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Dilation operator in gauge theories

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The electromagnetic field is expanded in a series of O(4) eigenstates of total spin, and quantized by specifying commutators on surfaces of constant $x_\mu x^\mu = R^2$ in four-dimensional Euclidean space. It is demonstrated that, under an arbitrary gauge transformation, some of the O(4) eigenstates are invariant; these gauge-invariant states are labeled by SU(2)⊗SU(2) total (orbital plus internal) spin quantum numbers (A,B) and with $A \neq B$. Only these gauge-invariant states are nontrivial in the absence of sources, and are quantized. The leading-twist quantum states of the dilation field theory contain the minimum number of these dilation photons. The remaining spin degrees of freedom of the electromagnetic field are most simply written as a function of the form $\partial_\mu \phi(x) + x_\mu \psi(x)/R^2$. $\phi(x)$ is obviously devoid of physics while $\psi(x)$ is a classical field propagating between radial projections of two electrical currents $x_\mu J^\mu(x)$ and $y_\mu J^\mu(y)$ only if $x_\mu x^\mu = y_\mu y^\mu$. The quantization procedure described herein may be applied to non-Abelian theories. The procedure does not lead to a gauge-invariant decomposition of a non-Abelian field, but the identification of leading-twist quantum states is preserved in the zero-coupling limit.

I. INTRODUCTION

Dilation field theory is to conventional (time-evolution) quantum field theory as spherical capacitor problems are to parallel-plate problems in electrostatics. When solving Laplace's equation or d'Alembert's equation, separation in Cartesian coordinates yields solutions with simple translation properties, while separation in polar coordinates yields solutions with simple rotation and scaling properties.

Fubini, Hanson, and Jackiw[1] formulated a dilation field theory of massless scalars in Euclidean four-dimensional spacetime. They carried out a similar treatment for a class of two-dimensional theories, and outlined the procedures for quantizing massless Euclidean Dirac fields. Sommerfield[2] and DiSessa[3] applied the formalism to massive scalar fields in pseudo-Euclidean spaces of two and four dimensions, and studied the difficulties of extending the dilation-field-theory formalism outside the light cone. Gromes, Rothe, and Stech[4] dilation quantized the massive Dirac field in Minkowski space. Lovelace[5] developed a method for calculating perturbed dilation eigenvalues in dilation field theory. He showed how violation of scale invariance could be treated in a systematic adiabatic approximation. He applied the approximation method to ϕ^3 field theory in six dimensions and checked results against those found by solving the renormalized ladder Bethe-Salpeter equation.[6]

The present work formulates a dilation field theory of electromagnetism in four Euclidean dimensions. To do this, we must write the vector field $A_\mu(x)$ in terms of spherical harmonics of O(4), the group of rotations in four dimensions. O(4) is isomorphic to SU(2)⊗SU(2), i.e., the generators of O(4) may be written as two commuting SU(2) algebras, whose generators will be called L_a^+ and L_a^-, $a = 1,2,3$. We can construct all representations of O(4) from pairs of commuting SU(2) representations, using as labels the SU(2) Casimir eigenvalues (A,B) for the $(+)$ and $(-)$ algebras, respectively.

The spin degrees of freedom of the vector field $A_\mu(x)$ form a $(\frac{1}{2},\frac{1}{2})$ representation of O(4). The position dependence of each component of the field must be expanded in O(4) scalar spherical harmonics, a series of (l,l) representations with $l = 0, \frac{1}{2}, 1, \ldots$. We construct O(4) vector spherical harmonics using the Clebsch-Gordan series to combine the $(\frac{1}{2}\,\frac{1}{2})$ spin and each (l,l) orbital representation into (A,B) representations with $A,B = l \pm \frac{1}{2}$. Each vector harmonic is labeled with $\{A,B,l,a,b,r,s\}$, where $A \geq a \geq -A, B \geq b \geq -B$; $r,s = \pm\frac{1}{2}$ denote the spin indices of the vector field. We will see that the gradient of a scalar function $\partial_\mu \phi(x)$, when expanded, includes only vector harmonics with $A = B$. It follows that those harmonics in the expansion of the vector field $A_\mu(x)$ with $A \neq B$ are gauge invariant. Furthermore, only the harmonics with $A \neq B$ have physically significant nonzero solutions to the equation of motion in the absence of sources, so only the $A \neq B$ harmonics should be quantized. That part of $A_\mu(x)$ contributed by vector harmonics with $A \neq B$ will be called $\widetilde{A}_\mu(x)$. We will see that $\partial_\mu \widetilde{A}^\mu(x) = 0$, $x_\mu \widetilde{A}^\mu(x) = 0$.

To completely span the degrees of freedom of the vector field, vector harmonics with $A = B = l \pm \frac{1}{2}$ must also be included in its expansion. That part of $A_\mu(x)$ composed of $A = B$ harmonics may be written as

$$\partial_\mu \phi(x) + \frac{x_\mu}{R^2} \psi(x)$$

with $R^2 = x_\alpha x^\alpha$. Since $\partial_\mu A^\mu(x)$ and $x_\mu A^\mu(x)$ depend only on $\phi(x)$ and $\psi(x)$ we may contract the equation of motion

$$\Box A_\mu(x) - \partial_\mu[\partial_\alpha A^\alpha(x)] = J_\mu(x)$$

with x^μ and ∂^μ to obtain equations of motion for $\phi(x)$ and

$\psi(x)$. Obviously, $\phi(x)$ has no physical significance, and is unconstrained by the equations of motion. $\psi(x)$ satisfies

$$\frac{(x_\mu\partial_\nu-x_\nu\partial_\mu)(x_\mu\partial_\nu-x_\nu\partial_\mu)}{2R^2}\psi(x)=x_\mu J^\mu(x) .$$

$\psi(x)$ must be zero if $x_\mu J^\mu(x)=0$, and in any case $\psi(x)$ cannot propagate radially; its equation of motion contains no derivatives with respect to R, so $\psi(x)$ propagates away from $x_\mu J^\mu(x)$ along the surface $R^2=x_\mu x^\mu$. $\psi(x)$ is a classical field. Its contribution to the Lagrangian may be written as an interaction between the radial components of two currents, $x_\mu J^\mu(x)$ and $y_\nu J^\nu(y)$. The interaction potential is nonzero only if $x_\mu x^\mu=y_\nu y^\nu$. The analogy between time in the usual quantization procedure and $\ln(R)$ in dilation field theory[1,6] is reinforced here by the parallel between the classical interaction potential $\psi(x)$, "instantaneous" in R, and the instantaneous Coulomb potential of the electromagnetic field.

