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# Finite-temperature QCD at large $N$

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The deconfining and chiral phase transitions of an SU($N$) gauge theory are discussed at large $N$.

Though it remains insoluble, an SU($N$) gauge theory simplifies greatly in the limit of large $N$.[1–3] In this paper, I consider the large-$N$ theory at finite temperature. I presume that the essential physics is unchanged as $N\to\infty$; for instance, that the large-$N$ theory confines. From a few plausible assumptions such as this, the possible phase transitions at large $N$ can be understood. As I shall make clear, in some crucial aspects, the phase transitions at small $N$ are not like those at large $N$. Even so, from large $N$ qualitative insight into the finite-temperature behavior of an SU(3) gauge theory can be gained.

I start with the large-$N$ theory without quarks, showing that if the deconfining transition is of second order at infinite $N$, then universality requires the glueballs to have a mass spectrum of the Hagedorn form.[4] I then observe that at large $N$ the presence of quarks does not significantly affect the deconfining transition. If there are massless quarks, I extend results of Coleman and Witten[5] at zero temperature to show that, as the temperature is raised, chiral symmetry cannot be restored before deconfinement. I conclude by considering how the results at large $N$ might be reflected in hadronic matter.

## I. LARGE $N$ WITHOUT QUARKS

I begin by reviewing why one expects a phase transition at finite temperature in an SU($N$) gauge theory without quarks.[6] The operator $\Omega(\vec{x})$, the Wilson line, is defined as

$$\Omega(\vec{x})\sim \mathrm{tr}P\exp\left[i\int_0^\beta gA_0(\vec{x},t)dt\right], \tag{1}$$

where $P$ denotes path ordering, and $A_0(\vec{x},t)$ lies in the fundamental representation of SU($N$). In Euclidean space-time at a temperature $T$, I choose a gauge where the fields $A_\mu(\vec{x},t)$ are strictly periodic in the time $t$ with period $\beta=T^{-1}$.

The Wilson line is invariant under any local gauge transformation. Consider, however, the global gauge transformation generated by the function $\Lambda(\vec{x},t)$, where $\Lambda(\vec{x},\beta)=\widetilde{\Lambda}\Lambda(\vec{x},0)$, with $\widetilde{\Lambda}$ a constant element of $Z(N)$. Since $\widetilde{\Lambda}$ commutes with all SU($N$) matrices, $\Lambda(\vec{x},t)$ is an allowed gauge transformation in that the gauge-transformed $A_\mu(\vec{x},t)$ remains strictly periodic in time. In contrast, under this gauge transformation, $\Omega(\vec{x})\to\widetilde{\Lambda}\Omega(\vec{x})$ for all $\vec{x}$. Hence $\Omega(\vec{x})$ probes the response of the gluonic vacuum to global $Z(N)$ flux.

By the usual arguments, $\langle\Omega\rangle$ is related to the free energy of an infinitely massive quark, so confinement at zero temperature obliges $\langle\Omega\rangle$ to vanish. The deconfining transition occurs at $T_d$, where $T_d$ is the greatest temperature for which $\langle\Omega\rangle=0$. Above $T_d$, the gluonic vacuum spontaneously breaks the global $Z(N)$ symmetry, and sources of $Z(N)$ flux have finite free energy.

For finite temperatures in four space-time dimensions, the deconfining transition can be described by a Landau-type theory in three space dimensions:

$$L_\Omega=\tfrac{1}{2}(\partial_i\Omega^*)(\partial_i\Omega)+\tfrac{1}{2}c_2\Omega^*\Omega+\frac{c_4}{N^2}(\Omega^*\Omega)^2+\frac{c_6}{N^4}(\Omega^*\Omega)^3\,. \tag{2}$$

For stability at large $\Omega$, $c_6>0$. In $L_\Omega$, as $\Omega$ is a complex-valued field, the global symmetry is actually U(1)=O(2). To break it to $Z(N)$, the term

$$L_{Z(N)}=c_{Z(N)}[\Omega^N+(\Omega^*)^N] \tag{3}$$

must be added.

At zero temperature, $c_2$ must be large and positive to ensure $\langle\Omega\rangle=0$. As the temperature increases, $c_2$ decreases, eventually becoming negative. This is opposite to what happens in ordinary spin systems, and occurs because the $Z(N)$ symmetry is broken, not restored, with increasing temperature.

