proved, within the mean-field approximation, that confinement and CSB are appearing or disappearing together.[5]

One would like, however, to get a more transparent view of this connection. The potential approach to CSB (Refs. 3, 4, and 6) has in this respect a great intuitive appeal. The signal of CSB is the existence of a negative-energy bound state $q\bar{q}$ in the normal phase, leading to a new vacuum made out of condensed pairs. More precisely, Casher[6]—imposing the condition to have a tachyon in the Bethe-Salpeter equation—has found the following criterion: the operator $2p+V$ must have a negative eigenvalue. As we have shown in Ref. 4, Casher's instability criterion can also be obtained by the Bogoliubov-Valatin variational method, which we will use here. Similar criteria under other approximations have been proposed by Finger, Horn, and Mandula[3]—in the Tamm-Dancoff approximation—and by Banks and Raby[7] and ourselves[8] within the effective-potential method of field theory. From this form of the instability criterion it is then immediately seen that confinement by itself would not lead to CSB: a confining $V(r)$ like r^α $(\alpha>0)$, being positive everywhere, cannot give a negative eigenvalue.

However, we have shown[9] using the Bogoliubov-Valatin variational method that confinement implies indeed CSB, because of an overlooked possibility. The instability criterion of Casher's type was based on a Hamiltonian *normal ordered* relative to the massless fermion base. This was done to preserve chiral invariance and at the same time to avoid renormalization in the variational calculation. However, if the four-fermion interaction is not normal ordered—as we know according to field theory—it generates a fermion self-energy in the normal phase. This self-energy modifies the criterion of instability: the operator $2p+V$ is replaced by $2E(p)+V$, $E(p)$ being now given by the kinetic energy plus the massless fermion self-energy generated by the confining potential. We did show in Ref. 9 that, for a linear potential, this massless fermion self-energy is *negative* and finite, and big enough to dominate over the positive potential-energy contribution, making possible the appearance of a negative-energy bound state. It has already been noticed by Bars and Green[10] and Brout, Englert, and Frère[11] that two-dimensional QCD—which amounts to a non-normal-ordered four-fermion confining interaction—gives a negative fermion self-energy.

In this paper we want to make more explicit and complete the results of Ref. 9 in two directions. First, generalize, in three spatial dimensions, our result for the linear potential to any power r^α $(0<\alpha<3)$. As we will see, the proof extends to logarithmic potentials as $\alpha \to 0$. Second, we want to go beyond the proof of the instability of the chiral-invariant vacuum and show that there are stable solutions of the gap equation that break chiral symmetry and have lower energy than the chiral-invariant solution. We will solve the gap equation in the particular case of the harmonic-oscillator potential $\alpha=2$. In this case, the gap equation—a nonlinear integral equation in general—simplifies to a nonlinear second-order differential equation of the sine-Gordon type.

Since we want to study the dynamical breaking of chiral symmetry, we must choose a potential preserving chiral invariance. The simplest choice is the one of a vector force. And if it is a color octet, it would have indeed the desired property of being attractive both in the singlet $q\bar{q}$ and in the $\underline{3}$ qq channels, therefore binding mesons and baryons. We thus start from a confining interaction binding both mesons and baryons as is considered in the hadron spectroscopy, but we extend it to the second-quantization formalism, setting the quark mass to zero to preserve chiral invariance. This second-quantization formalism, leading to pair creation, has also been used in the context of Zweig-rule-allowed strong-interaction vertices.[12] However, a vector confining force alone is at odds with some conclusions of phenomenology concerning the spin dependence of the potential. Indeed, these would indicate also the presence of a Lorentz-scalar confining potential. The drawbacks of a vectorlike force are twofold. First, the spacelike components would lead to a long-range spin-spin force, namely, $(\Delta V/m^2)(\vec{\sigma}_1\cdot\vec{\sigma}_2)$ in the nonrelativistic approximation, and this is excluded by spectroscopy, which favors instead a pure Fermi contact interaction $\vec{\sigma}_1\cdot\vec{\sigma}_2\delta(\vec{r})$, which is provided by one-gluon exchange.[13] To this first drawback we can only remedy by the *ad hoc* prescription of retaining only the time component $\sim\gamma^0\gamma^0$.

But there still remains the problem of spin-orbit forces generated by the time component. In the nonrelativistic limit, one finds a spin-orbit force which, i.e., that fails to explain the splitting of P levels in charmonium and light mesons. A scalar potential is needed, at least for charmonium.[14] Analogously, in baryon spectroscopy, a scalar potential is needed to cancel the spin-orbit contribution of the one-gluon exchange.[15]

There is no clear answer to these objections. One must first emphasize that, to draw definitive conclusions, one should completely recalculate the spectrum predicted by vector forces in the situation of dynamical chiral-symmetry breaking. Second, one may expect to generate a scalar contribution to the effective forces through higher-order diagrams, once massive fermion propagators are introduced into internal fermion lines.

For the moment, we choose to keep to a vector potential and we shall include a discussion of the effect of the space-space components on the dynamical symmetry breaking. The heart of the above difficulties will be met when we calculate the meson spectrum.

Finally, let us say that we understand our chiral-invariant confining interaction as an approximation to the confining regime of QCD before chiral invariance has been spontaneously broken. There is a weakness in our approach: the instantaneous character of the interaction and the corresponding lack of covariance. Nobody knows the relative time dependence of the strong-coupling interaction between quarks, and we do not have an answer to this objection. We can say, however, that in a case where retardation effects can be computed, as one-gluon exchange, the qualitative picture is not changed when those are taken into account. The critical coupling beyond which there is instability of the chiral-invariant vacuum changes from $\frac{1}{3}$ (Coulomb gauge, instantaneous approximation) to $\pi/(3\pi-4)$ when taking into account retardation.[8] We simply hope on this basis that retardation will not restore the stability of the chiral-invariant vacuum in the case of confining potentials.

This paper is organized as follows. In Sec. II we write the Hamiltonian and outline the Bogoliubov-Valatin variational method. In Sec. III we study the gap equation, i.e., the condition of stationarity of the vacuum energy. In Sec. IV we address ourselves to a simpler problem than to solve the gap equation: to know if the chiral-invariant vacuum is unstable. In Sec. V we prove analytically that the chiral-invariant vacuum is unstable for a fourth-component-vector linear potential, independently of the coupling constant. In Sec. VI we extend this proof to any fourth-component-vector confining potential r^α $(0<\alpha<3)$. The proof applies also to logarithmic potentials but breaks down for $\alpha\geq 3$ (number of spatial dimensions). In Sec. VII we point out that a spin-spin interaction can modify these results, but there is a critical strength for this interaction below which there is still instability. In Sec. VIII we solve the gap equation for the harmonic-oscillator potential, $\alpha=2$. In this case it reduces to a nonlinear second-order-differential equation because the Fourier transform of $\vec{r}^{\,2}$ is proportional to $\Delta_{\vec{k}}\delta(\vec{k})$. We point out that there is an infinity of solutions breaking chiral invariance. In Sec. IX we finally discuss the shift in energy from the chiral-invariant vacuum to the new vacuum (corresponding to the solution of the gap equation with lowest energy), the expectation value of $\bar{\psi}\psi$, and the mass gap. In Sec. X we conclude. In Appendices A and B we give mathematical proofs of some of the results used in Sec. VI on vacuum instability.

II. BOGOLIUBOV-VALATIN TRANSFORMATION

Let us start from the chiral-invariant Hamiltonian for massless quark fields interacting through an instantaneous fourth-component color-confining potential,

$$\mathscr{H}=\sum_{\vec{x}}\psi^\dagger(\vec{x})(-i\vec{\alpha}\cdot\vec{\nabla})\psi(\vec{x}) +\frac{1}{2}\sum_{\vec{x},\vec{y},a}V(\vec{x}-\vec{y})\left[\psi^\dagger(\vec{x})\frac{\lambda^a}{2}\psi(\vec{x})\right] \times\left[\psi^\dagger(\vec{y})\frac{\lambda^a}{2}\psi(\vec{y})\right], \tag{2.1}$$

where $V(\vec{x})=-V_0^{\,1+\alpha}\,|\,\vec{x}\,|^{\,\alpha}$ $(\alpha>0)$. We need an overall minus sign to have an *attractive* force between qq in a color $\underline{3}$ and between $q\bar{q}$ in a color singlet: the effective potential is then $\frac{2}{3}V_0^{1+\alpha}|\vec{x}|^\alpha$ for a $\underline{3}$ qq and $\frac{4}{3}V_0^{1+\alpha}|\vec{x}|^\alpha$ for a $q\bar{q}$ singlet. The lattice formalism does not play an essential role here and is introduced to regularize infinities that appear in intermediate steps of the calculation. We consider n^3 sites with spacing a, the volume being $(an)^3$. At the end we will take without any problem the continuum infinite-volume limit

$$a^3\sum_{\vec{x}} \to \int d\vec{x},\quad \frac{1}{(an)^3}\sum_{\vec{k}} \to \int \frac{d\vec{k}}{(2\pi)^3}\ .$$

For the sake of simplifying the notations, we consider only one fermion flavor. The generalization to the realistic case of two massless flavors u,d, SU(2)×SU(2), is straightforward. We leave aside any discussion of the U(1) problem, which is beyond our phenomenological approach.

Instead of (2.1) we could start, as is sometimes done, from another chiral-invariant Hamiltonian, the normal ordering of $\mathscr{H}$ relative to the massless fermion base,

$$\mathscr{H}_n = N^{(0)}(\mathscr{H})\ . \tag{2.2}$$

$N^{(0)}$ means Wick ordering relative to the creation and annihilation operators of free massless fermions (normal ordering relative to any other base would break explicitly chiral symmetry). This means that in (2.2) we expand $\psi(\vec{x})$ in terms of free massless spinors:

$$\psi(\vec{x}) = \frac{1}{n^{3/2}}\sum_{\vec{k},s}[\,u_s^{(0)}(\vec{k})b_s^{(0)}(\vec{k}) + v_s^{(0)}(\vec{k})d_s^{(0)\dagger}(-\vec{k})]e^{i\vec{k}\cdot\vec{x}} \tag{2.3}$$

and we normal order relatively to the $b^{(0)}$ and $d^{(0)}$ operators. The main motivation for starting from (2.2) and not from (2.1) is, as we will see in a moment, to avoid divergences in the massless fermion self-energy—that would appear for instance for a Coulomb potential—and to normalize to zero the generally infinite energy of the chiral-invariant vacuum. We will see that, for a confining potential, this self-energy has no divergences and plays a crucial role in triggering CSB. CSB occurs starting from $\mathscr{H}$, but does not happen if we adopt $\mathscr{H}_n$. At each step we will compare the results for the choices $\mathscr{H}$ and $\mathscr{H}_n$.

Let us first perform a Bogoliubov-Valatin (BV) transformation: It consists in writing the quark fields in terms no longer of a massless spinor base, but in terms of arbitrary spinors u,v

$$\psi(\vec{x}) = \frac{1}{n^{3/2}}\sum_{\vec{k},s}[u_s(\vec{k})b_s(\vec{k}) + v_s(\vec{k})d_s^\dagger(-\vec{k})]e^{i\vec{k}\cdot\vec{x}}\ . \tag{2.4}$$

These spinors are not necessarily solutions of the Dirac equation, but obey the normalization conditions

$$\begin{aligned} u_{s'}^\dagger(\vec{k})u_s(\vec{k}) &= v_{s'}^\dagger(\vec{k})v_s(\vec{k}) = \delta_{ss'}\ ,\\ u_{s'}^\dagger(\vec{k})v_s(\vec{k}) &= v_{s'}^\dagger(\vec{k})u_s(\vec{k}) = 0 \end{aligned} \tag{2.5}$$

preserving in this way the canonical commutation relations. This BV transformation preserves automatically translational—because of the factorization of $\exp(i\vec{k}\cdot\vec{x})$—and baryonic number invariances. The BV method consists in writing the Hamiltonian in terms of the new creation and annihilation operators b, $b^\dagger$, d, $d^\dagger$ of the new spinor base. What characterizes the BV approximation is the *linear* character of the relation between the old and the new creation and annihilation operators, which follows from (2.3) and (2.4). The new spinors u and v are trial spinors to be varied to look for the stationary states of the theory. To obtain a useful expression, we need to rewrite the Hamiltonian in terms of normal-ordered operators relatively to the new base. One obtains, applying Wick's theorem,

$$\mathscr{H} = \mathscr{E} + :H_2: + :H_4:\ , \tag{2.6}$$

where

$$\mathscr{E} = 3\sum_{\vec{k}}\mathrm{Tr}[(\vec{\alpha}\cdot\vec{k})\Lambda_-(\vec{k})] + 4\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k},\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\mathrm{Tr}[\Lambda_+(\vec{k})\Lambda_-(\vec{k}')]\ , \tag{2.7}$$

$$H_2 = \frac{4}{3}\frac{1}{n^3}\frac{1}{2}\sum_{\vec{k},\vec{x},\vec{y}}V(\vec{x}-\vec{y})e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\{\psi^\dagger(\vec{x})[\Lambda_+(\vec{k})-\Lambda_-(\vec{k})]\psi(\vec{y})\} + \sum_{\vec{x}}\psi^\dagger(\vec{x})(-i\vec{\alpha}\cdot\vec{\nabla})\psi(\vec{x})\ , \tag{2.8}$$

$$H_4 = \frac{1}{2}\sum_{\vec{x},\vec{y},a}V(\vec{x}-\vec{y})\left[\psi^\dagger(\vec{x})\frac{\lambda^a}{2}\psi(\vec{x})\right]\left[\psi^\dagger(\vec{y})\frac{\lambda^a}{2}\psi(\vec{y})\right]\ . \tag{2.9}$$

$\mathscr{E}$ is the vacuum energy and H_2 and H_4 are the bilinear and quadrilinear terms in the quark fields, respectively. The algebraic factors come from color, and $\Lambda_\pm$ are the projectors

$$\begin{aligned} \Lambda_+(\vec{k}) &= \sum_s u_s(\vec{k})u_s^\dagger(\vec{k})\ ,\\ \Lambda_-(\vec{k}) &= \sum_s v_s(\vec{k})v_s^\dagger(\vec{k})\ . \end{aligned} \tag{2.10}$$

The preceding formulas follow easily from Wick's theorem taking into account the contractions

$$\begin{aligned} \overline{\psi_\alpha(\vec{x})\psi_\beta^\dagger}(\vec{y}) &= \frac{1}{n^3}\sum_{\vec{k}}[\Lambda_+(\vec{k})]_{\alpha\beta}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\ ,\\ \overline{\psi_\beta^\dagger(\vec{x})\psi_\alpha}(\vec{y}) &= \frac{1}{n^3}\sum_{\vec{k}}[\Lambda_-(\vec{k})]_{\alpha\beta}e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\ , \end{aligned} \tag{2.11}$$

where α,β refer to the $1,\ldots,4$ components of the Dirac fields. All the contractions needed reduce to (2.11) for the different components. This can be seen by making explicit the sums over the Dirac field components in $\mathscr{H}$.

The Fourier transform of $V(\vec{x})$ is

$$\tilde{V}(\vec{k})=a^3\sum_{\vec{x}}V(\vec{x})e^{i\vec{k}\cdot\vec{x}} . \tag{2.12}$$

For a confining potential, $\tilde{V}(\vec{k})$ is a distribution and will be defined with precision later.

If we had started from $\mathscr{H}_n$ instead of $\mathscr{H}$, we would have obtained, applying Wick's theorem to $\mathscr{H}$ relatively to the old (2.3) and the new base (2.4), the same expressions (2.7)—(2.9) but with $\Lambda_{\pm}(\vec{k})$ substituted by the difference of projectors

$$\Lambda_{\pm}^{(d)}(\vec{k})=\Lambda_{\pm}(\vec{k})-\Lambda_{\pm}^{(0)}(\vec{k}) , \tag{2.13}$$

where $\Lambda_{\pm}^{(0)}(\vec{k})$ are the free massless projectors

$$\Lambda_{\pm}^{(0)}(\vec{k})=\tfrac{1}{2}(1\pm\vec{\alpha}\cdot\hat{k}) . \tag{2.14}$$

$\Lambda_{\pm}^{(d)}$ is the shift of projectors from the free massless base to the arbitrary one. If all the invariances of the original interaction but chiral invariance are not spontaneously broken, one may see that $\Lambda_{-}(\vec{k})$ must be of the form

$$\Lambda_{-}(\vec{k})=\tfrac{1}{2}[1-\sin\varphi(k)\beta-\cos\varphi(k)\vec{\alpha}\cdot\hat{k}] , \tag{2.15}$$

where $\varphi(k)$ is a function of $k=|\vec{k}|$ since we assume rotational invariance to be preserved. We recover $\Lambda_{-}^{(0)}$ for $\varphi=0$. Note that although $\Lambda_{-}(\vec{k})$ [(2.15)] has a mass term, the Hamiltonian (2.6) in terms of $\Lambda_{-}(\vec{k})$ remains, of course, chiral invariant. This simply means that we are *trying* BV states that are not chiral invariant. To check the chiral invariance of $\mathscr{H}$ requires some calculation. $:H_4:$ is not chiral invariant because the normal product is taken relative to a general basis corresponding to massive particles. Under a chiral transformation, $:H_4:$ will change, but its change will be compensated by the explicit non-chiral-invariant piece in $:H_2:$. The constant $\mathscr{E}$ is chiral invariant, as expected from chiral degeneracy of the vacuum. $\mathscr{H}$ remains chiral invariant, but, of course, the states corresponding to the trial spinors are not in general chiral invariant.