In the next section we will give the definitions and properties of the O(4) harmonics used to expand arbitrary scalar- and vector-valued functions. By expressing $\partial_\mu\phi(x)$ and $x_\mu\psi(x)/R^2$ in terms of vector harmonics and expressing $\partial_\mu A^\mu(x)$ and $x_\mu A^\mu(x)$ in terms of scalar harmonics, the preceding assertions concerning the O(4) properties of $\widetilde{A}_\mu(x)$, $\partial_\mu\phi(x)$, and $x_\mu\psi(x)/R^2$ will be demonstrated. In Sec. III, solutions of the equation of motion will be found for the quantum field $\widetilde{A}_\mu(x)$. Quantization of $\widetilde{A}_\mu(x)$ follows Ref. 1 so closely that little will be said about this. A projection operator and propagator for $\widetilde{A}_\mu(x)$ will be constructed in Cartesian coordinates. In Sec. IV we find the equation of motion and propagator for $\psi(x)$. In Sec. V we will discuss complications resulting from the appearance of the origin of coordinates in the formalism, as well as the significance of the unperturbed dilation eigenvalues of the "photons" of this theory: field operators of leading twist have the smallest possible number of dilation photons. The possibility of applying this formalism to non-Abelian gauge fields will also be discussed.

II. CONVENTIONS

The rotation group in four dimensions has infinitesimal operators

$$L^{\mu\nu}=i(x^\mu\partial^\nu-x^\nu\partial^\mu),\quad \mu,\nu=1,2,3,4 \tag{1}$$

usually combined to form

$$L_a^\pm=-\frac{1}{2}\left[\frac{\epsilon_{abc}L^{bc}}{2}\pm L^{4a}\right],\quad a,b,c=1,2,3 \tag{2}$$

which satisfy

$$[L_a^\pm,L_b^\pm]=i\epsilon_{abc}L_c^\pm,\quad [L_a^\pm,L_b^\mp]=0 .$$

Thus representations of O(4) can be made from a direct product of two representations of SU(2), each labeled by the eigenvalues (l^+,l^-) of the Casimir operators:

$$(\vec{L}^\pm)^2=(L_1^\pm)^2+(L_2^\pm)^2+(L_3^\pm)^2=l^\pm(l^\pm+1) .$$

We define "bispherical" coordinates and O(4) scalar harmonics following Keam's work,[7] except for a permutation of coordinate definitions:

$$\begin{aligned}
&x_1=R\sin\nu\cos\phi ,\\
&x_2=R\sin\nu\sin\phi ,\\
&x_3=R\cos\nu\sin\omega ,\\
&x_4=R\cos\nu\cos\omega ,\\
&{x_1}^2+{x_2}^2+{x_3}^2+{x_4}^2=R^2 ,\\
&\int d^4x\,\theta({R_0}^2-x^2)\\
&\quad=\int_0^{2\pi}d\phi\int_0^{2\pi}d\omega\int_0^{\pi}d\nu\sin\nu\cos\nu\int_0^{R_0}R^3dR\\
&\quad\equiv\int d\Omega\int_0^{R_0}R^3dR .
\end{aligned} \tag{3}$$

A complete, orthonormal set of functions on the unit sphere, grouped into (l,l) representations of O(4), may be written

$$Z_{mn}^l(\Omega)=(-1)^{l-n}\left[\frac{2l+1}{2\pi^2}\right]^{1/2}[\exp[i\mathscr{L}_3(\omega+\phi)]\exp[2i\mathscr{L}_1\nu]\exp[i\mathscr{L}_3(\omega-\phi)]]_{m,-n} , \tag{4}$$

where $\mathscr{L}_{1,3}$ are $(2l+1)\times(2l+1)$ SU(2) matrices. After writing the infinitesimal operators (2) in terms of R,ν,ϕ,ω, one may verify the following identities for $(l,l)=(\frac{1}{2},\frac{1}{2})$, and $\mathscr{L}_{1,3}=\sigma_{1,3}/2$ ($\sigma_{1,3}$ are Pauli matrices):

$$L_3^+Z_{mn}^l(\Omega)=mZ_{mn}^l(\Omega),\quad L_3^-Z_{mn}^l(\Omega)=nZ_{mn}^l(\Omega) , \tag{5}$$

$$(\vec{L}^\pm)^2Z_{mn}^l(\Omega)=l(l+1)Z_{mn}^l(\Omega) , \tag{6}$$

$$(L_1^+\pm iL_2^+)Z_{mn}^l(\Omega)=[(l\mp m)(l\pm m+1)]^{1/2}Z_{m\pm1n}^l(\Omega) , \tag{7}$$

$$(L_1^-\pm iL_2^-)Z_{mn}^l(\Omega)=[(l\mp n)(l\pm n+1)]^{1/2}Z_{mn\pm1}^l(\Omega) . \tag{8}$$

The Clebsch-Gordan series for SU(2) rotation matrices can be applied to the $Z_{mn}^l(\Omega)$, ensuring that (5)—(8) are true for $l>\frac{1}{2}$:

$$Z_{bc}^A(\Omega)Z_{ef}^D(\Omega)=\sum_{Ghk}C(ADGbeh)C(ADGcfk)\left[\frac{(2A+1)(2D+1)}{2\pi^2(2G+1)}\right]^{1/2}Z_{hk}^G(\Omega) . \tag{9}$$

Since $\mathscr{L}_1, \mathscr{L}_3$ are real and symmetric, the definition (4) implies

$$[Z^l_{mn}(\Omega)]^* = (-1)^{m-n} Z^l_{-m\,-n}(\Omega) . \tag{10}$$

These harmonics are complete and orthonormal on the unit sphere:

$$\int_0^{2\pi} d\phi \int_0^{2\pi} d\omega \int_0^{\pi} d\nu \sin\nu \cos\nu [Z^A_{bc}(\Omega)]^* Z^D_{ef}(\Omega) = \delta_{AD}\delta_{be}\delta_{cf} . \tag{11}$$

As with all such orthonormal functions on the unit sphere[8]

$$\sum_{-l<m,n<l} [Z^l_{mn}(\Omega')]^* Z^l_{mn}(\Omega) = \frac{2l+1}{2\pi^2} C^1_{2l}(\Omega'-\Omega) , \tag{12}$$

where the $C^1_{2l}(x)$ are Gegenbauer polynomials. Also,

$$\sum_{\substack{l=0,1/2,1,\ldots \\ -l\le m,n\le l}} [Z^l_{mn}(\Omega')]^* Z^l_{mn}(\Omega) = \delta^3(\Omega'-\Omega) . \tag{13}$$