$N=3$ is unique, in that because of $L_{Z(N)}$ alone, the deconfining transition is of first order.[6] For $N=2$ or $N\geq 4$, if $c_4>0$, there will be a second-order transition as $c_2\to 0$; if $c_4<0$, there will be a first-order transition for some value of $c_2>0$. I henceforth *assume* that $c_4>0$ for all $N$; equivalently, that the deconfining transition is of second order whenever $N\neq 3$. Numerical simulations indicate this is true for $N=2$,[7] while $N\geq 4$ remains an open question. For $N\geq 4$, as $L_{Z(N)}$ is a marginal or irrelevant operator, the critical behavior about $T_d$ will be that of an O(2) theory in three dimensions.

I take it for granted that, as has been proven for the lattice theory,[8] in the continuum, $T_d$ remains finite as $N\to\infty$.[9] That is, on a scale in which the glueball masses are $\sim O(1)$, $T_d\sim O(1)$. By the usual rules of the $N^{-1}$ expansion,[3] at large $N$ any connected $2m$-point function of the $\Omega$'s is $\sim N^{-2(m-1)}$, so in Eq. (2) the $c$'s are all of $O(1)$. Above $T_d$, $\langle\Omega\rangle\sim N$, and receives contributions from an infinite number of planar diagrams.

On the other hand, it is not at all apparent how there could be a second-order phase transition at infinite $N$.[10] For large $N$, any glueball decay amplitude is $\sim N^{-1}$, while any scattering amplitude is $\sim N^{-2}$.[3] At infinite $N$, the mass spectrum consists entirely of massive glueballs which are both stable and noninteracting. But how can an *ideal* gas of glueballs develop the thermodynamic singularities of a second-order phase transition?

For a single glueball of mass $m$, its contribution to the free energy is

$$F(m)=T\int \frac{d^3p}{(2\pi)^3}\ln\{1-\exp[-\beta(p^2+m^2)^{1/2}]\}$$

$$\underset{m\gg T}{\sim} -m^{3/2}T^{5/2}\exp(-m/T) . \tag{4}$$

The total free energy is

$$F_0(T)=\int_0^\infty \rho_G(m)F(m)dm , \tag{5}$$

where $\rho_G(m)$ is the density of glueball states. Since the glueballs behave as an ideal gas at infinite $N$, Eq. (5) is *exact.*

The contribution of any finite number of glueballs gives a free energy which is analytic for all $T$, so only the behavior of $\rho_G(m)$ as $m\to\infty$ matters. If $\rho_G(m)\sim\exp(m^a)$ as $m\to\infty$, then for $a<1$, $F_0(T)$ remains analytic for all $T$, while if $a>1$, $F_0(T)$ is infinite for any nonzero temperature. Thus, asymptotically $\rho_G(m)$ *must* be of the Hagedorn form

$$\rho_G(m) \underset{m\to\infty}{\sim} m^{-9/2+\alpha_G}\exp(m/T_H) , \tag{6}$$

where $T_H$ is the Hagedorn temperature for glueballs. From Eqs. (4)—(6), it is easy to show that[4,11,12]

$$F_0(T) \underset{T\to T_H^-}{\sim} -(T_H-T)^{2-\alpha_G}+\cdots , \tag{7}$$

where the terms neglected, such as those from light glueballs, are more regular as $T\to T_H^-$. Clearly, at infinite $N$ the deconfining transition occurs entirely because of the Hagedorn spectrum, with $T_H=T_d$. Further, from the form of Eq. (7), $\alpha_G=\alpha_{O(2)}$, where $\alpha_{O(2)}$ is the critical index for the specific heat of the O(2) transition. The best theoretical estimate for this O(2) index is[13] $\alpha_{O(2)}=-0.007\pm0.006$.

Statements about the theory above $T_d$ can also be made. At large $N$, the total free energy can be written as

$$F(T)=N^2F_2(T)+NF_1(T)+F_0(T)+\cdots . \tag{8}$$

For a theory without quarks, at large $N$ the expansion parameter is $N^{-2}$,[1] so all terms odd in $N$ ($F_1, F_{-1},\ldots$) vanish.