III. GAP EQUATION

The gap equation is just the condition of stationarity of $\mathscr{E}$, the vacuum energy. $\mathscr{E}$ is a functional of the projector $\Lambda_{-}(k)$. Differentiating relatively to it, we get

$$\delta\mathscr{E}=3\sum_{\vec{k}}\mathrm{Tr}\left[\delta\Lambda_{-}(\vec{k})\left[\vec{\alpha}\cdot\vec{k}+\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[1-2\Lambda_{-}(\vec{k}')]\right]\right] \tag{3.1}$$

with $\delta\Lambda_{-}(\vec{k})$ satisfying the projector constraint

$$(\Lambda_{-}+\delta\Lambda_{-})^2=\Lambda_{-}+\delta\Lambda_{-} ,$$

i.e., neglecting the quadratic term,

$$\Lambda_{-}\delta\Lambda_{-}+\delta\Lambda_{-}\Lambda_{-}=\delta\Lambda_{-} . \tag{3.2}$$

We see that $\delta\Lambda_{-}$ is antidiagonal relative to Λ_{-}. In a base in which Λ_{-} is diagonal by blocks, $\delta\Lambda_{-}$ is antidiagonal by blocks.

The condition of extremum is then that the operator multiplying $\delta\Lambda_{-}(\vec{k})$ in (3.1) must be diagonal by blocks. Then, the gap equation can be written as the two coupled equations

$$H(\vec{k})=\vec{\alpha}\cdot\vec{k}+\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[1-2\Lambda_{-}(\vec{k}')] , \tag{3.3a}$$

$$[\Lambda_{-}(\vec{k}),H(\vec{k})]=0 . \tag{3.3b}$$

We use the notation $H(\vec{k})$ because this operator corresponds simply to the Hamiltonian of a Dirac particle, the bilinear part of expression (2.6). Using the explicit forms (2.15) and (3.3a), $H(\vec{k})$ can be written in the form

$$H(\vec{k})=A(k)\beta+B(k)\vec{\alpha}\cdot\hat{k} \tag{3.4}$$

with

$$A(k)=\frac{1}{2}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\sin\varphi(k') ,$$
$$B(k)=k+\frac{1}{2}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\cos\varphi(k')(\hat{k}\cdot\hat{k}') . \tag{3.5}$$

The second condition (3.3b) implies then

$$A(k)=E(k)\sin\varphi(k),\quad B(k)=E(k)\cos\varphi(k) \tag{3.6}$$

and the gap equation can be written in the familiar form of two coupled nonlinear integral equations:

$$A(k)=\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\frac{A(k')}{E(k')} ,$$
$$B(k)=k+\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\frac{B(k')}{E(k')}(\hat{k}\cdot\hat{k}') , \tag{3.7}$$

with the condition $[E(k)]^2=[A(k)]^2+[B(k)]^2$. This system of equations is just the Schwinger-Dyson equation for the self-mass

$$\Sigma(\vec{k})=[B(k)-k]\vec{\gamma}\cdot\hat{k}+A(k)$$

in the instantaneous ladder approximation (Fig. 1):

$$\Sigma(\vec{k})=\tfrac{4}{3}\int\frac{d\vec{k}\,'}{(2\pi)^3}\tilde{V}(\vec{k}-\vec{k}\,')\gamma^0 S(\vec{k}\,')\gamma^0 \ . \qquad (3.8)$$

$S(\vec{k})$ is the fermion propagator once the trivial integration over k^0 has been performed:

$$S(\vec{k})=\int\frac{dk^0}{(2\pi)}\frac{i}{k^0\gamma^0-\vec{k}\cdot\vec{\gamma}-\Sigma(\vec{k})}$$
$$=\frac{A(k)-B(k)(\vec{\gamma}\cdot\hat{k})}{2\{[A(k)]^2+[B(k)]^2\}^{1/2}} \ . \qquad (3.9)$$

FIG. 1. The Schwinger-Dyson equation for the self-mass in the ladder approximation.

A straightforward calculation gives the system (3.7). This system can be reduced to a single nonlinear integral equation for $\varphi(k)$ since $A/B=\tan\varphi$:

$$\frac{1}{2}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}\,'}\tilde{V}(\vec{k}-\vec{k}\,')[\sin\varphi(k')\cos\varphi(k)-\cos\varphi(k')\sin\varphi(k)(\hat{k}\cdot\hat{k}\,')]=k\sin\varphi(k) \ . \qquad (3.10)$$

The gap equation can be written as a single equation for $\varphi(k)$ since the vacuum energy depends only on $\varphi(k)$ through $\Lambda_-(\vec{k})$. Once a solution of (3.10) is known, $A(k)$ and $B(k)$ are given by (3.5), and the energy of a fermion is then given by

$$E(k)=\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}\,'}\tilde{V}(\vec{k}-\vec{k}\,')\frac{\sin\varphi(k')}{\sin\varphi(k)} \ . \qquad (3.11)$$

We will see in Sec. VIII that in the case of a harmonic-oscillator potential, $\alpha=2$, the gap equation reduces to a nonlinear differential equation for $\varphi(k)$, since the Fourier transform of the potential is just the Laplacian of a delta function, $\tilde{V}(\vec{k})\sim\Delta_{\vec{k}}\delta(\vec{k})$. This fact will allow us to solve the gap equation and establish the existence of chiral-noninvariant solutions.

What happens if instead of starting from $\mathscr{H}$ we begin with $\mathscr{H}_n$? The gap equations are exactly of the same form (3.3) but $[1-2\Lambda_-(\vec{k}\,')]$ must now be substituted by $-2\Lambda_-^{(d)}(\vec{k}\,')$ in (3.3a). The first equation (3.7) is not changed, but the second becomes

$$B(k)=k+\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}\,'}\tilde{V}(\vec{k}-\vec{k}\,')\times\left[\frac{B(k')}{E(k')}-1\right](\hat{k}\cdot\hat{k}\,') \ . \qquad (3.12)$$

The corresponding Schwinger-Dyson equation has the same form (3.8) with $S(\vec{k})$ substituted by $S(\vec{k})-S^{(0)}(\vec{k})$, $S^{(0)}$ being the massless fermion propagator integrated over k^0.

The gap equation always has a chiral-invariant solution, stable or unstable. If we adopt $\mathscr{H}$ we can see from (3.7) that it corresponds to

$$A^{(0)}(k)=0 \ ,$$
$$B^{(0)}(k)=k+\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}\,'}\tilde{V}(\vec{k}-\vec{k}\,')(\hat{k}\cdot\hat{k}\,') \ . \qquad (3.13)$$

There is a dynamically generated kinetic term, the lowest-order self-energy of a massless fermion, which corresponds to the bilinear term H_2 [(2.8)] when we make $\Lambda_\pm\to\Lambda_\pm^{(0)}$. This is the term that one wants to avoid by adopting the normal-ordered form $\mathscr{H}_n$, since, from (3.12) we get, in this case, the chiral-invariant solution

$$A^{(0)}(k)=0, \quad B^{(0)}(k)=k \ . \qquad (3.14)$$

In the case of one-gluon exchange, the second term of (3.13) is infinite and needs renormalization. To our knowledge, in QCD, the role of the self-energy of a fermion in the context of vacuum instability is still an open question. In this paper we will consider a confining potential, and in this case the self-energy is ultraviolet finite and can be computed explicitly.

IV. CONDITION OF INSTABILITY OF THE CHIRAL-INVARIANT VACUUM

The ideal thing is to study if there is a nontrivial solution of the gap equation corresponding to a minimum. This problem is too difficult in general and we will solve it in Sec. VIII for the harmonic-oscillator potential. We will try to see first if the chiral-invariant vacuum is unstable, i.e., if the second derivative of the vacuum energy is negative. The gap equation is nonlinear, but the study of the stability of a given stationary state is a simpler problem since it amounts to looking for a direction in which a quadratic form becomes negative. We need to determine the principal axes of a quadratic form, and this is a linear problem.

Let us formulate in general the condition of instability of the chiral-invariant vacuum. Since we start from $\mathscr{H}$, the unbroken vacuum has a nonvanishing energy $\mathscr{E}^{(0)}$ corresponding to make $\Lambda_\pm\to\Lambda_\pm^{(0)}$ in the expression of $\mathscr{E}$. Therefore, the shift in energy from the chiral-invariant vacuum to the new one defined by the BV transformation will be

$$\Delta\mathscr{E}=\mathscr{E}-\mathscr{E}^{(0)}$$
$$=3\sum_{\vec{k}}\mathrm{Tr}\{(\vec{\alpha}\cdot\vec{k})[\Lambda_{-}(\vec{k})-\Lambda_{-}^{(0)}(\vec{k})]\}+4\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k},\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\mathrm{Tr}[\Lambda_{-}(\vec{k})\Lambda_{+}(\vec{k}')-\Lambda_{-}^{(0)}(\vec{k})\Lambda_{+}^{(0)}(\vec{k}')] . \quad (4.1)$$

Calling $H^{(0)}(\vec{k})$ the expression (3.3a) for the case of the chiral-invariant solution,

$$H^{(0)}(\vec{k})=\vec{\alpha}\cdot\vec{k}+\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[1-2\Lambda_{-}^{(0)}(\vec{k}')] , \quad (4.2)$$

we obtain, after some algebraic manipulations,

$$\Delta\mathscr{E}=3\sum_{\vec{k}}\mathrm{Tr}[H^{(0)}(\vec{k})\Lambda_{-}^{(d)}(\vec{k})]+4\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k},\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\mathrm{Tr}[\Lambda_{-}^{(d)}(\vec{k})\Lambda_{+}^{(d)}(\vec{k}')] , \quad (4.3)$$

where $\Lambda_{\pm}^{(d)}(\vec{k})$ were defined in (2.13). $H^{(0)}(\vec{k})$ is just equal to

$$H^{(0)}(\vec{k})=B^{(0)}(k)(\vec{\alpha}\cdot\hat{k}) \quad (4.4)$$

with $B^{(0)}(k)$ given by (3.13). $H^{(0)}(\vec{k})$ is the Hamiltonian of a free massless fermion modified by the self-energy. We have seen that (3.13) corresponds to a stationary state. Therefore, $\Delta\mathscr{E}$ should be quadratic in $\Lambda_{-}^{(d)}(\vec{k})$. It is easy to see that this is the case, making use of the condition of projector for $\Lambda_{\pm}$ and $\Lambda_{\pm}^{(0)}$:

$$\Lambda_{-}^{(d)}=\Lambda_{-}^{(d)}\Lambda_{-}^{(0)}+\Lambda_{-}^{(0)}\Lambda_{-}^{(d)}+(\Lambda_{-}^{(d)})^2 . \quad (4.5)$$

We get

$$\mathrm{Tr}[(\Lambda_{-}^{(d)})^2+2\Lambda_{-}^{(d)}\Lambda_{-}^{(0)}]=\mathrm{Tr}\Lambda_{-}^{(d)}=0$$

and from

$$\Lambda_{-}^{(0)}(\vec{k})(\vec{\alpha}\cdot\hat{k})=-k\Lambda_{-}^{(0)}(\vec{k})$$

we obtain $\Delta\mathscr{E}$ as a quadratic form:

$$\Delta\mathscr{E}=3\sum_{\vec{k}}2B^{(0)}(k)\mathrm{Tr}\{[\Lambda_{-}^{(d)}(\vec{k})]^2\Lambda_{+}^{(0)}(\vec{k})\}+4\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k},\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\mathrm{Tr}[\Lambda_{-}^{(d)}(\vec{k})\Lambda_{+}^{(d)}(\vec{k}')] . \quad (4.6)$$

Since we have $\Delta\mathscr{E}$ as a quadratic form, we can treat $\Lambda_{-}^{(d)}$ as a differential and linearize the projector constraint,

$$\Lambda_{-}^{(d)}\Lambda_{-}^{(0)}+\Lambda_{-}^{(0)}\Lambda_{-}^{(d)}=\Lambda_{-}^{(d)} . \quad (4.7)$$

To linearize it amounts to approximating $\Lambda_{-}^{(d)}$ by

$$\Lambda_{-}^{(d)}\cong\varphi(k)\beta . \quad (4.8)$$

We get, therefore, since $\Lambda_{+}^{(d)}=-\Lambda_{-}^{(d)}$,

$$\Delta\mathscr{E}=6\sum_{\vec{k}}2B^{(0)}(k)[\varphi(k)]^2 - 8\frac{1}{(an)^3}\sum_{\vec{k},\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\varphi(k)\varphi(k') . \quad (4.9)$$

We are reduced to studying the nature of the stationary point $\Lambda_{-}^{(d)}=0$, i.e., $\varphi(k)=0$. We will have a sufficient condition of instability if we can find a normalizable function $\varphi(k)$ such that $\Delta\mathscr{E}<0$. We will proceed in this way in proving the instability, but let us first define more precisely the instability. Since we are interested in finding a function $\varphi(k)$ such that the quadratic form (4.9) is negative, we can fix its length and look for a negative value of (4.9) on the sphere

$$\frac{1}{(an)^3}\sum_{\vec{k}}[\varphi(k)]^2=1 . \quad (4.10)$$

If there is such a negative value, *a fortiori* the minimum on the sphere will be negative. We have then a new extremum problem: to find the stationary values on the sphere. The condition of instability will be that at least one of these values is negative. Using λ as Lagrange multiplier for the normalization constraint, the condition of extremum will be

$$\delta(\Delta\mathscr{E}+\mathscr{M})=0 , \quad (4.11)$$

$$\mathscr{M}=6(an)^3\lambda\left[\frac{1}{(an)^3}\sum_{\vec{k}}[\varphi(k)]^2-1\right] .$$

This gives

$$2B^{(0)}(k)\varphi(k)-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\varphi(k')=\lambda\varphi(k) . \quad (4.12)$$

This is the equation of instability. Multiplying by $\varphi(k)$ we see that we can rewrite $\Delta\mathscr{E}$ in the form

$$\Delta\mathscr{E}=6\lambda\sum_{\vec{k}}[\varphi(k)]^2 \quad (4.13)$$

and we see that the condition of instability amounts to find a negative eigenvalue of the basic equation (4.12). We can rewrite it in the form [from (3.13)]

$$2k\varphi(k)+\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[\varphi(k)(\hat{k}\cdot\hat{k}')-\varphi(k')] = \lambda\varphi(k) \,. \quad (4.14)$$

This equation can be viewed as a bound-state equation for the eigenvalues and eigenfunctions λ and $\varphi(k)$. The first term corresponds to the kinetic energy, the second to the self-energy, and the third to the potential energy. The condition of instability is $\lambda<0$; but, physically, λ does not strictly correspond to the energy of a $q\bar{q}$ pair, but includes effects of the creation and annihilation of two pairs, specific to the BV approximation.[4] The sum of the self-energy and the potential energy should be negative and dominate over the kinetic energy. Note that if we had started from $\mathscr{H}_n$ we would find as an equation of instability the expression (4.14) dropping in it the $\hat{k}\cdot\hat{k}'$ term, i.e., Casher's equation.[6] Fortunately, it is not necessary to solve completely the eigenvalue problem. It is enough to find a trial function $\varphi(k)$ for which (4.9) is negative. Let us go back to the expression (4.14). Multiplying by $\varphi(k)$ normalized to one, and going to the continuum limit, the minimal eigenvalue λ_0 will be given by

$$\lambda_0=\min_{\varphi}\left[\int\frac{d\vec{k}}{(2\pi)^3}2k[\varphi(k)]^2+\tfrac{4}{3}\int\frac{d\vec{k}}{(2\pi)^3}\frac{d\vec{k}'}{(2\pi)^3}\tilde{V}(\vec{k}-\vec{k}')\{(\hat{k}\cdot\hat{k}')[\varphi(k)]^2-\varphi(k)\varphi(k')\}\right] . \quad (4.15)$$

Symmetrizing this expression, we get

$$\lambda_0=\min_{\varphi}\left[\int\frac{d\vec{k}}{(2\pi)^3}2k[\varphi(k)]^2+\tfrac{4}{3}\tfrac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{d\vec{k}'}{(2\pi)^3}\tilde{V}(\vec{k}-\vec{k}')((\hat{k}\cdot\hat{k}')\{[\varphi(k)]^2+[\varphi(k')]^2\}-2\varphi(k)\varphi(k'))\right] . \quad (4.16)$$

To prove the instability it is enough to find a test function $\varphi(k)$ such that the quadratic form in (4.16) is negative. This will imply $\lambda_0<0$, and hence, instability. We see that the sign of the second term is far from being obvious since we have

$$\hat{k}\cdot\hat{k}'\le 1,\quad [\varphi(k)]^2+[\varphi(k')]^2\ge 2\varphi(k)\varphi(k') \,. \quad (4.17)$$

Before proving the instability for $\mathscr{H}$, let us see that if we start from $\mathscr{H}_n$, the normal ordering relative to the massless base, we cannot have vacuum instability due to the positivity of the confining potential. Expression (4.16) becomes, in this case, since we just drop the $\hat{k}\cdot\hat{k}'$ term,

$$\lambda_0=\min_{\varphi}\left[\int\frac{d\vec{k}}{(2\pi)^3}2k[\varphi(k)]^2-\tfrac{4}{3}\int\frac{d\vec{k}}{(2\pi)^3}\frac{d\vec{k}'}{(2\pi)^3}\tilde{V}(\vec{k}-\vec{k}')\varphi(k)\varphi(k')\right] . \quad (4.18)$$

Expressing the second term in configuration space, we obtain, since the convolution product becomes an ordinary product,

$$\lambda_0=\min_{\varphi}\left[\int\frac{d\vec{k}}{(2\pi)^3}2k[\varphi(k)]^2+\tfrac{4}{3}{V_0}^{1+\alpha}\int d\vec{r}[\tilde{\varphi}(r)]^2|\vec{r}|^{\alpha}\right] , \quad (4.19)$$

where we have used

$$\tilde{V}(\vec{k})=-{V_0}^{1+\alpha}\int d\vec{r}\,|\vec{r}|^{\alpha}e^{i\vec{k}\cdot\vec{r}} \quad (4.20)$$

and

$$\tilde{\varphi}(r)=\frac{1}{(2\pi)^3}\int d\vec{k}\,e^{-i\vec{k}\cdot\vec{r}}\varphi(k) \,. \quad (4.21)$$

The right-hand side of (4.19) is positive definite, and we cannot have, therefore, instability in this case.[6]

It is worth noting here that, if we start from $\mathscr{H}_n$—normal ordered Hamiltonian relatively to the massless base —and we consider a confining potential, but not positive, with a constant $U>0$ subtracted from it, we can have vacuum instability beyond some critical value $U^{\rm crit}$. For a potential

$$V(r)=-{V_0}^{1+\alpha}r^{\alpha}+U \quad (4.22)$$

($U>0$) we get, instead of Eq. (4.19),

$$\lambda_0=\min_{\varphi}\left[\int\frac{d\vec{k}}{(2\pi)^3}2k[\varphi(k)]^2+\tfrac{4}{3}{V_0}^{1+\alpha}\int d\vec{r}[\tilde{\varphi}(r)]^2r^{\alpha}-\tfrac{4}{3}U\int d\vec{r}[\tilde{\varphi}(r)]^2\right] . \quad (4.23)$$

It is clear that, given V_0, for large enough U, we can always find a test function $\varphi(k)$ such that the quantity in brackets in (4.23) becomes negative. But this term U in (4.22) does not play a role in the instability when we adopt the Hamiltonian $\mathscr{H}$. In this case, U does not appear in the right-hand side of (4.16). In $\vec{k}$ space, the constant term is proportional to $U\delta(\vec{k})$ and it cancels between the self-energy and the potential energy. The gap equation (3.10) is also independent of this type of terms added to the potential. However, as we will see in Sec. IX, even if $\varphi(k)$ is independent of this term, the functions $A(k)$, $B(k)$, and $E(k)$ depend on it.