We will write an arbitrary scalar-valued function $\psi(x)$ as

$$\psi(x) = \sum_{\substack{l \\ m}} \psi^l_{mn}(R) Z^l_{mn}(\Omega) . \tag{14}$$

A tensor-valued function $T^{RS}_{rs}(x)$ whose spin degrees of freedom make an (R,S) representation of O(4) could be written as a multiplet of scalar-valued functions and expanded in terms of scalar harmonics:

$$T^{RS}_{rs}(x) = \sum_{\substack{l \\ mn}} T^{lRS}_{mnrs}(R) Z^l_{mn}(\Omega)$$

or, equivalently, in terms of O(4) tensor harmonics,

$$T^{RS}_{rs}(x) = \sum_{\substack{ABl \\ ab}} T^{ABl}_{ab}(R) Z^{ABlRS}_{abrs}(\Omega) \tag{15}$$

with the tensor harmonics defined as

$$Z^{ABlRS}_{abrs}(\Omega) \equiv \sum_{m,n} C(lRAmra) C(lSBnsb) Z^l_{mn}(\Omega) . \tag{16}$$

And the $\{T^{ABl}_{ab}(R)\}$ uniquely determined,

$$T^{ABl}_{ab}(R) = \sum_{m,n,r,s} C(lRAmra) C(lSBnsb) T^{lRS}_{mnrs}(R) .$$

Defining $J^{\pm}_a = L^{\pm}_a + S^{\pm}_a$, $a=1,2,3$ where S^+_a, S^-_a are the spin matrices for the tensor degrees of freedom, one may verify that $Z^{ABlRS}_{abrs}(\Omega)$ is an eigenfunction of $(\vec{J}^+)^2$, $(\vec{J}^-)^2$, $(\vec{L}^+)^2$, $(\vec{L}^-)^2$, $(\vec{S}^+)^2$, $(\vec{S}^-)^2$, J^+_3, J^-_3, S^+_3, and S^-_3 with respective eigenvalues $A(A+1)$, $B(B+1)$, $l(l+1)$, $l(l+1)$, $R(R+1)$, $S(S+1)$, a, b, r, and s. From (16) and (10) it follows that

$$[Z^{ABlRS}_{abrs}(\Omega)]^* = (-1)^P Z^{ABlRS}_{-a\,-b\,-r\,-s}(\Omega) \tag{17}$$

with

$$P = A + a + B + b + R + r + S + s .$$

Orthonormality of the tensor harmonics is

$$\sum_{rs} \int d\Omega [Z^{ABlRS}_{abrs}(\Omega)]^* Z^{CDkRS}_{cdrs}(\Omega) = \delta_{AC}\delta_{BD}\delta_{lk}\delta_{ac}\delta_{bd} . \tag{18}$$

The addition theorem for tensor harmonics is

$$\sum_{\substack{A,B,l \\ a,b}} [Z^{ABlRS}_{abrs}(\Omega')]^* Z^{ABlRS}_{abtu}(\Omega) = \delta_{rt}\delta_{su}\delta^3(\Omega'-\Omega) , \tag{19}$$

where

$$\int d\Omega' \delta^3(\Omega'-\Omega) f(\Omega') = f(\Omega) .$$

A four-vector makes a $(\frac{1}{2},\frac{1}{2})$ representation of O(4). We rewrite the Cartesian coordinates x_μ, $\mu=1,2,3,4$ as x_{rs}, $r,s=\pm\frac{1}{2}$:

$$\begin{aligned} x_{\frac12\frac12} &= \frac{-i}{\sqrt{2}}(x_1 - ix_2) , \\ x_{\frac12-\frac12} &= \frac{-1}{\sqrt{2}}(x_4 - ix_3) , \\ x_{-\frac12-\frac12} &= \frac{i}{\sqrt{2}}(x_1 + ix_2) , \\ x_{-\frac12\frac12} &= \frac{1}{\sqrt{2}}(x_4 + ix_3) , \end{aligned} \tag{20}$$

so that

$$x_{rs} = \sqrt{2}\,\pi R Z^{00\frac12\frac12\frac12}_{00\,rs}(\Omega) = \frac{\pi R}{\sqrt{2}} [Z^{1/2}_{rs}(\Omega)]^* . \tag{21}$$

The metric tensor which satisfies

$$R^2 = x_{rs} g_{rstu} x_{tu}$$

is

$$g_{rstu} = (-1)^{r-s} \delta_{r,-t}\delta_{s,-u} . \tag{22}$$

Since we will deal exclusively with vector and scalar harmonics, henceforth we write

$$V^{ABl}_{ab\,rs}(\Omega) = Z^{ABl\frac12\frac12}_{ab\,rs}(\Omega) .$$

We will suppress the angle argument Ω, writing $V^{ABl}_{ab\,rs}$ and Z^l_{mn} for vector and scalar harmonics when no ambiguity arises.

Given a scalar-valued function

$$\psi(x) = \psi(R) Z^l_{mn} ,$$

we may express $x_\mu \psi(x)$ as $x_{rs}\psi(x)$ using (19). Then

$$x_{rs}\psi(R) Z^l_{mn} = R\psi(R) \left[\left[\frac{l+1}{2l+1} \right]^{1/2} V^{lll+1/2}_{mnrs} + \left[\frac{l}{2l+1} \right]^{1/2} V^{lll-1/2}_{mnrs} \right] , \tag{23}$$

where the $V^{lll\pm1/2}_{mnrs}$ are vector harmonics as defined above, but with total spin eigenvalues $A=B=l$ and orbital spin eigenvalues $l\pm\frac{1}{2}$. This can be shown using (21), (10), (16), and the identity

$$C(RSTrst) = (-1)^{R-T+s} \left[\frac{2T+1}{2R+1} \right]^{1/2} C(TSRt-sr) . \tag{24}$$

Since

$$\Box x_\mu - x_\mu \Box = 2\partial_\mu \tag{25}$$

and

$$\Box = \frac{1}{R^2}\left\{\frac{\partial}{\partial\rho}\left[\frac{\partial}{\partial\rho}+2\right]-2[(\vec{L}^{\,+})^2+(\vec{L}^{\,-})^2]\right\},$$
$$\frac{\partial}{\partial\rho}\equiv R\frac{\partial}{\partial R}\,. \tag{26}$$