It is now necessary to observe that while the total number of glueballs is infinite, the degeneracy of a given glueball will be finite as $N\to\infty$. Then $\rho_G(m)\sim O(1)$, and as indicated by Eq. (5), below $T_d$ there are only contributions to $F_0$:[12]

$$F_2(T)=0, \quad T\le T_d . \tag{9}$$

In essence, this condition is what is meant by confinement—the $\sim N^2$ gluons are not asymptotic states, and contribute to the free energy only through their bound states. A more formal argument for Eq. (9) can be constructed by considering the series of gauge-invariant, local operators which can be used to excite glueballs, such as $\mathrm{tr}(G_{\alpha\beta})^2$, $\mathrm{tr}D_\delta(G_{\alpha\beta})^2$, $\mathrm{tr}D_\delta D_\gamma(G_{\alpha\beta})^2$, etc. There is an infinite string of these operators, but they correspond to glueballs with increasingly greater spin. For a glueball with a given spin, the number of independent operators which can excite it is inevitably finite, and essentially independent of $N$, as $N\to\infty$.

At very high temperatures, by asymptotic freedom the free energy will approach that for an ideal gas of $(N^2-1)$ gluons, and so $F_2(T)\neq0$ for $T>T_d$. Since $F_2$ is the dominant contribution to the free energy for $T>T_d$, by universality, the manner in which $F_2$ vanishes as $T\to T_d^+$ must mimic that as $T\to T_d^-$:

$$F_2(T) \underset{T\to T_d^+}{\sim} -(T-T_d)^{2-\alpha_{O(2)}} . \tag{10}$$

I note that if the deconfining transition is of first order at infinite $N$, then

$$F_2(T) \underset{T\to T_d^+}{\sim} f_2(T-T_d) . \tag{11}$$

The latent heat of the transition, $f_2T_dN^2+O(1)$, is determined solely by the free energy above $T_d$ to $\sim N^{-2}$. For a first-order transition, there is no interesting restriction on $\rho_G(m)$.

Whatever the order of the transition at infinite $N$, while all glueballs should be massive below $T_d$, this will not be true above $T_d$. When $N=\infty$, as the vacuum spontaneously breaks a continuous $[Z(\infty)]$ symmetry above $T_d$, there will be a single massless state, a "Goldstone glueball." The Goldstone glueball is excited by the Wilson line:

$$\langle\Omega^*(\vec{x})\Omega(0)\rangle-|\langle\Omega\rangle|^2 \underset{\substack{\vec{x}\to\infty\\ T>T_d}}{\sim} \frac{1}{|\vec{x}|} . \tag{12}$$

It is worth emphasizing that confinement below $T_d$ was essential to obtaining constraints on the mass spectra. For example, consider a self-coupled scalar field $\varphi$, where $\varphi$ is an $N\times N$ matrix. In a confining theory, $\rho(m)\sim O(1)$ continues to grow as $m\to\infty$. In the unconfined $\varphi$ theory there are only a few states with different mass, but one of them, say at $m=m'$, has $\rho(m')\sim N^2$ as $N\to\infty$. The $\varphi$ self-interactions, $\sim N^{-2}$ at large $N$, are balanced by the $N^2$ states at $m=m'$ to give effects of $\sim O(1)$. Likewise, for the $\varphi$ theory, $F_2\neq0$ at all temperatures.

By smoothness of the limit $N\to\infty$, a Hagedorn spectrum should persist at large but finite $N$. Since at large $N$ interactions are small ($\sim N^{-2}$), $T_H$ must provide an absolute upper bound to $T_d$; if the deconfining transition remains of second order, most likely $T_H=T_d$. Interactions will alter the relation between $\alpha_G$ and $\alpha_{O(2)}$, so $\alpha_G=\alpha_{O(2)}+O(N^{-2})$. The $Z(N)$ symmetry is discrete for finite $N$, so Eq. (12) $\sim\exp(-m_G|\vec{x}|)$ with $m_G$ the mass of the (almost) Goldstone glueball. The (mass)$^2$ of nearly Goldstone particles are typically linear in the parameter which breaks the symmetry. For the pure glue theory, this parameter is $N^{-2}$, so $m_G{}^2\sim N^{-2}$ at large $N$. This is similar to the $\eta'$, for which $m_{\eta'}{}^2\sim N^{-1}$.[14]