We cannot go any further without specifying the potential. We will see that indeed the chiral-invariant vacuum is unstable for a powerlike potential r^α $(0<\alpha<3)$. The proof is relatively simple and enlightening for the linear potential. We will first perform the calculation for this case.[16] For a general powerlike potential, we will give a proof in Sec. VI.

V. PROOF OF THE INSTABILITY FOR A LINEAR POTENTIAL

To compute the Fourier transform $\tilde{V}(\vec{k})$ we will adopt as an expression for $|\vec{r}|$ the limit

$$r=\lim_{m\to 0}\frac{2}{m^2}\frac{(e^{-mr}-1+mr)}{r}\ . \tag{5.1}$$

We consider r as the limit of the class of potentials of Fig. 2 as $m\to 0$. We will make the calculation for $m\neq 0$ and take the limit $m\to 0$ at the end. $\tilde{V}(\vec{k})$ is given by

$$\tilde{V}(\vec{k})=-V_0^2\int d\vec{r}\,|\vec{r}|\,e^{i\vec{k}\cdot\vec{r}}$$
$$=\lim_{m\to 0}8\pi V_0^2\left[\frac{1}{\vec{k}^2(\vec{k}^2+m^2)}-\frac{2\pi^2}{m}\delta(\vec{k})\right]\ . \tag{5.2}$$

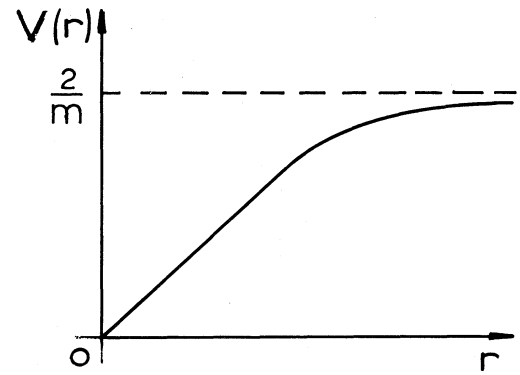

FIG. 2. The regularized potential (5.1). The linear potential is recovered as $m\to 0$.

The problem is solvable analytically by choosing for $\varphi(k)$ a Gaussian

$$\varphi(k)=(4\pi R^2)^{3/4}\exp(-R^2k^2/2) \tag{5.3}$$

normalized to unity

$$\int\frac{d\vec{k}}{(2\pi)^3}[\varphi(k)]^2=1\ . \tag{5.4}$$

The minimal eigenvalue λ_0 will be smaller than the value of the right-hand side of (4.16) for the Gaussian. We will now see that it is possible to choose the parameter R^2 such that this expression becomes negative. Therefore, the minimal eigenvalue is negative and we have instability. For the Gaussian, the last term in (4.16) is proportional to

$$I=\lim_{m\to 0}\int d\vec{k}\,d\vec{k}\,'\left[\frac{1}{(\vec{k}-\vec{k}\,')^2[(\vec{k}-\vec{k}\,')^2+m^2]}-\frac{2\pi^2}{m}\delta(\vec{k}-\vec{k}\,')\right]\exp[-R^2(k^2+k'^2)/2] \tag{5.5}$$

and we obtain, going to configuration space (4.21),

$$I=-\frac{2\pi^3}{R^2}\ . \tag{5.6}$$

The self-energy term can also be computed, and we get

$$F(\vec{k})=\int\frac{d\vec{k}\,'}{(2\pi)^3}\tilde{V}(\vec{k}-\vec{k}\,')(\hat{k}\cdot\hat{k}\,')$$
$$=\lim_{m\to 0}8\pi V_0^2\int\frac{d\vec{k}\,'}{(2\pi)^3}\left[\frac{1}{(\vec{k}-\vec{k}\,')^2[(\vec{k}-\vec{k}\,')^2+m^2]}-\frac{2\pi^2}{m}\delta(\vec{k}-\vec{k}\,')\right](\hat{k}\cdot\hat{k}\,')=-\frac{4}{\pi}V_0^2\frac{1}{k}\ . \tag{5.7}$$

This infrared singular behavior is a reflection of the growth of the linear potential at large distances. Averaging over the Gaussians this last result, we obtain, taking into account the sign of $\tilde{V}$ in Eq. (5.2),

$$\lambda_0\leq\frac{4}{R\sqrt{\pi}}\left[1-\frac{2V_0^2R^2}{3}\left[\frac{4}{\pi}-1\right]\right]\ . \tag{5.8}$$

The first term corresponds to the kinetic energy [first term in (4.16)], the second to the self-energy (term in $\hat{k}\cdot\hat{k}\,'$), and the third to the potential energy. We see that the self-energy has the right sign and dominates over the potential energy.

Since the kinetic term and the terms dependent on the potential scale differently, it is always possible to find an R^2 such that $\lambda_0 < 0$, for any value of the coupling constant $V_0{}^2$. We have therefore a sufficient condition of instability of the unbroken vacuum for a linear potential and any value of the coupling constant. If we had started from the Hamiltonian $\mathscr{H}_n$, we would have only the first and third term of (5.8), the right-hand side of (5.8) is then positive, as we know it is for any test function, according to expression (4.19). There is no instability in this case.

Let us now give some details of the calculation of I and $F(\vec{k})$. I does not have any particular problem since the quantity in brackets is just the Fourier transform of $-r/8\pi$. Performing the Fourier transform of the Gaussians, we get

$$\lim_{m\to 0} \int d\vec{k}\, d\vec{k}\,' \left[\frac{1}{(\vec{k}-\vec{k}\,')^2[(\vec{k}-\vec{k}\,')^2+m^2]} - \frac{2\pi^2}{m}\delta(\vec{k}-\vec{k}\,') \right] \varphi(k)\varphi(k') = -\frac{(2\pi)^6}{8\pi} \int d\vec{r}\, [\tilde{\varphi}(r)]^2 \, |\vec{r}\,| \tag{5.9}$$

with $\tilde{\varphi}(r)$ given by (4.21) normalized to unity,

$$\tilde{\varphi}(r) = (\pi R^2)^{-3/4} \exp(-r^2/2R^2), \quad \int d\vec{r}\, [\tilde{\varphi}(r)]^2 = 1 , \tag{5.10}$$

since the Fourier transform of a convolution of functions is the product of the Fourier transforms.

The calculation of $F(\vec{k})$ is more involved but without problem. Integrating over the angles,

$$F(k) = \frac{V_0{}^2}{\pi^2} \lim_{m\to 0} \left\{ \frac{\pi}{2k^2} \int_0^\infty dk' \left[\frac{(k^2+k'^2)}{m^2} \ln\frac{(k+k')^2}{(k+k')^2+m^2} - \frac{k^2+k'^2}{m^2} \ln\frac{(k-k')^2}{(k-k')^2+m^2} - \ln\frac{(k+k')^2+m^2}{(k-k')^2+m^2} \right] - \frac{2\pi^2}{m} \right\} . \tag{5.11}$$

Under the integral we have a singularity in $(k-k')^{-2}$ if we set $m=0$, but the last term, coming from the δ function, acts as regularization counterterm since we can write it, in the small-m limit,

$$-\frac{2\pi^2}{m} = -\frac{\pi}{k} + \frac{\pi}{2k^2} \int_0^\infty dk' \frac{2k^2}{m^2} \ln\frac{(k-k')^2}{(k-k')^2+m^2} + O(m) \tag{5.12}$$

and we have

$$F(k) = \frac{V_0{}^2}{\pi^2} \lim_{m\to 0} \left\{ -\frac{\pi}{k} + \frac{\pi}{2k^2} \int_0^\infty dk' \left[\frac{k^2+k'^2}{m^2} \ln\frac{(k+k')^2}{(k+k')^2+m^2} + \frac{(k^2-k')^2}{m^2} \ln\frac{(k-k')^2}{(k-k')^2+m^2} - \ln\frac{(k+k')^2+m^2}{(k-k')^2+m^2} \right] \right\} . \tag{5.13}$$

The second term under the integral is still singular, although less severely, in $(k-k')^{-1}$, when we set $m=0$. The limit $m\to 0$ exists and amounts to take the principal value

$$\lim_{m\to 0} \int_0^\infty dk' \frac{(k^2-k'^2)}{m^2} \ln\frac{(k-k')^2}{(k-k')^2+m^2} = -P \int_0^\infty dk' \left[\frac{k+k'}{k-k'} \right] . \tag{5.14}$$

We have

$$F(k) = \frac{V_0{}^2}{\pi^2} \left\{ -\frac{\pi}{k} + \frac{\pi}{2k^2} P \int_0^\infty dk' \left[-\frac{k^2+k'^2}{(k+k')^2} - \frac{k+k'}{k-k'} - 2\ln\left| \frac{k+k'}{k-k'} \right| \right] \right\} . \tag{5.15}$$

Splitting the integral $\int_0^\infty = \int_0^1 + \int_1^\infty$, reducing to a single integral between 0 and 1 by a change of variables, and using

$$\int_0^1 \frac{d\sigma}{\sigma} \left[\frac{1+\sigma^2}{2\sigma} \ln\left| \frac{1+\sigma}{1-\sigma} \right| - 1 \right] = 1 \tag{5.16}$$

we get (5.7).

In conclusion, we have shown that there is instability of the chiral-invariant vacuum for a linear fourth-component vector potential for any value of the coupling constant. The self-energy of massless quarks plays the crucial role, giving the right sign to produce instability, and dominates over the potential energy that has the wrong sign—being positive everywhere—to bind a negative-energy $q\bar{q}$ bound state.

Let us now make precise this point of the positivity of the potential and the relative role of the potential and the self-energies. A potential with a negative region, like (4.22), will shift—by the same amount—both the self and the potential energies. The difference will be the same, but their separate absolute values will be different. An extreme case is to consider the confining potential as the limit of the negative potentials of Fig. 3, that vanish as $r\to\infty$. This means that we take the confining potential to be the limit of potentials of the type [without the constant term in (5.1)]

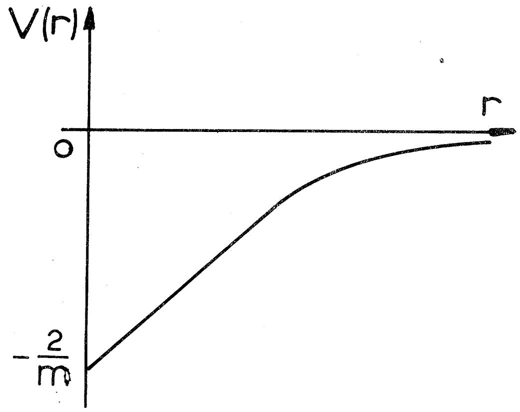

FIG. 3. The regularized potential (5.17), vanishing as $r \to \infty$. As $m \to 0$, one recovers the linear potential shifted by an infinite negative constant.

$$\lim_{m\to 0} \frac{2}{m^2} \frac{(e^{-mr}-1)}{r} . \tag{5.17}$$

The Fourier transform will be given now by the expression (5.2) dropping the δ-function term. This will mean that, in computing the self-energy we are adding to the result (5.7) a positive constant term that goes to infinity as $m \to 0$. Inversely, we are subtracting the same infinite constant to the potential term, the sum of both being invariant. The relative sign of the potential and the self-energies depend on how much we shift the potential allowing a negative piece, but the sum of both is the same as for the positive potential we have considered.

VI. PROOF OF THE INSTABILITY FOR A POWERLIKE POTENTIAL

A. Calculation of the self-energy

Let us first compute the self-energy

$$F(\vec{k}) = \int \frac{d\vec{k}'}{(2\pi)^3} \tilde{V}(\vec{k}-\vec{k}')(\hat{k}\cdot\hat{k}') \tag{6.1}$$

with

$$\tilde{V}(\vec{k}) = \int d\vec{r}\, V(r) e^{i\vec{k}\cdot\vec{r}} \tag{6.2}$$

and $V(r)$ regularized with an exponential (Fig. 4)

$$V(r) = -V_0{}^{1+\alpha} r^\alpha e^{-mr} . \tag{6.3}$$

The calculation is simpler with this regularization for a general powerlike potential. For a linear potential we will see that we recover the result of Sec. V. $\tilde{V}(\vec{k})$ is a distribution since $V(r) \to \infty$ as $r \to \infty$. To show more clearly the nature of the problems involved in the calculation of

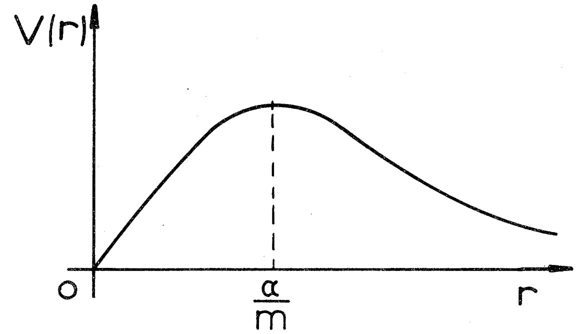

FIG. 4. The regularized potential (6.3). One gets a powerlike potential r^α as $m \to 0$.

the self-energy, we give in Appendix A an alternative way of computing it, without regularizing $V(\vec{r})$ and making use of the theory of distributions.