We can go from (23) to the expression for the gradient of a scalar-valued function,

$$\partial_{rs}\phi(R)Z^l_{mn} = \frac{1}{R}\left[\frac{l+1}{2l+1}\right]^{1/2} V^{lll+1/2}_{mnrs}\left[\frac{\partial}{\partial\rho}-2l\right]\phi(R) + \frac{1}{R}\left[\frac{l}{2l+1}\right]^{1/2} V^{lll-1/2}_{mnrs}\left[\frac{\partial}{\partial\rho}+2l+2\right]\phi(R)\,. \tag{27}$$

Using (16), (20), (21), (10), (9), (25), and orthogonality of the Clebsch-Gordan coefficients we can show that

$$x_{rs}g_{rstu}A^{ABl}_{ab}(R)V^{ABl}_{abtu} = \frac{\delta_{AB}RA^{ABl}_{ab}(R)}{\sqrt{2}}\left[\frac{2l+1}{2A+1}\right]^{1/2} Z^A_{ab}\,, \tag{28}$$

$$\partial_{rs}g_{rstu}A^{ABl}_{ab}(R)V^{ABl}_{abtu} = \frac{\delta_{AB}}{\sqrt{2}R}\left[\frac{2l+1}{2A+1}\right]^{1/2} Z^A_{ab} \times\left[\frac{\partial}{\partial\rho}-2(A-l)[2A+1-4(A-l)]\right]A^{ABl}_{ab}(R)\,. \tag{29}$$

Equation (27) demonstrates that $\partial_\mu\phi(x)$ is expressed in terms of O(4) vector harmonics V^{ABl}_{abrs} with $A=B$. Therefore, only that part of the vector field $A_\mu(x)$ made up of vector harmonics with equal total spin quantum numbers can be changed by a gauge transformation $A_\mu(x)\to A_\mu(x)+\partial_\mu\phi(x)$; the rest of $A_\mu(x)$, proportional to vector harmonics with $A\neq B$, is gauge invariant. Furthermore, Eqs. (28) and (29) show that this gauge-invariant part of the vector field is annihilated by contracting with either ∂_μ or x_μ. These observations allow one to show that the equation of motion for the gauge-invariant part of $A_\mu(x)$ is just the Klein-Gordon equation for massless fields.

III. EQUATION OF MOTION—DILATION QUANTIZATION

To expand the vector field $A_\mu(x)$ and its source $J_\mu(x)$ in vector harmonics we replace each x_μ, $\mu=1,2,3,4$ in (20) with the corresponding Cartesian component of the vector functions to define $A_{rs}(x)$, $J_{rs}(x)$, $r,s=\pm\frac{1}{2}$. Following (15) with $R=S=\frac{1}{2}$, $A,B=l\pm\frac{1}{2}$,

$$A_{rs}(x)\equiv\frac{\mathscr{A}_{rs}(x)}{R}=\frac{1}{R}\sum_{\substack{ABl\\ab}}\frac{\mathscr{A}^{ABl}_{ab}(R)}{(4l+2)^{1/2}}V^{ABl}_{abrs}(\Omega)\,, \tag{30}$$

$$J_{rs}(x)\equiv\frac{\mathscr{J}_{rs}(x)}{R^3}=\frac{1}{R^3}\sum_{\substack{ABl\\ab}}\frac{\mathscr{J}^{ABl}_{ab}(R)}{(4l+2)^{1/2}}V^{ABl}_{abrs}(\Omega)\,. \tag{31}$$

We insert these expansions into the equation of motion

$$\Box A_\mu(x)-\partial_\mu[\partial_\nu A^\nu(x)]=J_\mu(x) \tag{32}$$

to get equations of motion for the $\mathscr{A}^{ABl}_{ab}(R)$. The result for harmonics with $A\neq B$ is particularly simple. By (29), $\partial_\mu A^\mu(x)$ involves no $\mathscr{A}^{ABl}_{ab}(R)$ with $A\neq B$. By (27), $\partial_\mu[\partial_\nu A^\nu(x)]$ creates only harmonics with $A=B$. Therefore if we equate the coefficients of $V^{ABl}_{abrs}(\Omega)$ on both sides of (32), we find that for $A\neq B$, (32) becomes

$$A\neq B:\ \Box\frac{1}{R}\mathscr{A}^{ABl}_{ab}(R)V^{ABl}_{abrs}=\frac{1}{R^3}\mathscr{J}^{ABl}_{ab}(R)V^{ABl}_{abrs}\,,$$

or, by (26)

$$A\neq B:\ \left[\left[\frac{\partial}{\partial\rho}\right]^2-(2l+1)^2\right]\mathscr{A}^{ABl}_{ab}(R)=\mathscr{J}^{ABl}_{ab}(R)\,. \tag{33}$$

If we define $\tilde{A}_\mu(x)$,$\tilde{J}_\mu(x)$ as the sum of those terms in (30) and (31) with $A\neq B$, (32) becomes the massless Klein-Gordon equation

$$\Box\tilde{A}_\mu(x)=\tilde{J}_\mu(x)\,. \tag{34}$$

At this point, the dilation quantization of quantum electrodynamics looks like scalar field theory, as treated by Fubini, Hanson, and Jackiw.[1] The similarity is most obvious in comparing our Eq. (33) with Eq. (3.40) of their work. We see that the orbital quantum number l found in their work is equivalent to $2l$ as used in (33).

The Green's function for Eq. (34) must be the inverse d'Alembertian $(2\pi|x-y|)^{-2}$ combined with a projection operator for the $A\neq B$ harmonics. We construct this projection operator by making the following observations.