There is some indirect evidence for a Hagedorn spectrum at small $N$. Numerical simulations of an SU(2) lattice gauge theory[7] indicate that $T_d\sim200$ MeV is much lower than the mass of the lightest glueball, $m_0 \sim 1200$ MeV.[15] This could happen due to extremely strong interactions between the glueballs, but as the Boltzmann factor $\exp(-m_0/T_d) \sim 10^{-3}$ for the lightest glueball, it could be that $T_H = T_d$ for all $N \neq 3$. (For $N=3$, presumably $T_d < T_H$: A first-order transition occurs before the second-order transition controlled by the Hagedorn spectrum.) Even at large $N$, while $T_H$ and $m_0$ are each $\sim O(1)$, there is no reason why, numerically, $T_H$ could not be significantly less than $m_0$; indeed, this is rather natural for a rapidly increasing density of states. If $T_H$ is much smaller than $m_0$, as the SU(2) data suggest, then even the light glueballs are very rare about $T_d$. While the probability of an individual, very massive glueball is vanishingly small, there are so many heavy glueballs that they end up dominating the free energy below $T_d$.

## II. LARGE $N$ WITH QUARKS

For a finite number of colors, confinement is merely a qualitative notion in the presence of quarks. Quarks, which I take to lie in the fundamental representation of SU($N$), themselves carry $Z(N)$ flux, and so external sources of $Z(N)$ flux, screened by virtual $\bar{q}q$ pairs, will have finite free energy. As a heuristic example, the quark bilinear $\mathrm{tr}\bar{\psi}(\vec{x},\beta)\psi(\vec{x},0)$ is like the Wilson line, in that it is invariant under local gauge transformations, while under the global transformation $\Lambda(\vec{x},t)$ of Sec. I, it becomes $\tilde{\Lambda}^* \mathrm{tr}\bar{\psi}(\vec{x},\beta)\psi(\vec{x},0)$.

In contrast, even with quarks, confinement can be precisely characterized if there are an infinite number of colors. The coupling to quarks of any glueball, or that of gluonic operator such as the Wilson line, is suppressed at least by $\sim N^{-1}$, and vanishes at infinite $N$.[1-3] Thus $T_d$, the order of the deconfining transition, the glueball mass spectrum, etc., are all unaffected by the presence of quarks.[16] This is simply a consequence of there being $\sim N^2$ gluons but only $\sim N$ quarks at large $N$.

With quarks, besides glueballs, there will also be mesons, whose masses are $\sim O(1)$. (Baryons[3] have masses $\sim N$, and so can be ignored.) Below $T_d$, the total free energy is a sum of glueball and mesonic contributions,

$$F_0(T) = F_0^G(T) + F_0^M(T) , \tag{13}$$

where $F_0^G$ is given by Eq. (5), and

$$F_0^M(T) = \int_0^\infty \rho_m(m) F(m) dm \tag{14}$$

with $\rho_M(m) \sim O(1)$ the mesonic density of states. Since glueballs and mesons decouple at large $N$, however, there is no significant restriction on the asymptotic form of $\rho_M(m)$. It is plausible that mesons do have a Hagedorn spectrum with the same Hagedorn temperature as for glueballs, but this is not necessary.

With quarks, $F_1(T) \neq 0$ above $T_d$, while like $F_2(T)$, $F_1(T) = 0$ for $T \leq T_d$. $F_1(T)$ is not the dominant contribution to the free energy above $T_d$, so all that can be said about how $F_1(T)$ vanishes as $T \to T_d^+$ is that $F_1(T) \sim (T - T_d)^{2-\alpha_M}$, $\alpha_M \leq 1$.

At large but finite $N$, the effect quarks have on the deconfining transition can easily be estimated. As sources of $Z(N)$ flux, quarks will induce a value of $\langle \Omega \rangle \neq 0$, and so act as a background magnetic field for $\Omega$:[17]

$$I_q = h_q \Omega . \tag{15}$$

Relative to the $L_\Omega$ of Eq. (2), $L_q$ gives a $\langle \Omega \rangle \sim O(1)$ if $h_q \sim O(1)$. This is correct: without quarks, $\langle \Omega \rangle \sim N$ above $T_d$. At large $N$, each quark loop gives a factor of $N^{-1}$, so diagrams with a single quark loop yield $\langle \Omega \rangle \sim O(1)$ at all temperatures.