Integrating over the angles, we obtain for the Fourier transform $\tilde{V}(k)$,

$$\begin{aligned} \tilde{V}(\vec{k}) &= 2\pi \int_0^\infty r^2 dr \int_{-1}^{+1} du\, V(r) e^{ikru} \\ &= -V_0{}^{1+\alpha} \frac{4\pi}{k} \mathrm{Im} \int_0^\infty dr\, r^{\alpha+1} e^{-(m-ik)r} \\ &= -V_0{}^{1+\alpha} \frac{4\pi\Gamma(\alpha+2)}{k} \mathrm{Im} \frac{1}{(m-ik)^{\alpha+2}} . \end{aligned} \tag{6.4}$$

Let us now integrate over the angles the self-energy $F(\vec{k})$ [(6.1)]:

$$F(k) = \frac{1}{(2\pi)^2} \int_0^\infty k'^2 dk' \times \int_{-1}^{+1} du\, \tilde{V}[(k^2+k'^2-2ukk')^{1/2}]u . \tag{6.5}$$

Making the change of variables $v^2 = k^2+k'^2-2ukk'$, we get

$$F(k) = \frac{1}{8\pi^2} \frac{1}{k^2} \int_0^\infty dk' \int_{|k-k'|}^{k+k'} dv\, \tilde{V}(v) v (k^2+k'^2-v^2) . \tag{6.6}$$

It is more convenient to integrate over k' first. The integration limits $k'>0$, $k+k'>v>|k-k'|$ are equivalent to $v>0$, $k+v>k'>|k-v|$, so that

$$F(k) = \frac{1}{8\pi^2} \frac{1}{k^2} \int_0^\infty dv\, v\tilde{V}(v) \int_{|k-v|}^{k+v} dk' (k^2+k'^2-v^2) . \tag{6.7}$$

We get, performing the integration over k',

$$F(k) = \frac{1}{6\pi^2} \frac{1}{k^2} \left[\int_0^k dv\, v\tilde{V}(v)(3vk^2-v^3) + 2k^3 \int_k^\infty dv\, v\tilde{V}(v) \right] . \tag{6.8}$$

Making explicit $\tilde{V}(k)$ [(6.4)],

$$F(k)=-V_0{}^{1+\alpha}\frac{2\Gamma(\alpha+2)}{3\pi}\frac{1}{k^2}\mathrm{Im}\left[\int_0^k dv\frac{3vk^2-v^3}{(m-iv)^{\alpha+2}}+2k^3\int_k^\infty dv\frac{1}{(m-iv)^{\alpha+2}}\right]. \tag{6.9}$$

The integrand can be expressed as a combination of powers of $(m-iv)$ through $v=i(m-iv-m)$ and we obtain

$$F(k)=-V_0{}^{1+\alpha}\frac{2\Gamma(\alpha+2)}{3\pi}\frac{1}{k^2}\mathrm{Im}\left[\frac{1}{(\alpha-2)}\frac{1}{(m-ik)^{\alpha-2}}-\frac{3m}{(\alpha-1)}\frac{1}{(m-ik)^{\alpha-1}}\right.$$
$$\left.+3\frac{(m^2+k^2)}{\alpha}\frac{1}{(m-ik)^{\alpha}}-\frac{m^3+3mk^2-2ik^3}{(\alpha+1)}\frac{1}{(m-ik)^{\alpha+1}}\right]. \tag{6.10}$$

We can simplify this expression writing it in terms of powers of $(m-ik)$ setting $k=i(m-ik-m)$ and we obtain, finally,

$$F(k)=-V_0{}^{1+\alpha}\frac{4\Gamma(\alpha)}{\pi}\frac{1}{k^2}\mathrm{Im}\left[\frac{1}{(\alpha-2)}\frac{1}{(m-ik)^{\alpha-2}}-\frac{m}{(\alpha-1)}\frac{1}{(m-ik)^{\alpha-1}}\right] \tag{6.11}$$

or

$$F(k)=-V_0{}^{1+\alpha}\frac{4\Gamma(\alpha)}{\pi}\frac{1}{k^2}\left\{\frac{1}{(\alpha-1)}\frac{\sin[(\alpha-2)\mathrm{argtan}(k/m)]}{(m^2+k^2)^{(\alpha-2)/2}}-\frac{m}{(\alpha-1)}\frac{\sin[(\alpha-1)\mathrm{argtan}(k/m)]}{(m^2+k^2)^{(\alpha-1)/2}}\right\}. \tag{6.12}$$

There is no problem in taking the limit $m\to 0$, and we obtain

$$F(k)=-V_0{}^{1+\alpha}\frac{4\Gamma(\alpha)}{\pi}\frac{1}{k^\alpha}\frac{\sin[(\pi/2)(\alpha-2)]}{(\alpha-2)}. \tag{6.13}$$

Using now the relations

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z},\qquad \Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\tfrac{1}{2}), \tag{6.14}$$

the expression (6.13) can be written

$$F(k)=-V_0{}^{1+\alpha}\frac{1}{k^\alpha}\frac{2^\alpha}{\sqrt{\pi}}\frac{\Gamma((1+\alpha)/2)}{\Gamma((4-\alpha)/2)}. \tag{6.15}$$

The infrared singularity of this expression as $k\to 0$ is a reflection of the large-distance behavior. For $\alpha=1$ we obtain the expression (5.7) for a linear potential, independently of the regularization. This formula is valid for $\alpha>-1$, but will give an infrared divergence when $\alpha\geq 3$ within the integrals because of the $k^{-\alpha}$ behavior. For $\alpha=-1$ we find just the ultraviolet divergence of the self-energy for a Coulomb potential.

B. Gaussian test functions

Let us adopt again as test functions the Gaussians (5.3) and (5.10), and call Q the quadratic form in the right-hand side of (6.16):

$$Q=\int\frac{d\vec{k}}{(2\pi)^3}2k[\varphi(k)]^2+\tfrac{4}{3}\tfrac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{d\vec{k}'}{(2\pi)^3}\tilde{V}(\vec{k}-\vec{k}')\times((\hat{k}\cdot\hat{k}')\{[\varphi(k)]^2+[\varphi(k')]^2\}-2\varphi(k)\varphi(k')). \tag{6.16}$$

We will see that, given the potential strength V_0, Q can be made negative for some R^2, and therefore the minimal eigenvalue will be negative. In terms of the self-energy $F(k)$, we have

$$Q=\int\frac{d\vec{k}}{(2\pi)^3}2k[\varphi(k)]^2+\tfrac{4}{3}V_0{}^{1+\alpha}\int dr[\tilde{\varphi}(r)]^2r^\alpha+\tfrac{4}{3}\int\frac{d\vec{k}}{(2\pi)^3}F(k)[\varphi(k)]^2. \tag{6.17}$$

The first term in (6.17) has already been computed in Sec. V, and it scales like R^{-1}. Since the terms dependent on the potential scale differently than the kinetic term, like R^α $(\alpha>0)$, we only need to explore the sign of these terms.

Calling

$$\langle r^\alpha\rangle\equiv\int d\vec{r}[\tilde{\varphi}(r)]^2r^\alpha,\qquad \langle k^{-\alpha}\rangle\equiv\int\frac{d\vec{k}}{(2\pi)^3}[\varphi(k)]^2k^{-\alpha}, \tag{6.18}$$

we obtain

$$Q=2\langle k\rangle+\tfrac{4}{3}V_0{}^{1+\alpha}\left[\langle r^\alpha\rangle-\frac{2^\alpha}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)}{\Gamma((4-\alpha)/2)}\langle k^{-\alpha}\rangle\right]. \tag{6.19}$$

We need to investigate the sign of the quantity in square brackets. Let us compute the mean values (6.18) in terms of the Gaussians. From the definition of the Euler function Γ, we get

$$\int d\vec{r}\,r^\alpha[\tilde{\varphi}(r)]^2=\frac{2}{\sqrt{\pi}}R^\alpha\Gamma\left[\frac{\alpha+3}{2}\right],\qquad \int\frac{d\vec{k}}{(2\pi)^3}k^{-\alpha}[\varphi(k)]^2=\frac{2}{\sqrt{\pi}}R^\alpha\Gamma\left[\frac{3-\alpha}{2}\right], \tag{6.20}$$

and Q can be written in the form

$$Q=\frac{4}{(\pi R^2)^{1/2}}+\tfrac{4}{3}V_0{}^{1+\alpha}\frac{2}{\sqrt{\pi}}R^{\alpha}\Gamma\left[\frac{\alpha+3}{2}\right]\times\left[1-\frac{2^{\alpha}}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)\Gamma((3-\alpha)/2)}{\Gamma((4-\alpha)/2)\Gamma((3+\alpha)/2)}\right]. \quad (6.21)$$

For $\alpha=1$ this expression reduces to (5.8). Using the relations (6.14) and $\Gamma(z+1)=z\Gamma(z)$, the function

$$G(\alpha)=\frac{2^{\alpha}}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)\Gamma((3-\alpha)/2)}{\Gamma((4-\alpha)/2)\Gamma((3+\alpha)/2)} \quad (6.22)$$

can be written in the form

$$G(\alpha)=\frac{8}{\pi}\frac{1}{(\alpha+1)}\frac{[\Gamma((3-\alpha)/2)]^2}{\Gamma(3-\alpha)}. \quad (6.23)$$

For integer α, G takes the values $F(0)=1$, $G(1)=4/\pi$, $G(2)=\frac{8}{3}$, and $G\to+\infty$ for $\alpha\to3$. For $\alpha=1$ (linear potential) we recover the result of Sec. V, the right-hand side of inequality (5.8). However, $G(\alpha)$ is not a monotonically increasing function for $0<\alpha<3$. Its shape is drawn in Fig. 5. We see that we have proved the vacuum instability for $0.3\lesssim\alpha<3$ since $G(\alpha)>1$ in this range, making Q negative by conveniently choosing R^2. However, we must improve the method to extend this result to $0<\alpha\lesssim0.3$. This is interesting not only to have a general result for any powerlike potential r^{α}, $0<\alpha<3$, but also for $\alpha\to0$ we can extend the result to logarithmic potentials, as we will now see.

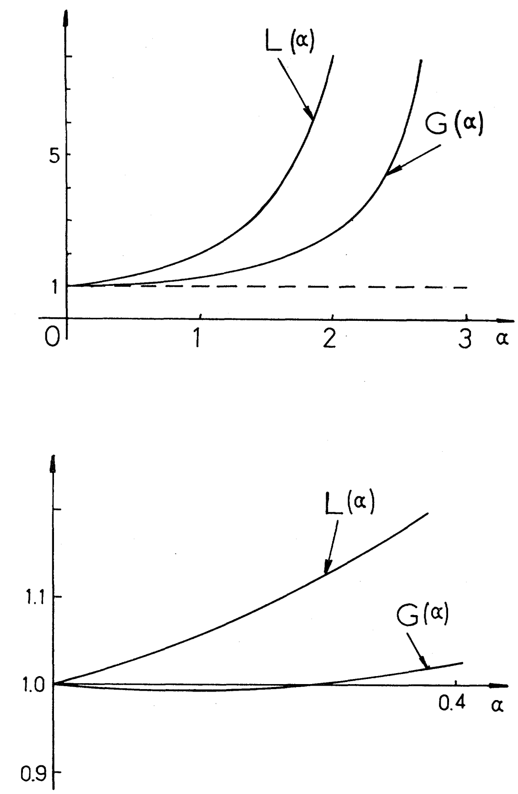

FIG. 5. The functions $G(\alpha)$ [(6.23)] and $L(\alpha)$ [(6.27)] of the exponent of the confining potential r^{α}.

C. General proof of the vacuum instability

Let us consider the quadratic form (6.19) for general test functions, not only Gaussians. Calling

$$H(\alpha)=\frac{2^{\alpha}}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)}{\Gamma((4-\alpha)/2)}\frac{\langle k^{-\alpha}\rangle}{\langle r^{\alpha}\rangle}, \quad (6.24)$$

we will prove that we can obtain $H(\alpha)>1$ for $0<\alpha<3$. If we take Gaussians, $H(\alpha)$ becomes equal to $G(\alpha)$.

In Appendix C we will prove the following equality:

$$\sup_{\varphi}\left[\frac{\langle k^{-\alpha}\rangle}{\langle r^{\alpha}\rangle}\right]=2^{-\alpha}\left[\frac{\Gamma((3-\alpha)/4)}{\Gamma((3+\alpha)/4)}\right]^2. \quad (6.25)$$

The mean value is understood in terms of normalizable functions $\varphi(k)$ whose domain is specified in Appendices B and C. In the right-hand side of (6.25), $3=d$ stands for the number of spatial dimensions.

From (6.25) we will have

$$\sup_{\varphi}H(\alpha)=L(\alpha) \quad (6.26)$$

with

$$L(\alpha)=\frac{1}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)}{\Gamma((4-\alpha)/2)}\left[\frac{\Gamma((3-\alpha)/4)}{\Gamma((3+\alpha)/4)}\right]^2. \quad (6.27)$$

From (6.26) we see that if we prove that $L(\alpha)>1$ for $0<\alpha<3$ we can always choose a convenient test function $\varphi(k)$ such that $H(\alpha)>1$ and we have, therefore, proved the instability.

Let us now give a heuristic way of obtaining the right-hand side of (6.25). The rigorous proof of (6.25) will be detailed in the Appendices.

Let us adopt as test functions, instead of Gaussians, the form

$$\tilde{\varphi}_{\beta}(r)=C_{\beta}r^{\beta}\exp(-r^2/2R^2) \quad (6.28)$$

and let us compute the ratio $\langle k^{-\alpha}\rangle/\langle r^{\alpha}\rangle$. The normalization constant will be fixed by

$$C_{\beta}{}^{-2}=\int d\vec{r}\,r^{2\beta}e^{-r^2/R^2}=(2\pi)R^{2\beta+3}\Gamma(\beta+\tfrac{3}{2}). \quad (6.29)$$

The mean value $\langle r^{\alpha}\rangle$ is easy to compute:

$$\langle r^{\alpha}\rangle=R^{\alpha}\frac{\Gamma(\beta+(\alpha+3)/2)}{\Gamma(\beta+\frac{3}{2})}. \quad (6.30)$$

$\langle k^{-\alpha}\rangle$ is more difficult to estimate since we need the Fourier transform of (6.28),

$$\varphi_{\beta}(k)=\int d\vec{r}\,e^{-i\vec{k}\cdot\vec{r}}\tilde{\varphi}_{\beta}(r), \quad (6.31)$$

normalized to unity according to

$$\int\frac{d\vec{k}}{(2\pi)^3}[\varphi_{\beta}(k)]^2=1. \quad (6.32)$$

In terms of $\tilde{\varphi}_{\beta}(r)$, $\langle k^{-\alpha}\rangle$ writes

$$\langle k^{-\alpha}\rangle=\int d\vec{r}\,d\vec{r}\,'\tilde{\varphi}_\beta^\dagger(r)\times\left[\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{k^\alpha}e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\right]\tilde{\varphi}_\beta(r')\;. \quad (6.33)$$

Computing the Fourier transform of $k^{-\alpha}$, we obtain

$$\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{k^\alpha}e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')}=\gamma_{\alpha,3}\frac{1}{|\vec{r}-\vec{r}\,'|^{3-\alpha}} \quad (6.34)$$

with

$$\gamma_{\alpha,3}=\frac{2^{(3-\alpha)/2}}{2^{\alpha/2}}\frac{\Gamma((3-\alpha)/2)}{\Gamma(\alpha/2)}\;. \quad (6.35)$$

Making the change of variable $\vec{r}\,'=r\vec{\rho}$ we obtain

$$\langle k^{-\alpha}\rangle=\frac{\gamma_{\alpha,3}}{(2\pi)^{3/2}}C_\beta^2\int d\vec{r}\,d\vec{\rho}\,\frac{r^{2\beta+\alpha}\rho^\beta e^{-r^2(1+\rho^2)/2R^2}}{|\hat{u}-\vec{\rho}\,|^{3-\alpha}}\;. \quad (6.36)$$

In this relation $\hat{u}$ is a fixed unit vector. Superficially we have $\hat{u}=\hat{r}$ in the integrand, but by rotational invariance we know that the integral is independent of the direction $\hat{r}$. Making now the change of variables

$$\frac{\vec{r}(1+\vec{\rho}\,^2)}{2R^2}=\vec{\lambda} \quad (6.37)$$

and integrating over $\vec{\lambda}$ we obtain

$$\langle k^{-\alpha}\rangle=\frac{\gamma_{\alpha,3}}{(2\pi)^{3/2}}\frac{\Gamma((3+\alpha)/2+\beta)}{\Gamma(\beta+\frac{3}{2})}R^\alpha\int d\vec{\rho}\,\frac{1}{\rho^{(3+\alpha)/2}}\frac{1}{|\hat{u}-\vec{\rho}\,|^{3-\alpha}}\left[\frac{2\rho}{1+\rho^2}\right]^{(3+2\beta+\alpha)/2}. \quad (6.38)$$

Therefore, we have for the ratio

$$\frac{\langle k^{-\alpha}\rangle}{\langle r^\alpha\rangle}=\frac{\gamma_{\alpha,3}}{(2\pi)^{3/2}}\int d\vec{\rho}\,\frac{1}{\rho^{(3+\alpha)/2}}\frac{1}{|\hat{u}-\vec{\rho}\,|^{3-\alpha}}\left[\frac{2\rho}{1+\rho^2}\right]^{(3+2\beta+\alpha)/2}. \quad (6.39)$$

Taking the limit $\beta\to-(3+\alpha)/2$ we obtain

$$\left[\frac{\langle k^{-\alpha}\rangle}{\langle r^\alpha\rangle}\right]_{\beta=-(3+\alpha)/2}=\frac{\gamma_{\alpha,3}}{(2\pi)^{3/2}}\int d\vec{\rho}\,\frac{1}{\rho^{(3+\alpha)/2}}\frac{1}{|\hat{u}-\vec{\rho}\,|^{3-\alpha}}\;. \quad (6.40)$$

This integral can be computed by successive Fourier transforms:

$$\left[\frac{\langle k^{-\alpha}\rangle}{\langle r^\alpha\rangle}\right]_{\beta=-(3+\alpha)/2}=\frac{1}{(2\pi)^{3/2}}\int d\vec{\rho}\,\frac{1}{\rho^{(3+\alpha)/2}}\int d\vec{p}\,\frac{e^{i\vec{p}\cdot(\hat{u}-\vec{\rho})}}{p^\alpha}$$

$$=\frac{\gamma_{(\alpha+3)/2,3}}{(2\pi)^{3/2}}\int d\vec{p}\,\frac{1}{p^{(\alpha+3)/2}}e^{i\vec{p}\cdot\hat{u}}=(\gamma_{(\alpha+3)/2,3})^2=2^{-\alpha}\left[\frac{\Gamma((3-\alpha)/4)}{\Gamma((3+\alpha)/4)}\right]^2. \quad (6.41)$$

The fact that this expression is just the $\mathrm{Sup}(\langle k^{-\alpha}\rangle/\langle r^\alpha\rangle)$ will be proved in Appendices B and C.

There is a problem, however, in the present form of the calculation: for $\beta\to-(3+\alpha)/2$ the functions (6.28) are not normalizable, as we can see from (6.29), since $\alpha>0$. We will show in Appendix B that our result is correct by conveniently choosing the domain of functions $\tilde{\varphi}(r)$ that differ from the simple form (6.28) at the limits $r\to0$ and $r\to\infty$.