Applying (28) to (30) gives

$$x_\mu A^\mu(x)=\frac{1}{2}\sum_{\substack{A=0,1/2,1,\dots\\-A\le a,b\le A}}\frac{1}{(2A+1)^{1/2}}[\mathscr{A}^{AAA+1/2}_{ab}(R)+\mathscr{A}^{AAA-1/2}_{ab}(R)]Z^A_{ab}\,. \tag{35}$$

While (29) applied to (30) gives

$$\partial_\mu A^\mu(x)=\frac{1}{2R^2}\sum_{A,a,b}\frac{1}{(2A+1)^{1/2}}\left[\left[\frac{\partial}{\partial\rho}+2A+2\right]\mathscr{A}^{AAA+1/2}_{ab}(R)+\left[\frac{\partial}{\partial\rho}-2A\right]\mathscr{A}^{AAA-1/2}_{ab}(R)\right]Z^A_{ab}\,. \tag{36}$$

Of course, $\mathscr{A}^{AAA-1/2}_{ab}(R)=0$ for $A=0$. If we require $x_\mu A^\mu=0$ and $\partial_\mu A^\mu(x)=0$ everywhere, the coefficients of Z^A_{ab} in (35) and (36) must each be set to zero. Indeed, the conditions $\partial_\mu A^\mu(x)=0$, $x_\mu A^\mu(x)=0$ are necessary and sufficient to set all $\mathscr{A}_{ab}^{AAA\pm 1/2}(R)$ to zero so that $A_\mu(x)=\widetilde{A}_\mu(x)$. Thus a projection operator $P_{\mu\nu}$ with the properties $x^\mu P_{\mu\nu}=0$, $\partial^\nu P_{\mu\nu}=0$, $P_{\mu\nu}\widetilde{A}^{\nu}(x)=\widetilde{A}_\mu(x)$ will extract the $A\neq B$ harmonics from (30). Since $x_\mu \widetilde{A}^{\mu}(x)=0$, $\partial_\mu \widetilde{A}^{\mu}(x)=0$ by (28) and (29), the necessary properties are possessed by

$$P_{\mu\nu}(x)=g_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}+\left[x_\mu-\frac{\partial_\mu}{\Box}\left[\frac{\partial}{\partial\rho}+4\right]\right]\left[\frac{L_{\alpha\beta}L^{\alpha\beta}}{2}\right]^{-1}\left[\Box x_\nu-\left[\frac{\partial}{\partial\rho}+2\right]\partial_\nu\right] . \tag{37}$$

The Green's function solving Eq. (34) is

$$\widetilde{D}_{\mu\nu}(x,y)=P_{\mu\nu}(x)\frac{1}{4\pi^2(x-y)^2} , \tag{38}$$

$$\widetilde{A}_\mu(x)=-\int d^4y\, \widetilde{D}_{\mu\nu}(x,y)J_\nu(y) . \tag{39}$$

To quantize $\widetilde{A}_\mu(x)$ we could set all $A=B$ harmonics to zero in (29) and (30), substitute these field expansions into the Lagrangian

$$L\equiv\int\frac{dR}{R}\mathscr{L}(R)=\int R^3dR\, d\Omega\,[\tfrac{1}{2}\widetilde{A}_\mu(x)\Box\widetilde{A}^{\mu}(x)-A_\mu(x)\widetilde{J}^{\mu}(x)] , \tag{40}$$

and follow the procedure of Fubini, Hanson, and Jackiw to quantize the coefficient functions $\mathscr{A}_{ab}^{ABl}(R)$. Neglecting the interaction, the outcome of the quantization is an expansion of $\widetilde{A}_\mu(x)$ in terms of free-field creation and annihilation operators,

$$\widetilde{\mathscr{A}}_{rs}(x)\equiv R\widetilde{A}_{rs}(x)=\sum_{A\neq B}\left[\frac{\mathscr{A}_{ab}^{(+)ABl}}{(4l+2)^{1/2}}e^{(2l+1)\rho}V_{abrs}^{ABl}+\frac{\mathscr{A}^{(-)ABl}}{(4l+2)^{1/2}}e^{-(2l+1)\rho}[(-1)^{r-s}V_{ab-r-s}^{ABl}]^*\right] , \tag{41}$$

where $\rho=\ln R$. We postulate the commutators

$$[\mathscr{A}_{ab}^{(\pm)ABl},\mathscr{A}_{cd}^{(\pm)CDk}]=0 ,$$
$$[\mathscr{A}_{ab}^{(-)ABl},\mathscr{A}_{cd}^{(+)CDk}]=\delta_{AC}\delta_{BD}\delta_{lk}\delta_{ac}\delta_{bd} . \tag{42}$$

The vacuum is defined by

$$\mathscr{A}_{ab}^{(-)ABl}|0\rangle=0,\quad \langle 0|0\rangle=1 ,$$
$$\langle 0|\mathscr{A}_{ab}^{(+)ABl}=0 . \tag{43}$$

The free-field dilation operator is

$$\Delta_0=\sum_{A\neq B}(2l+1)\mathscr{A}_{ab}^{(+)ABl}\mathscr{A}_{ab}^{(-)ABl} , \tag{44}$$

$$[\Delta_0,\widetilde{\mathscr{A}}_{rs}(x)]=\frac{d}{d\rho}\widetilde{\mathscr{A}}_{rs}(x) . \tag{45}$$

Using (41) and (42) we may compute equal-R commutators:

$$x^2=y^2:\quad\left[\frac{\partial}{\partial\rho}\widetilde{\mathscr{A}}_{rs}(x),\widetilde{\mathscr{A}}_{tu}(y)\right]=\frac{-1}{2}\sum_{\substack{A\neq B,l\\ a,b\\ v,w=\pm 1/2}}\{V_{abrs}^{ABl}(\Omega_x)[V_{abvw}^{ABl}(\Omega_y)]^*g_{vw,tu}+g_{rs,vw}[V_{abvw}^{ABl}(\Omega_x)]^*V_{abtu}^{ABl}(\Omega_y)\} . \tag{46}$$

Here Ω_x,Ω_y denote the triplet of angles specifying the unit vectors $x_\mu/|x|$ and $y_\mu/|y|$, respectively. We recall that $g_{vw,tu}$, $g_{rs,vw}$ are metric tensors defined in (22). Instead of explicitly limiting the sum in (46) to $A\neq B$ harmonics, we will sum over all $A,B=l\pm 1/2$ and remove $A=B$ harmonics with the projection operator $P_{\mu\nu}(x)$, applied to $V_{abrs}^{ABl}(\Omega_x)$. Of course, we must combine the tensor components of $P_{\mu\nu}(x)$, $\mu,\nu=1,2,3,4$ in accordance with (20) to construct $P_{de,fg}(x)$, $d,e,f,g=\pm\frac{1}{2}$:

$$x^2=y^2:\quad\left[\frac{\partial}{\partial\rho}\widetilde{\mathscr{A}}_{rs}(x),\widetilde{\mathscr{A}}_{tu}(y)\right]=-\tfrac{1}{2}\sum_{f,g=\pm 1/2}P_{rs,fg}(x)\sum_{\substack{ABl\\ ab\\ v,w=\pm 1/2}}V_{abfg}^{ABl}(\Omega_x)[V_{abvw}^{AB}(\Omega_y)]^*g_{vw,tu}$$
$$-\tfrac{1}{2}\sum_{f,g=\pm 1/2}P_{rs,fg}(x)\sum_{\substack{ABl\\ ab}}[V_{abfg}^{ABl}(\Omega_x)]^*V_{abtu}^{AB}(\Omega_y) . \tag{47}$$