At finite $N$, if the deconfining phase transition is of second order without quarks, there is, in the thermodynamic sense, no true phase transition with quarks.[17] This is because the background magnetic field, due to the quarks, prevents the critical fluctuations from diverging as $T \to T_d$. For large $N$, $h_q \sim O(1)$ is small, and this only happens very near $T_d$. To estimate it, I use mean-field theory. With $c_2 \sim -t$, where $t$ is the reduced temperature $t = (T - T_d)/T_d$, $\langle \Omega \rangle \sim N t^{1/2}$ as $T \to T_d^+$. Then $L_\Omega \sim N^2 t^2$, $L_q \sim N t^{1/2}$, and quarks do not cut off the critical fluctuations until $L_\Omega \sim L_q$, or $t \sim N^{-2/3}$. Hence, while there is no true phase transition, at large $N$ there is a tremendous qualitative change in the theory, between a free energy of $\sim O(1)$, and a free energy of $\sim N^2$. This change occurs at a temperature which, to $\sim N^{-2/3}$, is that of $T_d$ in the pure glue theory.

If the deconfining transition is of first order at infinite $N$, the presence of quarks can only reduce the latent heat, $\sim N^2$ without quarks, by an amount $\sim N$. Then at large $N$, while there is no precise measure of confinement, there is still a (first-order) deconfining phase transition.

To derive more interesting results, I assume there are $N_f$ flavors of massless quarks. Since the $\eta'$ is a Goldstone boson for infinite $N$,[14] the global flavor symmetry is $U(N_f) \times U(N_f)$. Under very plausible assumptions, Coleman and Witten showed that, at zero temperature, the $N = \infty$ vacuum must spontaneously break $U(N_f) \times U(N_f)$ to $U(N_f)$.[5] I assume there is a finite temperature $T_{ch}$ at which the $U(N_f) \times U(N_f)$ symmetry is restored to the vacuum, and extend the arguments of Coleman and Witten to show $T_{ch} \geq T_d$. While there are reasons for believing $T_{ch} \geq T_d$ for any $N$,[18] it is worth knowing that this is unavoidable at large $N$.[19]

The logic of Coleman and Witten proceeds in two steps. The first is the observation that diagrams with a single quark loop dominate the effective potential for chiral-symmetry breaking at large $N$. This is valid for the effective potential at any temperature, and implies that either the chiral symmetry is broken completely to $U(N_f)$, or remains unbroken.[5] Thus there must be a single chiral transition, at which the full chiral symmetry is restored. [For a large number of flavors, it is easy to imagine that there might be several chiral transitions, with only partial restoration of the $U(N_f) \times U(N_f)$ symmetry until the last transition; this is ruled out at large $N$.]

The second step depends on the anomaly equation.[20] Consider the color-singlet, chiral current $j_\alpha = \bar{\psi} A (1 - \gamma_5) \gamma_\alpha \psi$, for some flavor matrix $A$. If $\Gamma_{\alpha\beta\delta}(p,q,r)$ is the three-point function for this current, where $p$, $q$, and $r$ are the three external momenta, $p + q + r = 0$, then[5]

$$r^{\delta}\Gamma_{\alpha\beta\delta}=\frac{N}{\pi^2}(\mathrm{Tr}A^3)\epsilon_{\alpha\beta\delta\gamma}p^{\delta}q^{\gamma}\ . \tag{16}$$

The anomaly equation remains valid at finite temperature.[21] Since the current $j_\alpha$ is a quark bilinear, in Euclidean space-time at a finite temperature, its momentum will be like that of a boson; e.g., if $p^\alpha=(p^0,\vec{p})$, $p^0=2\pi nT$, $n$ an integer, etc. In Eq. (16), I take each external momentum to have zero energy, and $\alpha$ to be timelike. Precisely as at zero temperature,[5] from the permutation symmetry of $\Gamma$ in the external momenta, at nonzero temperatures, $\Gamma$ will not be analytic at $\vec{p}=\vec{q}=\vec{r}=0$.

The crux of the matter is that $\Gamma$ is a physically measurable quantity, being a correlation function of gauge-invariant quantities. In order that $\Gamma$ not be analytic at $\vec{p}=\vec{q}=\vec{r}=0$, there must then be, as physical excitations, massless particles which couple to $j_\alpha$.[22]

If this massless particle is a scalar (i.e., a pion), by Goldstone's theorem the vacuum must spontaneously break the chiral symmetry; this happens for $T<T_{\rm ch}$. Without confinement, and with explicit chiral symmetry ($T>T_d$, $T_{\rm ch}$), massless quark ($\bar{q}q$) pairs will themselves saturate the anomaly at zero momentum.