Let us accept for the moment the result (6.26) and use it to prove the vacuum instability. Let us now see, indeed, that $L(\alpha)>1$ for $0<\alpha<3$. Since for $\alpha=0$ we have $L(0)=1$ it is enough to prove that the first derivative of $L(\alpha)$ is positive for $0<\alpha<3$. This is easy to see using the expression of the logarithmic derivative of $\Gamma(z)$,

$$\frac{\Gamma'(z)}{\Gamma(z)}=-\frac{1}{z}-\gamma-\sum_{n=1}^{\infty}\left[\frac{1}{z+n}-\frac{1}{n}\right], \quad (6.42)$$

where γ is Euler's constant. We will have

$$\frac{L'(\alpha)}{L(\alpha)}=\sum_{n=0}^{\infty}\left[-\frac{1}{1+\alpha+2n}+\frac{2}{3-\alpha+4n}-\frac{1}{4-\alpha+2n}+\frac{2}{3+\alpha+4n}\right]. \quad (6.43)$$

The right-hand side can be bounded by

$$\frac{L'(\alpha)}{L(\alpha)}\ge\sum_{n=0}^{\infty}\frac{\alpha(4-\alpha)(12n+9)}{(1+\alpha+2n)(4-\alpha+2n)(3-\alpha+4n)(3+\alpha+2n)} \quad (6.44)$$

for $0<\alpha<3$. Since $L'(\alpha)>0$, we see that $L(\alpha)$ is a monotonically increasing function from $L(0)=1$ to $L(\alpha)\to+\infty$ for $\alpha\to 3$ (Fig. 5). We have proved that $L(\alpha)>1$ for $0<\alpha<3$ and therefore vacuum instability. The method breaks down for $\alpha\geq 3$ due to the severe infrared singularities of such confining potentials.

D. Extension to logarithmic potentials

We can extend formally the proof of the instability to logarithmic potentials by taking the limit $\alpha\to 0$. We need to prove that the quantity in square brackets in (6.19),

$$\langle r^\alpha\rangle-\frac{2^\alpha}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)}{\Gamma((4-\alpha)/2)}\langle k^{-\alpha}\rangle\ , \tag{6.45}$$

is negative for $\alpha\to 0$. Making an expansion in powers of α ($r^\alpha\cong 1+\alpha\ln r+\cdots$), we must show that

$$\langle \ln r+\ln k\rangle-\langle \ln 2+\tfrac{1}{2}[\psi(\tfrac{1}{2})+\psi(2)]\rangle<0\ , \tag{6.46}$$

where the constant comes from the derivative at $\alpha=0$ of the function multiplying $\langle k^{-\alpha}\rangle$ in (6.45), and $\psi(z)=\Gamma'(z)/\Gamma(z)$.

We can now perform the same expansion in powers of the equality (6.26). We get

$$\mathrm{Inf}\langle \ln r+\ln k\rangle=\ln 2+\psi(\tfrac{3}{4})\ . \tag{6.47}$$

Therefore, we need

$$\psi(\tfrac{3}{4})<\tfrac{1}{2}[\psi(\tfrac{1}{2})+\psi(2)]\ . \tag{6.48}$$

This inequality holds, since

$$\psi(\tfrac{1}{2})=-\gamma-2\ln 2=-1.963\ ,$$

$$\psi(1)=-\gamma=-0.577\ ,$$

$$\psi(\tfrac{3}{4})=\frac{\pi}{2}-\gamma-3\ln 2=-1.085\ ,$$

and, from

$$\psi(z+1)=\frac{1}{z}+\psi(z)\ ,$$

$$\psi(2)=1+\psi(1)=0.423\ .$$

Our proof of the instability extends therefore to logarithmic confining potentials.

VII. SPIN-SPIN INTERACTION AND VACUUM INSTABILITY

Up to now, we have only considered a fourth-component-vector color potential. The main reason to adopt such a potential is twofold. On the one hand, we need indeed a vector color potential since it is attractive both in the singlet-$q\bar{q}$ and $\bar{3}$-qq channels and conserves chiral symmetry. On the other hand, as we have emphasized in the Introduction, we have a phenomenological reason to restrict ourselves to a fourth-component $(\bar{\psi}\gamma^0\psi)^2=(\psi^\dagger\psi)^2$ interaction. An interaction like $(\psi^\dagger\vec{\alpha}\psi)\cdot(\psi^\dagger\vec{\alpha}\psi)$ would lead to long-range spin-spin forces $\vec{\sigma}_1\cdot\vec{\sigma}_2\cdot\Delta V$ for massive quarks, since $\Delta r^\alpha\sim r^{\alpha-2}$. This spatial dependence is excluded by spectroscopy, which favors a pure Fermi contact interaction $\vec{\sigma}_1\cdot\vec{\sigma}_2\,\delta(\vec{r})$ attributed to gluon exchange. Of course, this is just a phenomenological argument, and one would like to know if the instability we have proved survives if a spin-spin term is present. Nobody knows theoretically the Lorentz structure of the confining potential, and we only have phenomenological hints about it. Let us assume, for our purpose of investigating the role of the spin-spin interaction in connection with vacuum instability, a Dirac structure

$$(\psi^\dagger\psi)(\psi^\dagger\psi)-\kappa(\psi^\dagger\vec{\alpha}\psi)\cdot(\psi^\dagger\vec{\alpha}\psi)\ , \tag{7.1}$$

where κ is some parameter.

Our formulas are modified in the following way. The vacuum energy $\mathscr{E}$, the one-fermion Hamiltonian, and the gap equation are now given by

$$\mathscr{E}=3\sum_{\vec{k}}\mathrm{Tr}[(\vec{\alpha}\cdot\vec{k})\Lambda_-(\vec{k})]+4\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k},\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\mathrm{Tr}[\Lambda_-(\vec{k})\Lambda_+(\vec{k}')-\kappa\vec{\alpha}\Lambda_-(\vec{k})\cdot\vec{\alpha}\Lambda_+(\vec{k}')]\ , \tag{7.2}$$

$$H(\vec{k})=\vec{\alpha}\cdot\vec{k}+\frac{1}{2}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\{[1-2\Lambda_-(\vec{k}')]-\kappa\vec{\alpha}\cdot[1-2\Lambda_-(\vec{k}')]\vec{\alpha}\}\ , \tag{7.3}$$

$$k\sin\varphi(k)=\frac{1}{2}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[(1+3\kappa)\sin\varphi(k')\cos\varphi(k)-(1+\kappa)\cos\varphi(k')\sin\varphi(k)(\hat{k}\cdot\hat{k}')]\ . \tag{7.4}$$

We see that the self-energy of a massless quark,

$$B^{(0)}(k)-k=\frac{1}{2}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')(1+\kappa)(\hat{k}\cdot\hat{k}')\ , \tag{7.5}$$

is affected by a different factor, $(1+\kappa)$, than the potential term $(1+3\kappa)$. The reason is that in the first case $\vec{\alpha}\cdot\vec{\alpha}$ projects on one-fermion state, and in the second on the $q\bar{q}\,0^{++}$ channel.

The instability equation (4.14) becomes now

$$2k\varphi(k)+\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[(1+\kappa)(\hat{k}\cdot\hat{k}')\varphi(k)-(1+3\kappa)\varphi(k')]=\lambda\varphi(k)\ . \tag{7.6}$$

We see that the relative role of the self-energy and the potential terms is changed by the spin-spin interaction. The quadratic form whose sign we must explore is now

$$Q=\int \frac{d\vec{k}}{(2\pi)^3}2k[\varphi(k)]^2+\tfrac{4}{3}V_0{}^{1+\alpha}\left[(1+3\kappa)\int d\vec{r}[\tilde{\varphi}(r)]^2r^{\alpha}-\frac{2^{\alpha}}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)}{\Gamma((4-\alpha)/2)}(1+\kappa)\int \frac{d\vec{k}}{(2\pi)^3}[\varphi(k)]^2k^{-\alpha}\right]. \tag{7.7}$$

We will have vacuum instability if the quantity in (7.7),

$$1-\frac{2^{\alpha}}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)}{\Gamma((4-\alpha)/2)}\frac{(1+\kappa)}{(1+3\kappa)}\frac{\langle k^{-\alpha}\rangle}{\langle r^{\alpha}\rangle}, \tag{7.8}$$

is negative. Since we have the relation (6.25), we will not have instability for all κ but only for κ satisfying the inequality

$$\frac{1+3\kappa}{1+\kappa}<\frac{1}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)}{\Gamma((4-\alpha)/2)}\left[\frac{\Gamma((3-\alpha)/4)}{\Gamma((3+\alpha)/4)}\right]^2. \tag{7.9}$$

For each value of α we will have a range of values for κ for which we will have still instability. For example, for

$$-1<\kappa<1 \quad (\alpha=1),$$
$$\kappa<-\tfrac{7}{5},\ \kappa>-1 \quad (\alpha=2).$$

We see, therefore, that for potentials growing with distance like a power r^{α} we can have a sizable spin-spin interaction for $\alpha \gtrsim 1$, having still vacuum instability. It is important to emphasize that for each power α there is a critical coupling κ^{crit} beyond which the stability of the chiral-invariant vacuum is restored. A covariant potential $(\bar{\psi}\gamma^{\mu}\psi)(\bar{\psi}\gamma^{\mu}\psi)$ corresponds to $\kappa=1$, the critical value for a linear potential. We must remember, however, that in QCD gauge invariance will relate the longitudinal and time components, leading presumably to an effective κ smaller than 1, as it happens for one-gluon exchange.

VIII. SOLUTIONS OF THE GAP EQUATION FOR THE HARMONIC-OSCILLATOR POTENTIAL (REF. 17)

The case of the harmonic-oscillator potential ($\alpha=2$) is of special interest because the gap equation is relatively easy to solve. The reason is as follows: The Fourier transform of $\vec{r}^{\,2}$ is just the Laplacian of a δ function in momentum space and the gap equation reduces in this case to a single nonlinear second-order differential equation.

$V(\vec{r})=-V_0{}^3\,\vec{r}^{\,2}$ has as Fourier transform

$$\tilde{V}(\vec{k})=-V_0{}^2\int d\vec{r}\,\vec{r}^{\,2}e^{-i\vec{k}\cdot\vec{r}}=V_0{}^3(2\pi)^3\Delta_{\vec{k}}\delta(\vec{k}). \tag{8.1}$$

If $f(k)$ is a radial function we have the relations

$$\Delta_{\vec{k}'}f(k')\Big|_{\vec{k}=\vec{k}'}=\frac{2}{k}f'(k)+f''(k),$$
$$\Delta_{\vec{k}'}f(k')(\hat{k}\cdot\hat{k}')\Big|_{\vec{k}=\vec{k}'}=-\frac{2}{k^2}f(k)+\frac{2}{k}f'(k)+f''(k). \tag{8.2}$$

Let us consider, for completeness, the general case with a spin-spin interaction, $\kappa\neq 0$.

The shift in vacuum energy *density* $\Delta\epsilon=\Delta\mathcal{E}/(an)^3$ will be given by

$$\Delta\epsilon=\frac{3}{2\pi^2}\int_0^{\infty}dk\{2k^3(1-\cos\varphi)-\tfrac{4}{3}V_0{}^3(1+\kappa)\sin\varphi+\tfrac{1}{2}\tfrac{4}{3}V_0{}^3k^2[(1+\kappa)+2\kappa\cos^2\varphi](\varphi')^2\} \tag{8.3}$$

and the gap equation

$$\tfrac{4}{3}V_0{}^3[(1+\kappa)+2\kappa\cos^2\varphi](k^2\varphi')'=2k^3\sin\varphi-\tfrac{4}{3}V_0{}^3(1+\kappa)\sin 2\varphi+\tfrac{4}{3}V_0{}^3\kappa k^2\sin 2\varphi(\varphi')^2 \tag{8.4}$$

with

$$A=E\sin\varphi,\quad B=E\cos\varphi, \tag{8.5}$$

and $E(k)$ given by

$$E(k)=\frac{1+3\kappa}{1+\kappa+2\kappa\cos^2\varphi}\left[k\cos\varphi-\tfrac{4}{3}V_0{}^3\frac{(1+\kappa)}{k^2}\cos^2\varphi-\tfrac{4}{3}V_0{}^3\tfrac{1}{2}(1+\kappa)(\varphi')^2\right]. \tag{8.6}$$

Let us solve the gap equation for $\kappa=0$, the pure fourth-component vector potential. We have, in this case,

$$\Delta\epsilon=\frac{3}{2\pi^2}\int_0^{\infty}dk[2k^3(1-\cos\varphi)-\tfrac{4}{3}V_0{}^3\sin^2\varphi+\tfrac{1}{2}\tfrac{4}{3}V_0{}^3k^2(\varphi')^2], \tag{8.7}$$

$$\tfrac{4}{3}V_0{}^3(k^2\varphi')'=2k^3\sin\varphi-\tfrac{4}{3}V_0{}^3\sin 2\varphi, \tag{8.8}$$

$$E=k\cos\varphi-\tfrac{4}{3}V_0{}^3\frac{1}{k^2}\cos^2\varphi-\tfrac{1}{2}\tfrac{4}{3}V_0{}^3(\varphi')^2. \tag{8.9}$$

Note that the gap equation (8.8) is very much reminiscent of the sine-Gordon equation. It is convenient from now on to perform the change of variables

$$(\tfrac{4}{3}V_0{}^3)^{-1/3}k \rightarrow k$$

with k now dimensionless. Our equations write

$$\Delta\epsilon = \frac{3}{2\pi^2}(\tfrac{4}{3}V_0{}^3)^{4/3} \times \int_0^\infty dk[2k^3(1-\cos\varphi)-\sin^2\varphi+\tfrac{1}{2}k^2(\varphi')^2] , \quad (8.10)$$

$$(k^2\varphi')' = 2k^3\sin\varphi - \sin2\varphi , \quad (8.11)$$

$$E = (\tfrac{4}{3}V_0{}^3)^{1/3}\left[k\cos\varphi - \frac{1}{k^2}\cos^2\varphi - \tfrac{1}{2}(\varphi')^2\right] , \quad (8.12)$$

and we see that E has dimensions of energy and $\Delta\epsilon$ dimensions of energy/volume = (energy)^4.

The gap equation (8.11) is a second-order differential equation; we need two conditions to fix a solution. For $k\rightarrow\infty$, the finiteness of the vacuum energy implies $(1-\cos\varphi)\rightarrow 0$. We have, asymptotically, since $k\rightarrow\infty$,

$$(k^2\varphi')' = 2k^3\varphi . \quad (8.13)$$

Therefore,

$$\varphi(k) \underset{k\rightarrow\infty}{\sim} \exp\left[-\frac{2\sqrt{2}}{3}k^{3/2}\right] . \quad (8.14)$$

This exponential behavior ensures the finiteness of the term in $(\varphi')^2$ in the vacuum energy (8.10).

Let us study the behavior as $k\rightarrow 0$. Let us try $\varphi\rightarrow 0$ as $k\rightarrow 0$. We have then

$$(k^2\varphi')' = -2\varphi \quad (8.15)$$

using a powerlike behavior, $\varphi\sim k^s$, we have $s(s+1)=-2$ and therefore there is no solution for $\mathrm{Re}\, s>0$. Trying now $\varphi\rightarrow\pi/2$ as $k\rightarrow 0$, we write $\varphi=\pi/2+\psi$, $\psi\sim k^s$. We obtain $s(s+1)=2$ and therefore $s=+1$ or $s=-2$. The behavior $s=+1$ leads to finite-energy density of the vacuum, and we will retain it.

In conclusion, we have as conditions at the limits for a well-behaved solution, for $k\rightarrow\infty$ the exponential behavior (8.14), and for $k\rightarrow 0$

$$\varphi(k) \underset{k\rightarrow 0}{\sim} \frac{\pi}{2} + ck . \quad (8.16)$$

With these two conditions we can integrate numerically the gap equation and we obtain the function $\varphi(k)$ given in Fig. 6. The slope at $k=0$ is $c=-2.0375$.

As we try steeper slopes we find other solutions that correspond to more and more nodes in k for the function $\varphi(k)$ (Fig. 7). We find, calling n the number of nodes,

$$\begin{aligned} c &= -2.0375 \quad (n=0) , \\ c &= -22.424 \quad (n=1) , \\ c &= -241.62 \quad (n=2) , \\ c &= -2597.8 \quad (n=3) , \\ c &= -27\,924 \quad (n=4) . \end{aligned} \quad (8.17)$$

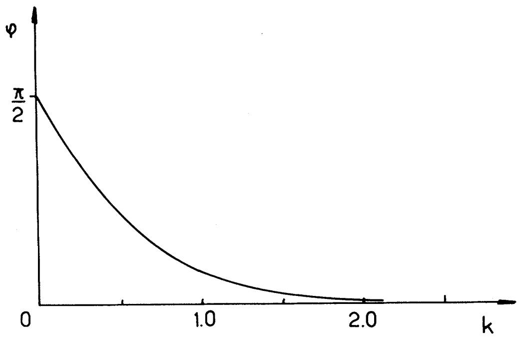

FIG. 6. The chiral-noninvariant solution, lowest in energy, of the gap equation for an r^2 potential, Eq. (8.11).

We have found an asymptotic formula of the slope c in terms of the number of nodes n that extrapolates very accurately even at small n:

$$c = -\exp\left[\frac{2}{\sqrt{7}}(0.9763+n\pi)\right] . \quad (8.18)$$

This formula gives

$$\begin{aligned} c &= -2.0918 \quad (n=0) , \\ c &= -22.485 \quad (n=1) , \\ c &= -241.69 \quad (n=2) , \\ c &= -2597.9 \quad (n=3) , \\ c &= -27\,926 \quad (n=4) \end{aligned} \quad (8.19)$$

to be compared with the numerical result (8.17). We have obtained the asymptotic formula by observing that the slope c increases very quickly with the number of nodes n.