The sums over vector harmonics may now be replaced with δ functions using the addition theorem (19);

$$x^2=y^2:\quad \left[\frac{\partial}{\partial\rho}\tilde{\mathscr{A}}_{rs}(x),\mathscr{A}_{tu}(y)\right]=-P_{rs,tu}(x)\delta^3(\Omega_x-\Omega_y)\ . \tag{48}$$

Similarly,

$$x^2=y^2:\quad [\tilde{\mathscr{A}}_{rs}(x),\mathscr{A}_{tu}(y)]=\left[\frac{\partial}{\partial\rho}\tilde{\mathscr{A}}_{rs}(x),\frac{\partial}{\partial\rho}\tilde{\mathscr{A}}_{tu}(y)\right]=0 \tag{49}$$

and

$$\theta(x^2-y^2)\langle 0\,|\,\tilde{A}_\mu(x)\tilde{A}_\nu(y)\,|\,0\rangle+\theta(y^2-x^2)\langle 0\,|\,\tilde{A}_\nu(y)\tilde{A}_\mu(x)\,|\,0\rangle=\tilde{D}_{\mu\nu}(x,y)\ . \tag{50}$$

IV. THE CLASSICAL FIELD

The equation of motion for vector harmonics with $A=B$ takes a simple form in Cartesian coordinates. From (23) and (27) it follows that any vector function of the form

$$A_{rs}(x)=\frac{1}{R}[\mathscr{A}_{ab}^{AAA-1/2}(R)V_{abrs}^{AAA-1/2}+\mathscr{A}_{ab}^{AAA+1/2}(R)V_{abrs}^{AAA+1/2}] \tag{51}$$

may be expressed as

$$A_{rs}(x)=\frac{x_{rs}}{R^2}\psi_{ab}^A(R)Z_{ab}^A+\partial_{rs}\phi_{ab}^A(R)Z_{ab}^A \tag{52}$$

with

$$\phi_{ab}^A(R)=\frac{\mathscr{A}_{ab}^{AAA-1/2}(R)}{[A(2A+1)]^{1/2}}-\frac{\mathscr{A}_{ab}^{AAA+1/2}(R)}{[(A+1)(2A+1)]^{1/2}}\ ,$$

$$\psi_{ab}^A(R)=\frac{1}{2}\left[\left[\frac{2A+1}{A}\right]^{1/2}\mathscr{A}_{ab}^{AAA-1/2}(R)+\left[\frac{2A+1}{A+1}\right]^{1/2}\mathscr{A}_{ab}^{AAA+1/2}(R)-2\left[\frac{\partial}{\partial\rho}+1\right]\phi_{ab}^A(R)\right]\ . \tag{53}$$

In the case $A=0$, setting $\mathscr{A}_{ab}^{AAA-1/2}(R)=0$ in (53) and neglecting $1/A$ terms will give the correct result. Thus the contribution of all vector harmonics with $A=B$ may be written

$$A_\mu(x)-\tilde{A}_\mu(x)=\partial_\mu\phi(x)+\frac{x_\mu}{R^2}\psi(x)\ . \tag{54}$$

Substituting the right-hand side of (54) into (32), we find

$$\Box\frac{x_\mu\psi(x)}{R^2}-\partial_\mu\left[\frac{\partial}{\partial\rho}+4\right]\frac{\psi(x)}{R^2}=J_\mu(x)-\tilde{J}_\mu(x)\ . \tag{55}$$

This satisfies current conservation and, of course, is independent of $\phi(x)$. Contracting x^μ into both sides of (55) and using (26) leads to

$$-\frac{L_{\mu\nu}L^{\mu\nu}}{2}\psi(x)=R^2x_\mu J^\mu(x)\ . \tag{56}$$

Substituting the harmonic expansion (14) for $\psi(x)$ and defining

$$R^2x_\mu J^\mu(x)=\sum_{\substack{l\\mn}}\mathscr{J}_{mn}^l(R)Z_{mn}^l \tag{57}$$

yields

$$-4l(l+1)\psi_{mn}^l(R)=\mathscr{J}_{mn}^l(R)\ . \tag{58}$$

With the exception of the $l=0$ term, we see that $\psi(x)=0$ when $x_\mu J^\mu(x)=0$. The exception is artificial, because $x_\mu\psi(R)/R^2$ can be rewritten as $\partial_\mu\phi(R)$ and is annihilated in (55). Therefore the expansion of $\psi(x)$ in scalar harmonics actually starts from $l=\frac{1}{2}$.

Since (56) contains no derivatives with respect to R, $\psi(x)$ does not propagate in a radial direction. We should then consider $\psi(x)$ to be a classical interaction potential between radial components of the current, analogous to the Coulomb-gauge electrostatic potential in time-evolution field theory. In addition, the equation of motion for $\psi(x)$ is gauge invariant. We solve Eq. (56) by writing

$$\psi(x)=-x^2\int d^4y\frac{\delta(\,|x|-|y|\,)}{|x|^3}\left[\sum_{\substack{l=1/2,1,3/2,\ldots\\-l,\le m,n\le l}}\frac{Z_{mn}^l(\Omega_x)Z_{mn}^{l*}(\Omega_y)}{4l(l+1)}\right]y_\mu J^\mu(y)\ .$$

By (12) the quantity in brackets is

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{n+1}{n(n+2)}C_{2n}^1(\cos\theta)=\frac{1}{4\pi^2}\int_0^1 dz\left[\frac{1}{z}+z\right]\sum_{n=1}^{\infty}z^nC_n^1(\cos\theta)$$

with

$$\cos\theta=\frac{x_\mu y^\mu}{|x|\ |y|}\ .$$

Since

$$\sum_{n=0}^{\infty} z^n C_n^\lambda(\cos\theta)=(1-2z\cos\theta+z^2)^{-\lambda}\ ,$$

we find

$$\psi(x)=\int d^4y\, G_c(x,y)y_\mu J^\mu(y)\ ,$$
$$G_c(x,y)=\frac{\delta(|x|-|y|)}{4\pi^2|x|}[\tfrac{1}{2}+(\theta-\pi)\cot\theta]\ . \tag{59}$$