Assume that $T_{\rm ch}$ were less than $T_d$.[23] For temperatures $T_{\rm ch}<T<T_d$, the nonanalyticity of $\Gamma$ cannot be due to scalar particles (as $T>T_{\rm ch}$), nor to bosons with spin 1 or greater,[5] nor to free quark pairs (as $T<T_d$). Therefore, when $T_{\rm ch}<T<T_d$, there *must* be at least one type of baryon, which I term a nucleon, which is exactly massless. I stress that the necessity of massless nucleons for $T_{\rm ch}<T<T_d$ follows only from the anomaly equation, and is valid for *any* $N$. In general, in a confining, chirally symmetric phase, all baryons need to be parity doubled. The anomaly equation shows something much stronger—that for some baryons, parity doubling must manifest itself via masslessness.

For two massless flavors, the appearance of massless baryons when $T_{\rm ch}<T<T_d$ can also be seen from a $\sigma$ model.[22,23] In a $\sigma$ model, $\langle\sigma\rangle\neq0$ below $T_{\rm ch}$, $\langle\sigma\rangle=0$ above $T_{\rm ch}$. Since $\langle\sigma\rangle$ is proportional to the nucleon's mass, the nucleons are massless above $T_{\rm ch}$. The anomaly equation demonstrates that this conclusion is not peculiar to the $\sigma$ model.

To satisfy the anomaly, however, the nucleons must not only be massless, but couple to $j_\alpha$. It is here that a contradiction is obtained at large $N$. $j_\alpha$ is a quark bilinear, while any baryon has $N$ quarks. As Witten observed,[3] if the probability for creating an additional $\bar{q}q$ pair from one $\bar{q}q$ pair is $x$ ($x<1$), the probability to produce $N-1$ extra pairs is $x^{N-1}$. At large $N$, the amplitude for $j_\alpha$ to pull any $\bar{B}B$ pair (massless or not) out of the vacuum vanishes exponentially at large $N$ $\sim\exp(-yN)$, $y=-\ln x>0$.

Therefore the anomaly equation cannot be satisfied if $T_{\rm ch}<T<T_d$, and so $T_{\rm ch}\geq T_d$ at large $N$. Since the gluons dominate the quarks at large $N$, prejudice favors $T_{\rm ch}=T_d$, but there is no simple reason why $T_{\rm ch}>T_d$ could not be true; rough arguments suggest $T_{\rm ch}$ cannot be much greater than $T_d$.[18]

What of the nature of the chiral transition? At large $N$, the $\epsilon$ expansion predicts that the chiral transition is of first order if there are two or more massless flavors.[24] Since the quarks only contribute $\sim N$ to the free energy, relative to $\sim N^2$ from the gluons, the chiral transition is unavoidably weakly first order.

I conclude this section by considering the large-$N$ theory in the presence of a quark chemical potential $\mu\neq0$. As has been seen, usually the effects of quarks are suppressed by $N^{-1}$. For any $N$, though, naively one expects that the phase diagram should be similar for $\mu$ and $T\neq0$, as when $\mu=0$, $T\neq0$; I argue that this is in fact true.

To generate a net density of quarks over antiquarks, a term $\sim\mu\psi^\dagger\psi$ is added to the quark Lagrangian. In a confining phase, $\mu\neq0$ can only manifest itself through a net density of baryons. While the baryon mass $m_B\sim N$,[3] since each quark color is affected by $\mu\neq0$, the natural scale for $\mu$ is when $N\mu\sim m_B$, or $\mu\sim O(1)$, as it is for the temperature.

Analogous to hadronic matter, I assume that the lightest baryons are those with the smallest possible total spin (e.g., spin $\frac{1}{2}$ for odd $N$, spin 0 for even $N$); then these baryons will be the first to condense. When $\mu\sim O(1)$, the density of quarks $d_q\sim N\mu^3\sim N$. If $p_f$ is the Fermi momentum of the baryons, the density of baryons is $d_B\sim p_f{}^3$. As each baryon is composed of $N$ quarks, $Nd_B\sim d_q$, and so the baryon $p_f\sim O(1)$. Note that while $d_B\sim O(1)$, the Fermi energy for the baryons is $\epsilon_f\sim p_f{}^2/2m_B\sim N^{-1}$, and the baryon condensate is highly nonrelativistic.