For a solution with some nodes we have three different regions in k (Fig. 8). There is a region (1) where $k^3 \ll 1$ and φ is appreciable. In this region we can approximate the gap equation by

$$(k^2\varphi')' = -\sin2\varphi \quad (8.20)$$

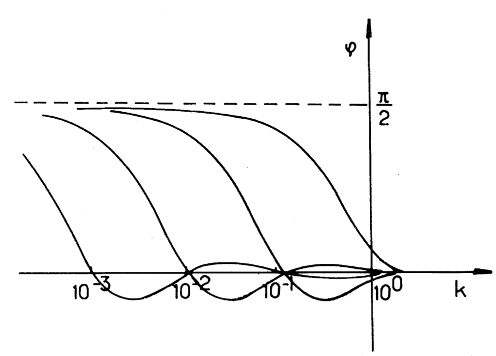

FIG. 7. Solutions of the gap equation (8.11) for an r^2 potential with 0, 1, 2, and 3 nodes.

with the condition as $k \to 0$, $\varphi^{(1)}(k) \cong \pi/2 + ck$. This equation is invariant by a scale transformation $k \to ck$: if $\varphi(k)$ is a solution, $\varphi(ck)$ is another solution.

We have a region (2) where, since the slope c grows very quickly with the number of nodes, k^3 remains still very small, $k^3 \ll 1$, and φ has become also small, $|\varphi| \ll 1$. In this region, the gap equation reduces to a linear differential equation,

$$(k^2\varphi')' = -2\varphi \ . \tag{8.21}$$

The solution in this region depends on two parameters:

$$\varphi^{(2)}(k) = \frac{\gamma}{\sqrt{k}} \cos\left[\frac{\sqrt{7}}{2} \ln k + \alpha\right] . \tag{8.22}$$

Finally, we have a third region (3) where k is large and φ oscillates remaining very small, $|\varphi| \ll 1$:

$$(k^2\varphi')' = 2(k^3 - 1)\varphi \ . \tag{8.23}$$

For $k \to \infty$ we recover in this region the asymptotic behavior (8.14). The conditions of continuity and derivability between the three regions will give us formula (8.18).

Since in region (1) the gap equation (8.20) is invariant by a scale transformation $k \to ck$, we can begin by imposing to this equation the condition as $k \to 0$, $\varphi(k) \cong \pi/2 - k$. We then solve numerically (8.20) imposing to have as asymptotic behavior when $k \to \infty$ a solution of the gap equation in region (2), i.e., $\varphi^{(2)}$ given by (8.22). We adjust α and γ to have the slope -1 when $k \to 0$. We obtain

$$\gamma = 0.701 \ , \quad \alpha = 0.2385 \ . \tag{8.24}$$

Therefore, a general solution of (8.20) with condition as $k \to 0$, $\varphi(k) \cong \pi/2 + ck$, will have the asymptotic behavior, as $k \to \infty$,

$$\varphi(k) \cong \frac{\gamma}{\sqrt{|c|k}} \cos\left[\frac{\sqrt{7}}{2} \ln|c|k + \alpha\right] \tag{8.25}$$

with α, γ given by (8.24).

We need now to solve equation (8.23) in region (3) with the behavior (8.25) at the limit $k \to 0$ and $\varphi \to 0$ as $k \to \infty$.

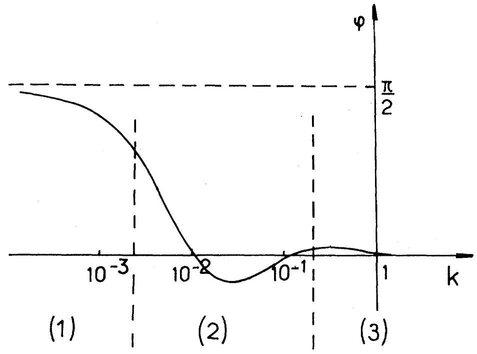

FIG. 8. The regions in k for a generic solution of the gap equation (8.11) defined in the text.

The behavior $\varphi \to 0$ as $k \to \infty$ fixes the constant inside the large parentheses of the cosines up to $n\pi$, and we get

$$\frac{\sqrt{7}}{2} \ln|c| + \alpha = 1.2148 + n\pi \tag{8.26}$$

(independently of γ) and from the value for α (8.24) we get formula (8.18).

IX. ENERGY DENSITY SHIFT, MASS GAP, AND VACUUM EXPECTATION VALUE OF $\bar{\psi}\psi$

We have enumerated the solutions of the gap equation for a harmonic-oscillator potential. Let us now see that the energy density shift $\Delta\epsilon$ between the chiral-invariant vacuum and any of these nontrivial solutions is negative, as we can already presume from the instability of the invariant solution. We will later see that the more stable solution, lowest in energy, is the one without any node. We will then compute the mass gap $A(k)$ and the vacuum expectation value of the operator $\bar{\psi}\psi$.

Let us come back to the general relations of the preceding sections, independent of the specific form of the potential. The energy shift from the chiral-invariant to a nontrivial solution is given by (4.3):

$$\begin{aligned}\Delta\mathscr{E} &= \mathscr{E} - \mathscr{E}^{(0)} \\ &= 3\sum_{\vec{k}} \mathrm{Tr}[H^{(0)}(\vec{k})\Lambda_-^d(\vec{k})] \\ &\quad + 4\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k},\vec{k}'} \tilde{V}(\vec{k}-\vec{k}')\mathrm{Tr}[\Lambda_-^{(d)}(\vec{k})\Lambda_+^{(d)}(\vec{k}')]\end{aligned} \tag{9.1}$$

with, from (3.13),

$$H^{(0)}(\vec{k}) = E^{(0)}(k)\vec{\alpha}\cdot\hat{k} \ , \tag{9.2}$$

$$E^{(0)}(k) = k + \frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k},\vec{k}'} \tilde{V}(\vec{k}-\vec{k}')(\hat{k}\cdot\hat{k}') \ . \tag{9.3}$$

Let us now see that if $\Lambda_-^{(d)}(\vec{k})$ satisfies the gap equation, we can express $\Delta\mathscr{E}$ in a simple and intuitive way. The gap equation tells us that the operator

$$H(\vec{k}) = \vec{\alpha}\cdot\vec{k} + \frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}'} \tilde{V}(\vec{k}-\vec{k}')[1 - 2\Lambda_-(\vec{k}')] \tag{9.4}$$

can be written as

$$H(\vec{k}) = E(k)[\sin\varphi(k)\beta + \cos\varphi(k)\vec{\alpha}\cdot\hat{k}] \tag{9.5}$$

if $\Lambda_-(\vec{k})$ is parametrized by (2.15). Since

$$H(\vec{k}) - H^{(0)}(\vec{k}) = \frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}'} \tilde{V}(\vec{k}-\vec{k}')2\Lambda_+^{(d)}(\vec{k}') \ , \tag{9.6}$$

we see that we can write the energy *density* shift in the form

$$\Delta\epsilon=\frac{3}{2}\frac{1}{(an)^3}\sum_{\vec{k}}\mathrm{Tr}\{[H(\vec{k})+H^{(0)}(\vec{k})]\Lambda_-^{(d)}(\vec{k})\} \quad (9.7)$$

and from (9.2), (9.5), and (2.15), we obtain

$$\Delta\epsilon=3\frac{1}{(an)^3}\sum_{\vec{k}}[E^{(0)}(k)-E(k)][1-\cos\varphi(k)] \quad (9.8)$$

with $E^{(0)}(k)$ given by (9.3) and $E(k)$ by (3.11). In terms of A, B, E, we can rewrite $\Delta\epsilon$ in the form

$$\Delta\epsilon=3\frac{1}{(an)^3}\sum_{\vec{k}}\frac{[E^{(0)}(k)-E(k)][A(k)]^2}{E(k)[E(k)+B(k)]} . \quad (9.9)$$

We see now the meaning of these expressions: A solution of the gap equation will correspond to a state lower in energy than the chiral-invariant one if, *on average*, $E^{(0)}-E<0$. $E^{(0)}$ and E are, respectively, the fermion energies corresponding to the chiral-invariant and the nonchiral-invariant vacuums.

The expectation value of $\bar{\psi}\psi$ will be given by

$$\langle\Omega\,|\,\bar{\psi}\psi\,|\,\Omega\rangle=\frac{3}{(an)^3}\sum_{\vec{k}}\mathrm{Tr}[\beta\Lambda_-(\vec{k})]$$
$$=-\frac{6}{(an)^3}\sum_{\vec{k}}\sin\varphi(k) , \quad (9.10)$$

where the factor 3 comes from color. Going to the continuum, infinite-volume limit, and restricting to the harmonic-oscillator potential, we have

$$\Delta\epsilon=\frac{3}{2\pi^2}\int_0^\infty k^2dk\,[E^{(0)}(k)-E(k)][1-\cos\varphi(k)] , \quad (9.11)$$

$$\langle\bar{\psi}\psi\rangle=-\frac{3}{\pi^2}\int_0^\infty k^2dk\,\sin\varphi(k) , \quad (9.12)$$

where $E(k)$ is given by (8.9), and we obtain $E^{(0)}$ making $\varphi=0$ in this expression. If we report (8.9) into (9.11), we do not get, at first sight, expression (8.7). One can easily see that both expressions are identical since we have

$$\int_0^\infty dk\{k^3\sin^2\varphi-(\tfrac{4}{3}V_0^2)[\sin^2\varphi-\tfrac{1}{2}k^2(\varphi')^2]\cos\varphi\}=0 . \quad (9.13)$$

This identity follows integrating by parts and making use of the gap equation (8.8) and the conditions at the limits $\varphi(0)=\pi/2$ and $\varphi(\infty)=0$.

Let us rewrite $\Delta\epsilon$ under the form

$$\Delta\epsilon=\frac{3}{2\pi^2}\int_0^\infty dk\left\{k^3(1-\cos\varphi)-\tfrac{4}{3}V_0^3\left[\sin\varphi-\frac{k^2}{2}(\varphi')^2\right]\right\}(1-\cos\varphi) . \quad (9.14)$$

Before performing numerical calculations, let us study qualitatively the behavior of $\Delta\epsilon$ and of $\langle\bar{\psi}\psi\rangle$ with the number of nodes or the slope as $k\to 0$, c [(8.16) and (8.18)]. The regions of integration in k that dominate for $\Delta\epsilon$ and $\langle\bar{\psi}\psi\rangle$ will be different. This is due to the fact that, for $\Delta\epsilon$, the measure k^2dk of the three-dimensional integration becomes dk because of the $1/k^2$ behavior of the self-energy. This does not happen for $\langle\bar{\psi}\psi\rangle$.

For $\Delta\epsilon$, the region (1) (Fig. 8) will dominate the integral. Making the very rough approximation $\varphi(k)\cong\pi/2+ck$, we have

$$\Delta\epsilon\cong\frac{3}{2\pi^2}(-\tfrac{4}{3}V_0^3)\int_0^{\pi/2|c|}dk\,\sin^2\varphi(k)[1-\cos\varphi(k)]$$
$$=-\frac{3}{2\pi^2}(\tfrac{4}{3}V_0^3)^{4/3}\frac{1}{|c|}\left[\frac{\pi}{2}-\frac{2}{3}\right] . \quad (9.15)$$

This approximation is of course bad mostly for the solution without nodes, but we find indeed, numerically, that $\Delta\epsilon$, being always negative, scales approximately like $1/|c|$. The lowest-energy solution corresponds to the solution without nodes, as expected on physical grounds. We obtain, numerically,

$$\Delta\epsilon=-\frac{3}{2\pi^2}(\tfrac{4}{3}V_0^3)^{4/3}\times 0.208\quad (n=0) ,$$
$$\Delta\epsilon=-\frac{3}{2\pi^2}(\tfrac{4}{3}V_0^3)^{4/3}\times 0.018\quad (n=1) . \quad (9.16)$$

We see from (9.12) that the region of integration dominating $\langle\bar{\psi}\psi\rangle$ will be the intermediate region (2) (Fig. 8) since high k will give the largest contribution. For this region, we have

$$\langle\bar{\psi}\psi\rangle=-\frac{3}{\pi^2}(\tfrac{4}{3}V_0^3)\int_{k_0}^{k_1}k^2dk\,\sin\left[\frac{\gamma}{\sqrt{|c|k}}\cos\left[\frac{\sqrt{7}}{2}\ln|c|k+\alpha\right]\right] , \quad (9.17)$$

where $k_0\cong-\pi/(2|c|)\cong 0$ for a large number of nodes. Since the sum $(\sqrt{7}/2)\ln|c|+\alpha$ is fixed by (8.26), we obtain, owing to the fact that k_0 is small, and k_1 independent of c,

$$\langle\bar{\psi}\psi\rangle\cong-\frac{3}{\pi^2}(\tfrac{4}{3}V_0^3)\frac{\gamma}{\sqrt{|c|}}\times\int_0^{k_1}k^2dk\frac{1}{\sqrt{k}}\cos\left[\frac{\sqrt{7}}{2}\ln k+\mathrm{const}\right] . \quad (9.18)$$

We find therefore that $\langle\bar{\psi}\psi\rangle$ will scale like $|c|^{-1/2}$. The numerical calculation confirms this behavior, even when we compare the solutions $n=0$ and $n=1$:

$$\langle\bar{\psi}\psi\rangle=-\frac{3}{\pi^2}(\tfrac{4}{3}V_0^3)0.3722\quad (n=0) ,$$
$$\langle\bar{\psi}\psi\rangle=-\frac{3}{\pi^2}(\tfrac{4}{3}V_0^3)0.1087\quad (n=1) . \quad (9.19)$$

Let us adopt a value for the strength of the harmonic-oscillator potential, $\frac{4}{3}V_0^3$. From Feynman, Kislinger, and Ravndal,[18]

$$\tfrac{4}{3}V_0{}^3 \cong (368\text{ MeV})^3 \tag{9.20}$$

we obtain

$$\langle \bar{\psi}\psi \rangle \cong -(178\text{ MeV})^3 \tag{9.21}$$

and

$$\Delta\epsilon \cong -(155\text{ MeV})^4 = -73\text{ MeV/fm}^3 \; . \tag{9.22}$$

The value obtained for $\langle \bar{\psi}\psi \rangle$ is to be compared with the recent estimation by Leutwyler,[19]

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle = -(250\text{ MeV})^3 \; . \tag{9.23}$$

We get the right magnitude, but about a factor 2 too small. This result is not disappointing: The harmonic oscillator is certainly a rough approximation for the confining region, and moreover the short-distance interaction can also contribute to spontaneous breaking of chiral symmetry and to the value of $\langle \bar{\psi}\psi \rangle$. On the other hand, the estimate (9.23) depends on the renormalization subtraction point and the related scale M for the QCD sum rules. It is not clear to us to which value of M we should compare our result. Leutwyler's results are consistent with $M \cong 1$ GeV.

Let us now discuss the mass gap $A(k)$. We adopt hereafter the lowest-energy solution $\varphi(k)$ without nodes. We find, from $E(k)$ [(3.11)] and $A = E\sin\varphi$, $B = E\cos\varphi$, the functions plotted in Fig. 9 together with $E(k)$ and $E^{(0)}(k)$, the energies of a massive and massless fermion. It is important to emphasize that the infrared singularity in $-1/k^2$ in $E^{(0)}(k)$ is removed for the chiral-noninvariant solution $E(k)$, since this term becomes $-\cos^2\varphi/k^2$, as we see in (8.9), and $\cos^2\varphi(k) \sim |c|^2k^2$ as $k \to 0$. We find $B(0)=0$ and $B(k) \to k$ as $k \to \infty$, and $A(0) \neq 0$, but negative. This is a surprising feature at first sight, but it can be interpreted easily. Our starting point was a positive-definite potential. We have obtained vacuum instability by the effect of the self-energy. All is clear up to now. But if we go now to the Bethe-Salpeter equation, where we have the same *positive potential* as kernel, the quadrilinear piece (2.9), we see that we cannot have the Goldstone realization of chiral symmetry, i.e., the existence in the spectrum of a massless pseudoscalar boson, bound state of *massive* quarks, unless the quarks have a dynamically generated *negative* mass. The negative mass of quarks is compensated by the positive binding energy of the confining potential, leading to a massless Goldstone boson. We have begun the study of the Bethe-Salpeter equation in our scheme, and proved that, as expected, the chiral invariance of the original Hamiltonian implies vacuum degeneracy and chiral invariance of the gap equation. This last property implies the existence of a massless boson in the Bethe-Salpeter equation. We postpone the study of the Bethe-Salpeter equation and the meson spectrum to another publication.

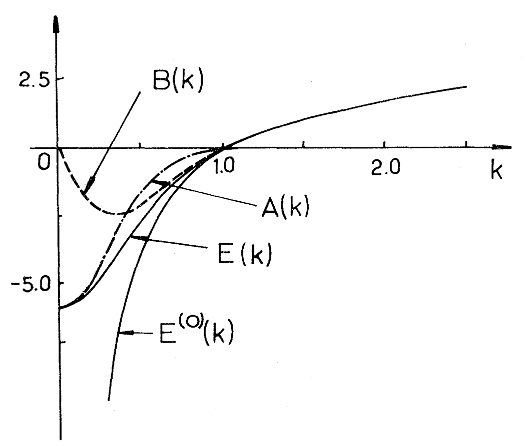

FIG. 9. The fermion mass term $A(k)$, kinetic term $B(k)$, and energy $E(k)$ for the lowest solution (Fig. 6) of the gap equation (8.11). For comparison, we plot $E^{(0)}(k)$, the energy of a massless fermion [Eq. (8.12) when $\varphi=0$]. $E^{(0)}(k)$ has a $-1/k^2$ singularity due to the self-energy [(6.15) for $\alpha=2$].