The classical field appears in both the kinetic part of the Lagrangian

$$\int d^4x\frac{x_\mu\psi(x)}{2R^2}[\Box g_{\mu\nu}-\partial_\mu\partial_\nu]\frac{x_\nu\psi(x)}{R^2}$$

and the interaction

$$-\int d^4x\frac{x_\mu\psi(x)}{R^2}J^\mu(x)\ .$$

Using (59) and (56), we find the net contribution is

$$L_c=-\tfrac{1}{2}\int d^4x\, d^4y\frac{x_\alpha J^\alpha(x)}{|x|}G_c(x,y)\frac{y_\beta J^\beta(y)}{|y|}\ . \tag{60}$$

V. DISCUSSION

The decomposition of the vector field described here is a natural consequence of applying the dilation-field-theory quantization procedure to the vector field. We cannot define a dilation momentum conjugate to $A_R(x)=x_\mu A^\mu/R$ because the Lagrangian density $\mathscr{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is independent of $(\partial/\partial R)A_R$. We may contrive a dilation-theory analog to the Coulomb-gauge condition $\vec{\nabla}\cdot\vec{A}(x)=0$,[9] if we decompose both ∂_μ and $A_\mu(x)$ according to

$$b_\mu\to b_\mu^R=\frac{x_\mu x_\alpha b^\alpha}{R^2},\ \ b_\mu^\perp=b_\mu-b_\mu^R\ .$$

Then the dilation "Coulomb" gauge condition is $\partial_\mu^\perp A^{\perp\mu}(x)=0$ or

$$\left[R\partial_\mu-(x_\alpha\partial^\alpha+3)\frac{x_\mu}{R}\right]A^\mu(x)=0\ . \tag{61}$$

Substituting (47) for $A^\mu(x)$ above gives

$$L_{\mu\nu}L^{\mu\nu}\phi(x)=0\ . \tag{62}$$

Imposing (61) as a gauge condition makes $x_\mu A^\mu(x)$ equivalent to $\psi(x)$, so $x_\mu A^\mu(x)$ is a classical field which cannot propagate radially. This completes the analogy to the Coulomb-gauge quantization in time-evolution field theory, wherein the timelike component of $A_\mu(x)$ is a classical field which propagates instantaneously.

Just as the Coulomb-gauge condition does not survive a Lorentz transformation, a translation breaks the dilation-gauge condition (61), by making $\phi(x)$ nonzero. This is demonstrated by resolving an infinitesimally translated solution of Eq. (32) into gauge-invariant ($A\neq B$), classical, and gauge terms:

$$b_\alpha\partial^\alpha\left[A_\mu(x)=\widetilde{A}_\mu(x)+\frac{x_\mu}{R^2}\psi(x)+\partial_\mu\phi(x)\right]$$
$$=\delta\widetilde{A}_\mu(x)+\frac{x_\mu}{R^2}\delta\psi(x)+\partial_\mu\delta\phi(x)\ . \tag{63}$$

We may contract both sides of (63) with x_μ, ∂_μ, requiring that $x_\mu\widetilde{A}^\mu(x)$, $\partial_\mu\widetilde{A}^\mu(x)$, $x_\mu\delta\widetilde{A}^\mu(x)$, and $\partial_\mu\delta\widetilde{A}^\mu(x)$ be zero. Similarly, $P_{\mu\nu}(x)$ may be contracted into both sides of (63), annihilating terms proportional to x_μ and ∂_μ. Thus we arrive at expressions for the terms on the right-hand side of Eq. (63);

$$\delta\phi(x)=b_\mu\partial^\mu\phi(x)+\delta\Phi(x)\ , \tag{64}$$

$$-\frac{L_{\mu\nu}L^{\mu\nu}}{2}\delta\Phi(x)=\left[\frac{\partial}{\partial\rho}+2\right]b_\mu\widetilde{A}^\mu(x)+\left[b_\mu\partial^\mu-\left[\frac{\partial}{\partial\rho}+4\right]\frac{b_\mu x^\mu}{R^2}\right]\psi(x)\ ,$$

$$\frac{L_{\mu\nu}L^{\mu\nu}}{2}\delta\psi(x)=R^2\Box b_\mu\widetilde{A}^\mu+\left[\frac{\partial}{\partial\rho}+\frac{L_{\mu\nu}L^{\mu\nu}}{2}\right]b_\mu\partial^\mu\psi(x)-\left[\frac{\partial}{\partial\rho}\left[\frac{\partial}{\partial\rho}+4\right]+\frac{L_{\mu\nu}L^{\mu\nu}}{2}\right]\frac{b_\mu x^\mu}{R^2}\psi(x)\ , \tag{65}$$

$$\delta\widetilde{A}_\mu(x)=b_\nu\partial^\nu\widetilde{A}_\mu(x)-\left[x_\mu\Box-\partial_\mu\left[\frac{\partial}{\partial\rho}+2\right]\right]\left[\frac{L_{\alpha\beta}L^{\alpha\beta}}{2}\right]^{-1}b_\nu\widetilde{A}^\nu(x)+P_{\mu\nu}b^\nu\frac{\psi}{R^2}(x)\ . \tag{66}$$

Equations (65) and (66) verify that $\delta\psi(x)$ and $\delta\widetilde{A}_\mu(x)$ are independent of $\phi(x)$, the gauge function before translation. Equation (65) demonstrates that $\psi(x)$ retains its status as a classical field under translations, since $\delta\psi(x)$ cannot propagate radially and is zero in the absence of sources [i.e., when $\widetilde{A}_\mu=0$ and $\psi(x)=0$]. In order to see that the equations of motion (34) and (56) survive a translation of the fields, one must determine the translation properties of the sources $\widetilde{J}_\mu(x)$ and $x_\mu J^\mu(x)$. The decomposition of the vector field used in Eqs. (54) and (63) depends only on the O(4) properties of $A_\mu(x)$ and is applicable to any vector-valued function, in particular $J_\mu(x)$. We may make current conservation an explicit constraint on that part of $J_\mu(x)$ proportional to V_{abrs}^{ABl} with $A=B$ by writing

$$J_\mu(x)=\widetilde{J}_\mu(x)+\left[\Box x_\mu-\partial_\mu\left[\frac{\partial}{\partial\rho}+4\right]\right]\frac{J(x)}{R^2}$$

with $J(x)$ a scalar-valued function. One can find the translation properties of

$$\delta J_\mu(x)=b_\alpha\partial^\alpha J_\mu(x)$$
$$=\delta\widetilde{J}_\mu(x)+\left[\Box x_\mu-\partial_\mu\left[\frac{\partial}{\partial\rho}+4\right]\right]\frac{1}{R^2}\delta J(x)$$

analogous to Eqs. (63)–(66) and verify the form invariance of the equations of motion (34) and (56),

$$\Box\delta\widetilde{A}_\mu(x)=\delta\widetilde{J}_\mu(x)\ ,\ -\frac{L_{\alpha\beta}L^{\alpha\beta}}{2}\delta\psi(x)=x_\mu\delta J^\mu(x)\ .$$