Because baryons are singlets under $Z(N)$, the $Z(N)$ symmetry is not broken merely by the presence of a baryon condensate. The effects of $\mu\neq0$ will manifest themselves indirectly, by the (virtual) interactions of the Wilson line with the baryon condensate. Following Witten,[3] the interactions of $\Omega$ with the condensate, like those of mesons or glueballs, are strong, $\sim O(1)$. For instance, single-gluon exchange is nominally $\sim N^{-1}$, but as the gluon can be exchanged with any one of $N$ quarks in the baryon, the cumulative effect is $\sim O(1)$. This is not in contradiction with the previous estimate of $\bar{B}B$ production from the vacuum, $\sim\exp(-yN)$. At $\mu\neq0$, the interactions of $\Omega$ with baryons are only with those in the condensate, and not with those in the vacuum *per se*. A typical process will be exciting a baryon, with momentum $p<p_f$ in the condensate, to a state with $p>p_f$, and so on; the momentum transfer $\sim O(1)$ is customary of glueball (or meson) scattering.

I showed above that for large $N$, it is meaningful to speak of a deconfining transition in the presence of quarks. The strong interactions of $\Omega$ with the baryon condensate should decrease $T_d$ by an amount $\sim O(1)$, for increasing $\mu\neq0$; similarly for $T_{\rm ch}$. When $\mu\neq0$, $T_{\rm ch}<T_d$ cannot be excluded as was possible at $\mu=0$. Nevertheless, this appears extremely unlikely for any $\mu$. Because of the anomaly equation (which of course remains valid for $\mu$ and $T\neq0$), if $T_{\rm ch}$ were $<T_d$ for some $\mu\neq0$, the baryons in the condensate would have to be massless for $T_{\rm ch}<T<T_d$. No matter how strong the interactions in and with the baryon condensate, it is difficult to imagine why, at $\mu\neq0$, a highly nonrelativistic condensate with $\epsilon_f\sim N^{-1}$ (for $T<T_{\rm ch}$), should become an extremely relativistic condensate with $\epsilon_f\sim O(1)$ (for $T_{\rm ch}<T<T_d$).

## III. HADRONIC MATTER AND LARGE $N$

The phase transitions at large $N$ may be a rough guide to those of hadronic matter, as described by SU(3) color with three light flavors.

There is experimental evidence for a Hagedorn spectrum in hadronic matter with $T_H \sim 160$ MeV.[4] Since the width of very high mass states is comparable to $T_H$, this value is not an absolute upper bound to whatever is meant by a deconfining transition at $T_d$. Even so, the value of $T_H$ suggests that, as for the pure glue SU(2) theory, $T_d(\sim 200$ MeV?) may be significantly less than the masses of most mesons and baryons. The observed value of the subleading power index for the Hagedorn spectrum, $\alpha_{\rm exp} \sim \frac{3}{2}$,[4] is not close to that for glueballs at infinite $N$, $\alpha_G = \alpha_{O(2)} \approx 0$. Granted the experimenal uncertainty in the determination of $\alpha_{\rm exp}$, and that $\alpha_{\rm exp}$ is measured from mesons and baryons, this does not necessarily mean that corrections in $N^{-1}$ to $\alpha_G \approx \alpha_{O(2)}$ are large at $N=3$.

To say the very least, in the present view, the Hagedorn spectrum is not the result of a limiting temperature, but that of a (more or less) commonplace phase transition in a confining theory.

Preliminary estimates indicate that the (first-order) deconfining transition in a pure glue SU(3) theory may be washed out by the quarks.[25] But at temperatures of $\sim 200$ MeV, surely there is a significant difference between a phase dominated by pions, and possibly the tail of Hagedorn spectrum, and a deconfined phase of quarks and gluons. After all, counting color and flavor degrees of freedom, in the former there are only three types of pions, while in the latter, there are eight gluons and six quarks. Perhaps the deconfining transition in hadronic matter is like that of a second-order deconfining transition at large $N$. That is, the free energy is analytic for all temperatures, but increases rapidly in value at some temperature which can be approximately characterized as $T_d$.

There remains the chiral phase transition, which almost certainly occurs in hadronic matter. In the end, the most dramatic effects may be associated with the chiral transition,[24] and not directly with deconfinement.

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[9]This will not hold below four space-time dimensions. In three space-time dimensions, as a discrete $Z(N)$, but not a continuous $Z(\infty)$, symmetry can be broken at finite temperatures, $T_d$, as defined in the text, will diverge as $N \rightarrow \infty$; see also Ref. 6. Similarly, in two space-time dimensions, the discrete $Z(N)$ symmetry cannot be broken at $T \neq 0$, so $T_d$ will be infinite for all $N$ (obviously, as without quarks the gluonic theory is trivial anyway).

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