One may feel uneasy about a negative A; crudely speaking, this means a negative quark mass, and one may fear to get unphysical results, for instance, unphysical magnetic moments, having a wrong sign. It is true that to draw definite conclusions, one should proceed to the actual calculation of matrix elements in the Bethe-Salpeter scheme. Anyway, a negative A is not a necessary feature of our scheme. We can subtract a constant term from the confining potential, i.e., we take as in (4.22)

$$V(r) = -(V_0{}^3r^2 - U) \; . \tag{9.24}$$

In the situation we consider, without normal ordering the Hamiltonian, U does not modify the instability equation. Moreover, Eq. (3.10) for φ is independent of U since to add a constant U in (9.24) amounts to add to $\tilde{V}(\vec{k}-\vec{k}\,')$ the δ-function piece $(2\pi)^3\delta(\vec{k}-\vec{k}\,')U$. We get the solution of the new gap equation by just shifting E by a constant:

$$E(k) = (\tfrac{4}{3}V_0{}^3)^{1/3}\left[k\cos\varphi - \frac{1}{k^2}\cos^2\varphi - \tfrac{1}{2}(\varphi')^2\right] + \tfrac{2}{3}U \; . \tag{9.25}$$

To ensure $A(0)=E(0)>0$, one has the condition

$$-\tfrac{3}{2}(\tfrac{4}{3}V_0{}^3)^{1/3}|c|^2 + \tfrac{2}{3}U > 0 \; , \tag{9.26}$$

i.e.,

$$U > 3300\text{ MeV} \; . \tag{9.27}$$

One last comment on $A(k)$. We see, from the behavior of $\varphi(k)$ at large k [(8.14)], that $A(k) \to 0$ exponentially as $k \to \infty$. This behavior comes from the confining character of the potential. A short-distance piece like a Coulomb potential, dominating at large k, would restore the $A(k) \sim \langle \bar{\psi}\psi \rangle / k^2$ behavior expected from QCD.[20]

X. CONCLUSION

In conclusion, we have shown that the chiral-invariant vacuum is unstable for a large class of confining potentials. We have made explicit the precise mechanism at work, and the role of the self-energy in it. Moreover, we have been able to solve the gap equation in the particular case of the harmonic-oscillator potential. We have shown that there is an infinite number of solutions, noninvariant under chiral transformations, that have lower energy than the chiral-invariant one. The physical parameters that we obtain for the lowest energy solution are in qualitative agreement with the phenomenology.

One may of course wonder to what extent the variational method used is reliable for confining potentials. We found a chiral-noninvariant state which is lower in energy than the trivial chiral-invariant one we started from. But it may happen that there exists an even lower energy state which is chiral invariant. Drell *et al.*[21] exhibit such a case in the Nambu—Jona-Lasinio (NJL) model on a lattice. They show that for large coupling constant, although the variational method yields a chiral-symmetry-broken "ground state," the configuration-space approximation yields a "better" ground state (lower in energy) which is chiral invariant. The latter configuration is achieved by filling each site with a quark-antiquark pair and since the NJL potential is negative for a $q\bar{q}$ pair at the same point, this lowers the energy density. We guess that for a positive-definite potential between q and $\bar{q}$ the latter configuration should not be lower in energy than the trivial chiral-invariant "vacuum," and thus should not alter our conclusion. However, in the case of a potential negative at the origin one should perform a more careful study. Anyhow, even when we can prove that the configuration with pairs at each site is not the ground state, there may still exist another one outside our approximation scheme. We must not forget the limitations of variational methods which use a one-loop approximation to the self-energy.

What is still left to be done of this program? Keeping strictly to our model of an instantaneous Lorentz-vector confining potential, it would be very interesting to solve the Bethe-Salpeter equation. For the harmonic oscillator, it will reduce to a system of coupled linear differential equations. To solve this system is not a hopeless task, and we are studying the problem. It would be very nice if we could understand a number of problems of the light-quark meson spectrum: not only the pion, that comes out with zero mass automatically, but the masses of the isoscalar ϵ, 0^{++}, the ρ mass, and the one of the A_1, etc. Maybe we can have some new insight on the different splittings, quite different indeed from the nonrelativistic quark model corrected with the one-gluon-exchange spin-dependent perturbation. Related to the pion and the A_1—and their radial excitations—are the matrix elements of the axial current, the calculation of f_π, etc. All this should be examined also.

Beyond the strict frame of our model, it would be very interesting to see if the precise physical mechanism at work in the spontaneous breakdown of chiral symmetry that we have pointed out appears also in other effective theories of the confinement regime of the strong interactions, as in dual models, in the strong-coupling limit of lattice gauge theories, or in the $1/N$ limit of QCD. We think that our calculation within our phenomenological ansatz and the Bogoliubov-Valatin variational method may shed new light on the problem of dynamical chiral symmetry breaking in these approaches to the confinement.

Note added in proof. We acknowledge an enlightening discussion with Professor H. Leutwyler on the magnitude of the vacuum energy shift that has allowed us to point out a numerical error in the preprint version of the paper.

Dr. H. Kleinert has pointed out to us an interesting work on spontaneous breaking of chiral symmetry in QCD [H. Kleinert, Phys. Lett. 62B, 77 (1976)].

APPENDIX A

In Sec. VI we computed the self-energy (6.1) and obtained (6.15). Since the potential grows with the distance, its Fourier transform $\tilde{V}(\vec{k})$ is a distribution. In Sec. VI we have *defined* this distribution by a limit procedure. The integral of $\tilde{V}(\vec{k})$ with a smooth function $f(\vec{k})$ was defined by

$$K=\int \frac{d\vec{k}}{(2\pi)^3}\tilde{V}(\vec{k})f(\vec{k})$$

$$=\lim_{m\to 0}\int \frac{d\vec{k}}{(2\pi)^3}\tilde{V}_m(\vec{k})f(\vec{k}) . \tag{A1}$$

$\tilde{V}_m(\vec{k})$ is the Fourier transform of the regularized potential vanishing as $r\to\infty$ [(6.3)]. It is clear that $\tilde{V}_m(\vec{k})$ does not present the infrared problems of $\tilde{V}(\vec{k})$, reflection of the behavior of $V(\vec{r})$ as $r\to\infty$.

Another way of doing this calculation, that makes more transparent the difficulties linked to the fact that $\tilde{V}(\vec{k})$ is a distribution, is the following. Instead of regularizing $\tilde{V}(\vec{k})$ we will split the integral we want to compute,

$$F(\vec{k})=\int \frac{d\vec{k}\,'}{(2\pi)^3}\tilde{V}(\vec{k}-\vec{k}\,')(\hat{k}\cdot\hat{k}\,') \tag{A2}$$

into two pieces

$$F(\vec{k})=\int_{|\vec{k}-\vec{k}\,'|<\epsilon} d\vec{k}\,'(\cdots)+\int_{|\vec{k}-\vec{k}\,'|>\epsilon} d\vec{k}\,'(\cdots) . \tag{A3}$$

The first piece will present the infrared behavior we have pointed out. This will manifest as inverse powers of the infrared cutoff ϵ. However, as we will see, the dependence on ϵ will disappear at the end when summing up both integrals (A3). To illustrate the method, let us first consider the integral (A1):

$$K=K_1+K_2 , \tag{A4}$$

$$K_1=\int_{|\vec{k}|<\epsilon}\frac{d\vec{k}}{(2\pi)^3}f(\vec{k})\tilde{V}(\vec{k}) , \tag{A5}$$

$$K_2=\int_{|\vec{k}|>\epsilon}\frac{d\vec{k}}{(2\pi)^3}f(\vec{k})\tilde{V}(\vec{k}) . \tag{A6}$$

The integral K_2 does not present any particular problem. As for K_1, since ϵ is supposedly infinitesimal, we can expand it in powers of ϵ:

$$K_1=\int_{|\vec{k}|<\epsilon}\frac{d\vec{k}}{(2\pi)^3}\tilde{V}(\vec{k})f(\vec{k})$$
$$=c_0(\epsilon)f(0)+\epsilon c_1^i(\epsilon)\partial_i f(0)$$
$$+\epsilon^2 c_2^{ij}(\epsilon)\partial_i\partial_j f(0)+\cdots \quad (A7)$$

The coefficients c_0, c_1^i, c_2^{ij},... will depend on ϵ. This equality means that we have expressed the distribution $\tilde{V}(\vec{k})$, restricted to $|\vec{k}|<\epsilon$, as a linear combination of δ functions and their derivatives. By rotational symmetry, the expansion in the right-hand side of (A7) reduces to

$$c_0(\epsilon)f(0)+\epsilon^2 c_2(\epsilon)\Delta f(0)+\cdots \quad (A8)$$

Let us compute $c_0(\epsilon)$, $c_2(\epsilon)$, etc. We will see later that all the dependence on ϵ cancels when summing K_1 and K_2. To compute $c_0(\epsilon)f(0)$ we see from (A7) that we must simply estimate

$$\int_{|\vec{k}|<\epsilon}\frac{d\vec{k}}{(2\pi)^3}\tilde{V}(\vec{k})f(0)\ . \quad (A9)$$

But, since $V(\vec{0})=0$, we will have, integrating over all space,

$$\int\frac{d\vec{k}}{(2\pi)^3}\tilde{V}(\vec{k})=0\ . \quad (A10)$$

Therefore,

$$c_0(\epsilon)=\int_{|\vec{k}|<\epsilon}\frac{d\vec{k}}{(2\pi)^3}\tilde{V}(\vec{k})$$
$$=-\int_{|\vec{k}|>\epsilon}\frac{d\vec{k}}{(2\pi)^3}\tilde{V}(\vec{k})\ . \quad (A11)$$

We can compute without problem $\tilde{V}(\vec{k})$ for $|\vec{k}|>\epsilon$ and we get from (B12) and (6.4)

$$\tilde{V}(\vec{k})=\int d\vec{r}\,V(\vec{r})e^{-i\vec{k}\cdot\vec{r}}$$
$$=-V_0^{1+\alpha}(2\pi)^{3/2}\gamma_{-\alpha,3}\frac{1}{|\vec{k}|^{\alpha+3}} \quad (A12)$$

with $\gamma_{\alpha,d}$ given by

$$\gamma_{\alpha,d}=\frac{2^{(d-\alpha)/2}}{2^{\alpha/2}}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}\ . \quad (A13)$$

We get, therefore,

$$c_0(\epsilon)=V_0^{1+\alpha}\gamma_{-\alpha,3}\sqrt{2/\pi}\frac{1}{\alpha\epsilon^\alpha}\ . \quad (A14)$$

Let us now compute $c_2(\epsilon)$. The term $c_2(\epsilon)\epsilon^2$ will give a power $1/\epsilon^{\alpha-2}$. Therefore, $c_2(\epsilon)\to 0$ when $\epsilon\to 0$ and $\alpha<2$, and we need only to restrict ourselves to $\alpha>2$. We make an expansion of $f(\vec{k})-f(0)$ in powers of $\vec{k}$:

$$f(\vec{k})-f(0)=\vec{k}\cdot\vec{\nabla}f(0)+\frac{k_ik_j}{2}\partial_i\partial_j f(0)\ . \quad (A15)$$

From the symmetry of $\tilde{V}(\vec{k})$ we can write

$$\int_{|\vec{k}|<\epsilon}[f(\vec{k})-f(0)]\tilde{V}(\vec{k})d\vec{k}$$
$$=\int_{\vec{k}|<\epsilon}\frac{k_i^2}{2}\partial_i^2 f(0)\tilde{V}(\vec{k})d\vec{k}$$
$$=\int_{|\vec{k}|<\epsilon}\frac{k^2}{6}\Delta f(0)\tilde{V}(\vec{k})d\vec{k}\ . \quad (A16)$$

We have then, for $\alpha>2$, since $\Delta_{\vec{x}}V(\vec{x})|_{\vec{x}=0}=0$,

$$c_2(\epsilon)\epsilon^2=\int_{|\vec{k}|<\epsilon}\frac{k^2}{6}\tilde{V}(\vec{k})d\vec{k}$$
$$=-\int_{|\vec{k}|>\epsilon}\frac{k^2}{6}\tilde{V}(\vec{k})d\vec{k}$$
$$=V_0^{1+\alpha}\sqrt{2/\pi}\,\gamma_{-\alpha,3}\frac{1}{6}\frac{1}{(\alpha-2)\epsilon^{\alpha-2}}\ . \quad (A17)$$

Let us now go back to the original integral (A2). $F(\vec{k})$ will be equal to

$$F(\vec{k})=c_0(\epsilon)(\hat{k}\cdot\hat{k}')|_{\vec{k}=\vec{k}'}+c_2(\epsilon)\epsilon^2\Delta_{\vec{k}'}(\hat{k}\cdot\hat{k}')|_{\vec{k}=\vec{k}'}$$
$$+\int_{|\vec{k}-\vec{k}'|>\epsilon}\frac{d\vec{k}'}{(2\pi)^3}\tilde{V}(\vec{k}-\vec{k}')(\hat{k}\cdot\hat{k}') \quad (A18)$$

with $c_0(\epsilon)$ and $c_2(\epsilon)$ given by (A14) and (A17), and

$$\Delta_{\vec{k}'}(\hat{k}\cdot\hat{k}')|_{\vec{k}=\vec{k}'}=-\frac{2}{k^2}\ . \quad (A19)$$

The integral (A18) does not present any infrared problem. We will compute it and prove that the dependence on ϵ coming from the lower limit of the integral, $|\vec{k}-\vec{k}'|>\epsilon$, cancels with the terms $c_0(\epsilon)$, $c_2(\epsilon)\epsilon^2$. We will see that we are left with our result (6.15) obtained regularizing $\tilde{V}(\vec{k})$.

The integral in (A18) is given by

$$\int_{|\vec{k}-\vec{k}'|>\epsilon}\frac{d\vec{k}'}{(2\pi)^3}\tilde{V}(\vec{k}-\vec{k}')(\hat{k}\cdot\hat{k}')$$
$$=-\frac{V_0^{1+\alpha}}{(2\pi)^{3/2}}\gamma_{-\alpha,3}\int_{|\vec{k}-\vec{k}'|>\epsilon}d\vec{k}'\frac{(\hat{k}\cdot\hat{k}')}{|\vec{k}-\vec{k}'|^{\alpha+3}}\ . \quad (A20)$$

Performing the change of variables $u=(\vec{k}-\vec{k}')^2$, we obtain

$$J=\int_{|\vec{k}'-\vec{k}|>\epsilon}d\vec{k}'\frac{(\hat{k}\cdot\hat{k}')}{|\vec{k}-\vec{k}'|^{\alpha+3}}$$
$$=\frac{2\pi}{4k^2}\int_0^\infty dk'\int_{\max\{(k-k')^2,\epsilon^2\}}^{(k+k')^2}du\frac{k^2+k'^2-u^2}{u^{(\alpha+3)/2}}\ . \quad (A21)$$

J will decompose into two pieces,

$$J=\frac{\pi}{2k^2}\left[\int_{(k-k')^2<\epsilon^2}dk'\int_{\epsilon}^{(k+k')^2}du\frac{(k^2+k'^2-u^2)}{u^{(\alpha+3)/2}}+\int_{(k-k')^2>\epsilon^2}dk'\int_{(k-k')^2}^{(k+k')^2}du\frac{(k^2+k'^2-u^2)}{u^{(\alpha+3)/2}}\right]. \tag{A22}$$

The integral over u gives

$$\int_a^{(k+k')^2}du\frac{(k^2+k'^2-u^2)}{u^{(\alpha+3)/2}}=2[(k+k')^{-\alpha-1}-a^{-\alpha-1}]\left[\frac{(k+k')^2}{\alpha-1}-\frac{(k+k')^2}{\alpha+1}\right] \tag{A23}$$

with $a=\epsilon^2$ or $(k-k')^2$.

We obtain, performing the last integrations over k' and splitting the terms dependent or independent of ϵ,

$$J=\frac{4\pi}{k^\alpha}\frac{(-2)}{(-\alpha-1)(-\alpha)(-\alpha+2)}+\frac{4\pi\epsilon^{-\alpha}}{\alpha}+\frac{4\pi\epsilon^{-\alpha+2}}{3(-\alpha+2)k^2} \tag{A24}$$

and we get, for the integral (A20),

$$\int_{|k-k'|>\epsilon}\frac{d\vec{k}'}{(2\pi)^3}\tilde{V}(\vec{k}-\vec{k}')(\hat{k}\cdot\hat{k}')=-V_0{}^{1+\alpha}\gamma_{-\alpha,3}\sqrt{2/\pi}\left[\frac{1}{k^\alpha}\frac{(-2)}{(-\alpha-1)(-\alpha)(-\alpha+2)}+\frac{\epsilon^{-\alpha}}{\alpha}+\frac{\epsilon^{-\alpha+2}}{3(-\alpha+2)k^2}\right]. \tag{A25}$$

We see from this expression that the terms in ϵ cancel in (A18) if we take into account (A19). We obtain, finally, for the self-energy $F(\vec{k})$,

$$F(\vec{k})=-V_0{}^{1+\alpha}\gamma_{-\alpha,3}\sqrt{2/\pi}\frac{1}{k^\alpha}\frac{(-2)}{(-\alpha-1)(-\alpha)(-\alpha+2)}. \tag{A26}$$

Taking into account the explicit expression for $\gamma_{-\alpha,3}$ [(A13)], we obtain

$$F(\vec{k})=-V_0{}^{1+\alpha}\frac{2^\alpha}{\sqrt{\pi}}\frac{1}{k^\alpha}\frac{\Gamma((\alpha+1)/2)}{\Gamma((4-\alpha)/2)}, \tag{A27}$$

i.e., the same expression we found in Sec. VI regularizing the potential. We have shown therefore that the result is independent of the explicit regularization we have used.