Lovelace[5] showed that, for a scalar field $\phi(x)$, the kets of the dilation field are "local products of field $\phi(x)$ and its derivatives, evaluated at $x_\mu=0$." His argument applies to the dilation field theory of quantum electrodynamics described here. Each term in the creation part of the vector field,

$$\widetilde{A}^{(+)}_{rs}(x)=\sum_{\substack{A\neq Bl\\ ab}}\mathscr{A}^{(+)ABl}_{ab}R^{2l}V^{ABl}_{abrs}(\Omega) \tag{67}$$

has position dependence which, tracked back through the definitions of the vector and scalar harmonics, is just a polynomial of order $2l$ in the Cartesian coordinates x_μ. Thus

$$\mathscr{A}^{(+)ABl}_{ab}\,|\,0\rangle$$

is the result of taking a $2l$th derivative of the vector field at the origin of coordinates. An N-photon state

$$\prod_{n=1}^{N}\mathscr{A}^{(+)A_nB_nl_n}_{a_nb_n}\,|\,0\rangle \tag{68}$$

has an unperturbed dilation eigenvalue

$$\zeta=\sum_n(2l_n+1) \tag{69}$$

and a maximum O(3) spin of

$$J=\sum_n(A_n+B_n)\ . \tag{70}$$

The importance of this state in an operator-product expansion on the light cone would be measured by its twist $\zeta-J$.[10] When applying the expansion to a field product with fixed scale dimension, the most singular coefficient functions are paired with the lowest-twist composite fields.[11] At first sight, composite fields with any number of photons have equal twist because the twist of $A_\mu(x)$ is zero. Lovelace[12] observed, however, that in dilation field theory the zero-twist part of the vector field does not contain dilation quanta; instead, a classical potential appears. The dilation quanta have $A\neq B$ so $A+B=2l$. Thus a noninteracting N-photon state like Eq. (68) has twist

$$\zeta-J=\sum_{n=1}^{N}(2l_n+1-A_n-B_n)=N\ . \tag{71}$$

Evidently the polarization vectors associated the spin components of $\widetilde{A}_\mu(x)$ have scale dimension zero. If dilation quanta could have total spins $A=B=l+\frac{1}{2}$ (e.g., in the Feynman gauge), zero-twist, many-photon states would reappear in dilation field theory. We have seen, however, that the potentially troublesome O(4) eigenstates with $A=B=l+\frac{1}{2}$ are found only in the classical field $x_\mu\psi(x)/R$ or the unimportant $\partial_\mu\phi(x)$. Dilation field theory thus offers justification for applying Lovelace's program[5,6] to a gauge field theory. The deep-inelastic structure functions could be calculated in a renormalized ladder Bethe-Salpeter approximation, and no many-vector intermediate states need be included. The renormalized ladder is trustworthy only for an asymptotically free-field theory, so the dilation quantization procedure would have to be applied to a non-Abelian gauge theory.

Although gauge invariance is not determined simply from O(4) quantum numbers in a non-Abelian theory, all added complications in this case are proportional to the asymptotically weakening coupling constant. Each gluon $B^a_\mu(x)$ satisfies

$$\Box B^a_\mu(x)-\partial_\mu[\partial^\alpha B^a_\alpha(x)]=J^a_\mu(x)\ , \tag{72}$$

so writing

$$B^a_\mu(x)=\widetilde{B}^a_\mu(x)+\partial_\mu\phi^a(x)+\frac{x_\mu}{R^2}\psi^a(x)$$

will eliminate $\phi^a(x)$ from the left-hand side of (74) and give us the same propagators for $\widetilde{B}^a(x)$ and $\psi^a(x)$ as were found in the Abelian case, and $\phi^a(x)$ can be determined by fixing the gauge. Enforcing (61) will set $\phi^a(x)$ to zero, but Faddeev-Popov ghosts will arise from this choice.[13] Setting $x^\mu B^a_\mu(x)=0$ will give no ghosts, but we must have

$$\frac{\partial}{\partial\rho}\phi^a(x)=-\psi_a(x)\ , \tag{73}$$

so $\partial^\mu(\partial/\partial\rho)^{-1}\psi^a(x)$ will appear on the right-hand side of (72). In either case, the isomultiplets $\{\widetilde{B}^a_\mu(x)\}$ and $\{\psi^a(x)\}$ are coupled by the equations of motion.

The QED Lagrangian

$$L=\int d^4x\,[\tfrac{1}{2}\widetilde{A}_\mu(x)\Box\widetilde{A}^\mu(x)-\widetilde{A}_\mu(x)\widetilde{J}^\mu(x)]$$
$$-\tfrac{1}{2}\int d^4x\,d^4y\,y_\mu J^\mu(y)G_c(x,y)x_\mu J^\mu(x) \tag{74}$$

is gauge invariant no matter what quantization procedure is used. Two gauge-invariant quantization procedures for quantum electrodynamics in Minkowski space have appeared in the literature. Weinberg[14] showed that a (1,0) or (0,1) representation of the Lorentz group may be used for the internal degrees of freedom of a massive spin-one field. He writes creation and annihilation operators for eigenstates of helicity and momentum. Then, in the zero-mass limit, the zero-helicity state disappears from the theory without producing an undefined propagator. The choice of gauge is not an issue in his approach, since the fundamental quantum fields are components of the electromagnetic tensor $F_{\mu\nu}$. Gambini and Hojman[15] showed that the equation of motion in momentum space,

$$k^2A_\mu(k)-k_\mu[k_\alpha A^\alpha(k)]=0\ ,$$

gives no constraint on $A_\mu(k)$ polarized along k_μ, so fields with this polarization can be ignored. In addition, when $k_\mu k^\mu=0$, the equation of motion requires that there can be no $A_\mu(k)$ with momentum $k_\mu=(k_0,\vec{k})$ and polarization in the direction of $\eta_\mu=(k_0,-\vec{k})$. If one simply ignores photons polarized along k_μ and η_μ everywhere in momentum space, the remaining 2 degrees of freedom in $A_\mu(k)$ can be quantized independent of gauge. Since this approach involves only quantum fields in the absence of sources, the classical potential does not appear.

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