APPENDIX B

We will prove here the relation

$$\operatorname*{Sup}_{\varphi\in L^2}\frac{\langle\varphi|r^{-\alpha/2}p^{-\alpha}r^{-\alpha/2}|\varphi\rangle}{\langle\varphi|\varphi\rangle}=2^{-\alpha}\left[\frac{\Gamma((d-\alpha)/4)}{\Gamma((d+\alpha)/4)}\right]^2, \tag{B1}$$

where d is the number of spatial dimensions. This relation can be found in Ref. 22 in a slightly different form. In this paper, the L^p norms of the operator $p^{-\alpha/2}r^{-\alpha/2}$ are calculated. The left-hand side of (B1) is just the square of the L^2 norm of this operator.

Later, in Appendix C, starting from (B1), we will prove the relation (6.25) used to demonstrate vacuum instability. Let us consider the norm of the operator

$$T_\alpha=\frac{1}{r^{\alpha/2}}\frac{1}{p^\alpha}\frac{1}{r^{\alpha/2}}, \tag{B2}$$

$$||T_\alpha||^2=\operatorname*{Sup}_{\substack{\varphi\in L^2\\ ||\varphi||=1}}\int|(T_\alpha\varphi)(\vec{r})|^2d\vec{r}. \tag{B3}$$

Let us consider functions of the form

$$\varphi_l(r)=f(r)P_l(\vec{r}), \tag{B4}$$

where $f(r)$ is a radial function and $P_l(\vec{r})$ is a homogeneous harmonic polynomial of degree l: $\Delta P_l(\vec{r})=0$. If the norm (B3) is defined relative to functions of the type (B4), we will first prove

$$\left|\left|\frac{1}{r^{\alpha/2}}\frac{1}{p^\alpha}\frac{1}{r^{\alpha/2}}\right|\right|_{l,d}=2^{-\alpha}\left[\frac{\Gamma((d+2l-\alpha)/4)}{\Gamma((d+2l+\alpha)/4)}\right]^2. \tag{B5}$$

Let us now show that to prove (B5) it is sufficient to restrict ourselves to radial functions. To see this it is enough to consider functions of the type

$$\varphi_{a,l}(\vec{r})=\exp\left[-\frac{ar^2}{2}\right]P_l(\vec{r}) \tag{B6}$$

since their linear combinations form a dense system. In what follows we will make use of this and other results from Fourier analysis in Euclidean spaces.[23] The Fourier transform of (B6) is

$$\begin{aligned}\tilde{\varphi}_{a,l}(\vec{p})&=\frac{1}{(2\pi)^{d/2}}\int\varphi_{a,l}(\vec{r})e^{-i\vec{p}\cdot\vec{r}}\\&=\frac{(-i)^l}{a^{l+d/2}}\exp(-\vec{p}^{\,2}/2a)P_l(\vec{p}).\end{aligned} \tag{B7}$$

Assuming $P_l(\vec{r})$ normalized,

$$\int d\vec{r}\,|P_l(\vec{r})|^2=1, \tag{B8}$$

we obtain, from the definition of the Γ function,

$$\int d\vec{r}\,r^\alpha\varphi^*_{b,l}(\vec{r})\varphi_{a,l}(\vec{r})=\frac{1}{2}\left[\frac{2}{a+b}\right]^{(d+2l+\alpha)/2}\Gamma\left[\frac{d+2l+\alpha}{2}\right], \tag{B9}$$

$$\int d\vec{p}\,p^{-\alpha}\tilde{\varphi}^*_{b,l}(\vec{p})\tilde{\varphi}_{a,l}(\vec{p})=\frac{1}{2}\frac{1}{(ab)^{l+d/2}}\left[\frac{2ab}{a+b}\right]^{(d+2l-\alpha)/2}\Gamma\left[\frac{d+2l-\alpha}{2}\right].$$

We see from these expressions that the result depends on d and l through the combination $d+2l$. When considering the mean values in (B5) or the norm in (B4), it is therefore sufficient to restrict ourselves to radial functions $\varphi(|\vec{r}|)$. We will now prove, restricting the definition of the norm to such functions,

$$\left|\left|\frac{1}{r^{\alpha/2}p^{\alpha}r^{\alpha/2}}\right|\right|=2^{-\alpha}\left[\frac{\Gamma((d-\alpha)/4)}{\Gamma((d+\alpha)/4)}\right]^2 \tag{B10}$$

and we obtain from it, making $d\to d+2l$, relation (B5). We will prove (B10) in two steps. First, we will show that

$$\left|\left|\frac{1}{r^{\alpha/2}p^{\alpha}r^{\alpha/2}}\right|\right|\le 2^{-\alpha}\left[\frac{\Gamma((d-\alpha)/4)}{\Gamma((d+\alpha)/4)}\right]^2, \tag{B11}$$

and, second, that the equality holds.

We will make use several times of the formula $(0<\alpha<d)$

$$\mathscr{F}\frac{1}{p^{\alpha}}=\frac{1}{(2\pi)^{d/2}}\int d\vec{p}\,\frac{e^{i\vec{p}\cdot\vec{r}}}{p^{\alpha}}=\gamma_{\alpha,d}\frac{1}{r^{d-\alpha}} \tag{B12}$$

with

$$\gamma_{\alpha,d}=\frac{2^{(d-\alpha)/2}}{2^{\alpha/2}}\frac{\Gamma((d-\alpha)/2))}{\Gamma(\alpha/2)} . \tag{B13}$$

Equation (B12) follows from

$$\mathscr{F}e^{-p^2/2}=e^{-r^2/2} \tag{B14}$$

and

$$\int\frac{d\vec{p}}{p^{\alpha}}e^{-\vec{p}^2/2}=\gamma_{\alpha,d}\int\frac{d\vec{r}}{r^{d-\alpha}}e^{-\vec{r}^2/2} . \tag{B15}$$

Note that the relation

$$\gamma_{\alpha,d}\gamma_{d-\alpha,d}=1 \tag{B16}$$

expresses the reciprocity of the Fourier transform.

Applying the operator T_α [(B2)] to a radial function $\varphi(r)$, we obtain

$$(T_\alpha\varphi)(\vec{r})=\frac{\gamma_{\alpha,d}}{(2\pi)^{d/2}}\int d\vec{r}\,'\frac{(rr')^{-\alpha/2}}{|\vec{r}-\vec{r}\,'|^{d-\alpha}}\varphi(r') . \tag{B17}$$

We make now the change of variables $\vec{r}\,'=r\vec{\rho}$ and we obtain

$$(T_\alpha\varphi)(\vec{r})=\frac{\gamma_{\alpha,d}}{(2\pi)^{d/2}}\int d\vec{\rho}\frac{\rho^{-\alpha/2}}{|\hat{u}-\vec{\rho}|^{d-\alpha}}\varphi(r\rho) . \tag{B18}$$

Now u is a fixed unit vector, since the operator T_α [(B2)] is rotational invariant, $\varphi(r)$ is a radial function and therefore $(T_\alpha\varphi)(\vec{r})$ is independent of the direction of $\vec{r}$. We apply now the integral form of the Minkowski inequality,

$$\left|\left|\int dx\, v_x\right|\right|\le\int dx\,||v_x|| \tag{B19}$$

and we obtain

$$\begin{aligned}||T_\alpha\varphi||&=\left[\int d\vec{r}\,|(T_\alpha\varphi)(\vec{r})|^2\right]^{1/2}\\&=\frac{\gamma_{\alpha,d}}{(2\pi)^{d/2}}\int d\vec{\rho}\frac{\rho^{-\alpha/2}}{|\hat{u}-\vec{\rho}|^{d-\alpha}}\\&\quad\times\left[\int d\vec{r}\,|\varphi(r\rho)|^2\right]^{1/2} .\end{aligned} \tag{B20}$$

By a change of variables, we get

$$\left[\int d\vec{r}\,|\varphi(r\rho)|^2\right]^{1/2}=\rho^{-d/2}||\varphi|| \tag{B21}$$

and, finally,

$$||T_\alpha||\le\frac{\gamma_{\alpha,d}}{(2\pi)^{d/2}}\int d\vec{\rho}\frac{1}{\rho^{(\alpha+d)/2}}\frac{1}{|\hat{u}-\vec{\rho}|^{d-\alpha}} . \tag{B22}$$

The integral is of convolution type and can be computed by successive Fourier transforms, as we have done in (6.41), and we get the right-hand side of (B11).

Let us now show that we have the equality in (B11). For a symmetric operator like T_α one can show

$$\begin{aligned}||T_\alpha||&=\left[\operatorname*{Sup}_{\varphi}\int|(T_\alpha\varphi)(\vec{r})|^2d\vec{r}\right]^{1/2}\\&=\operatorname*{Sup}_{\varphi}\int\varphi^{\dagger}(\vec{r})(T_\alpha\varphi)(\vec{r})d\vec{r}\\&=\operatorname*{Sup}_{\varphi}\langle\varphi|T_\alpha|\varphi\rangle .\end{aligned} \tag{B23}$$

Let us now show that

$$\operatorname*{Sup}_{\varphi}\langle\varphi|T_\alpha|\varphi\rangle=2^{-\alpha}\left[\frac{\Gamma((d-\alpha/4))}{\Gamma((d+\alpha)/4)}\right]^2 . \tag{B24}$$

This maximum value is precisely obtained by the functions $\varphi_\beta(r)$ [(6.28)] normalized according to (6.29). We will now see that when $\beta\to-d/2$, the mean value $\langle\varphi_\beta|T_\alpha|\varphi_\beta\rangle$ converges towards the bound. We obtain, from (B18),

$$\langle\varphi_\beta|T_\alpha|\varphi_\beta\rangle=\frac{\gamma_{\alpha,d}}{(2\pi)^{d/2}}\int d\vec{r}\,d\vec{\rho}\frac{\rho^{-\alpha/2}}{|\hat{u}-\vec{\rho}|^{d-\alpha}}\varphi_\beta(r)\varphi_\beta(r\rho) . \tag{B25}$$

The integral over $\vec{r}$ gives

$$\int d\vec{r}\,\varphi_\beta(r)\varphi_\beta(r\rho)=\rho^{-d/2}\left[\frac{2\rho}{1+\rho^2}\right]^{2\beta+d} \tag{B26}$$

and we get, finally,

$$\langle\varphi_\beta|T_\alpha|\varphi_\beta\rangle=\frac{\gamma_{\alpha,d}}{(2\pi)^{d/2}}\int d\vec{\rho}\frac{1}{\rho^{(\alpha+d)/2}}\frac{1}{|\hat{u}-\vec{\rho}|^{d-\alpha}}\left[\frac{2\rho}{1+\rho^2}\right]^{2\beta+d} . \tag{B27}$$

Since $2\rho\le(1+\rho^2)$, the integrand is bound by the integrable function $[\rho^{(\alpha+d)/2}|\hat{u}-\vec{\rho}|^{d-\alpha}]^{-1}$, and we can perform the limit under the integral $\beta\to-d/2$. We get thus the right-hand side of (B22). This means that, by choosing β close enough to $-d/2$, we can approach the equality in (B22) as much as we want. We have proved therefore (B10) and hence (B5) and (B1). This is the result we will use in Appendix C.

APPENDIX C

We will prove here the relation

$$\sup_{\varphi\in D} \frac{\langle \varphi | p^{-\alpha} | \varphi \rangle}{\langle \varphi | r^{\alpha} | \varphi \rangle} = 2^{-\alpha} \left[\frac{\Gamma((d-\alpha)/4)}{\Gamma((d+\alpha)/4)} \right]^2 \tag{C1}$$

for a suitable functional domain D, starting from the relation proved in Appendix B:

$$\sup_{\psi\in L^2} \frac{\langle \psi | r^{-\alpha/2} p^{-\alpha} r^{-\alpha/2} | \psi \rangle}{\langle \psi | \psi \rangle} = 2^{-\alpha} \left[\frac{\Gamma((d-\alpha)/4)}{\Gamma((d+\alpha)/4)} \right]^2 , \tag{C2}$$

where d is the spatial dimension and $0<\alpha<d$.

Let us first discuss what the problem is and which are the requirements on the domain D in Eq. (C1). At first sight, Eq. (C1) follows directly from Eq. (C2), because if we put $\varphi(\vec{r}) = r^{-\alpha/2}\psi(\vec{r})$, we have

$$\frac{\langle \varphi | p^{-\alpha} | \varphi \rangle}{\langle \varphi | r^{\alpha} | \varphi \rangle} = \frac{\langle \psi | r^{-\alpha/2} p^{-\alpha} r^{-\alpha/2} | \psi \rangle}{\langle \psi | \psi \rangle} . \tag{C3}$$

However, we must be careful because ψ in L^2 does not imply $r^{-\alpha/2}\psi$ in L^2. It is not quite clear that besides finiteness of $\langle \varphi | r^{\alpha} | \varphi \rangle$ and $\langle \varphi | p^{-\alpha} | \varphi \rangle$ we need finiteness of $\langle \varphi | \varphi \rangle$, but we need finiteness of $\langle \varphi | p | \varphi \rangle$ to prove the vacuum instability. This is in fact a stronger condition [integrability of $p\,|\tilde{\varphi}(\vec{p})|^2$ and $p^{-\alpha}|\tilde{\varphi}(\vec{p})|^2$ imply integrability of $|\tilde{\varphi}(\vec{p})|^2$]. To see this, let us recall what is involved in our proof of the vacuum instability.

First, we have to find a trial function φ for which ($c>0$)

$$\langle \varphi | r^{\alpha} - c p^{-\alpha} | \varphi \rangle < 0 \tag{C4}$$

or, equivalently,

$$\frac{\langle \varphi | p^{-\alpha} | \varphi \rangle}{\langle \varphi | r^{\alpha} | \varphi \rangle} > \frac{1}{c} , \tag{C5}$$

where $-cp^{-\alpha}$ is the self-energy due to the potential r^{α}. Given (C1), we can find such a φ in D provided

$$2^{-\alpha} \left[\frac{\Gamma((d-\alpha)/4)}{\Gamma((d+\alpha)/4)} \right]^2 > \frac{1}{c} . \tag{C6}$$

Then it is argued that, by a change of scale, the negative term $\langle \varphi | r^{\alpha} - cp^{-\alpha} | \varphi \rangle$ can always overcome the positive kinetic energy term $\langle \varphi | p | \varphi \rangle < \infty$.

As an example illustrating the difficulty, let us consider the functions $\psi_\beta = r^{\beta} e^{-r^2}$, $\beta > -d/2$, which, as shown in Appendix B, give the upper bound in (C2) in the limit $\beta \to -d/2$. The corresponding functions $\varphi_\beta = r^{\beta-\alpha/2} e^{-r^2}$ do not satisfy $\langle p \rangle < \infty$ when $\beta < -d/2 + (1+\alpha)/2$ and are not even square integrable when $\beta < -d/2 + \alpha/2$, which happens when β is close enough to $-d/2$.

In conclusion, we need to prove Eq. (C1) for some domain D such that φ in D implies the finiteness of $\langle r^{\alpha} \rangle$, $\langle p^{-\alpha} \rangle$, and $\langle p \rangle$. We shall use the following definition: $\varphi \in D$ means

$$\begin{aligned} &(1)\ \varphi(\vec{r}) \text{ is infinitely derivable (smooth)},\\ &(2)\ \varphi(\vec{r}) \text{ vanishes for } |\vec{r}| \text{ big enough },\\ &(3)\ \varphi(\vec{r}) \text{ vanishes for } |\vec{r}| \text{ small enough }. \end{aligned} \tag{C7}$$

Some comments are in order. (1) and (2) imply $\varphi \in L^2$; (1) implies finiteness of $\langle p \rangle$ by rapid decrease of $\tilde{\varphi}(\vec{p})$; (2) implies finiteness of $\langle r^{\alpha} \rangle$ and finiteness of $\langle p^{-\alpha} \rangle$ for $\alpha < d$ by smoothness of $\tilde{\varphi}(\vec{p})$. Thus, D satisfies our requirements. The condition (3) only simplifies the proof of the following.

Proposition. Formula (C1) is valid with the functional domain D defined by conditions (C7).

Proof. The proof is in two steps; first we show that the set $r^{\alpha/2}D$ is dense in L^2 and next we show that the proposition follows from this fact.

One sees that a function φ has the properties (C7) if and only if the function $r^{\alpha/2}\varphi$ has the properties (C7), that is $r^{\alpha/2}D = D$, due to the smoothness of $r^{\alpha/2}$ and $r^{-\alpha/2}$ for $\vec{r} \neq 0$ [this is the why of condition (3): to make harmless the singularity of $r^{\alpha/2}$ and $r^{-\alpha/2}$ at $\vec{r}=0$]. We are reduced to show that D is dense in L^2, and this is standard practice in L^2 theory. First, the set D_1 of functions satisfying (2) and (3) is dense because one has, for $\psi \in L^2$,

$$\lim_{R\to\infty} \int_{|\vec{r}|>R} |\psi(\vec{r})|^2 d\vec{r} = \lim_{r_0\to 0} \int_{|\vec{r}|<r_0} |\psi(\vec{r})|^2 d\vec{r} = 0 . \tag{C8}$$

Next, by convolution with a smooth function with compact support, we can transform any function in D_1 into an arbitrary close function in D.

We have proved that $r^{\alpha/2}D$ is dense in L^2. Now, due to the continuity of the operator $r^{-\alpha/2}\, p^{-\alpha} r^{-\alpha/2}$, we may restrict ψ in Eq. (C2) to any dense subset of L^2, so that

$$\sup_{\psi\in r^{\alpha/2}D} \frac{\langle \psi | r^{-\alpha/2} p^{-\alpha} r^{-\alpha/2} | \psi \rangle}{\langle \psi | \psi \rangle} = 2^{-\alpha} \left[\frac{\Gamma((d-\alpha/4))}{\Gamma((d+\alpha)/4)} \right]^2 . \tag{C9}$$

Considering Eq. (C3), this is equivalent to Eq. (C1).

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