Renormalization group at finite temperature

H. Matsumoto, Y. Nakano, and H. Umezawa
Department of Physics, The University of Alberta, Edmonton, Alberta, Canada, T6G 2J1
(Received 22 September 1983)

The renormalization-group equation in quantum field theory at finite temperature is investigated. Owing to the freedom of the renormalization procedure, one can scale the temperature as well as the momentum in choices of renormalization points. The result is an extended version of the renormalization group at zero temperature. Its Lie differential form defines two types of sets of renormalization-group coefficients. Several examples of the applications include the high-momentum limit (deep-inelastic limit), the high-temperature limit, the low-temperature limit, and the critical behavior near a phase transition point.

I. INTRODUCTION

The renormalization group emerges from the intrinsic freedom which exists in our choices for the renormalization procedure;[1] the change of the renormalization points induces the redefinition of the renormalized parameters without changing structures of Feynman diagrams. Its applications are abundant in many areas of physics; in high-energy physics, it supplies us with a powerful method for analyzing the deep-inelastic limit.[2] In statistical physics, it has become the most powerful method for calculating the critical exponents.[3] However, it is common in the latter application that the temperature is introduced *ad hoc* through mass and other parameters. This leads us to the following anticipation: the entire formulation might acquire more solid ground and need fewer assumptions if the usual quantum field theory is extended to acquire temperature.

In recent years, the authors have developed a quantum field theory at finite temperature (thermo field dynamics).[4-6] There the usual quantum field theory is straightforwardly extended to situations at finite temperature through a process of doubling the field operators. All the machinery of quantum field theory with a real time coordinate is usable, including the operator formalism, causal formulation, and the renormalization procedure. Specifically, the Feynman diagram method can be used with real time, and the structure of diagrams is not different from the one at zero temperature. Since temperature is included in the theory, faithfully reflecting the (grand) canonical ensemble, we can study the renormalization group at finite temperature in this formalism.

It is common to use the Matsubara Green's-function method[7] to take into account the temperature effect in the quantum field system. The extension of this method to relativistic field theory and gauge theories is studied in Refs. 8 and widely used among high-energy physicists.[9] Therefore it may be useful for readers who are not familiar with the thermo field dynamics (TFD) to mention briefly the relations and differences between TFD and the Matsubara Green's-function method.

In the Matsubara method, the time variable is taken to be imaginary ($t=-iu$) in the finite region [$0\le u\le\beta$, $\beta=1/(k_BT)$], and the periodic (or antiperiodic) boundary condition on the time variable is imposed to Matsubara Green's functions because of the trace nature of the (grand) canonical ensemble average. Thus the Matsubara method requires a modification of the Feynman diagram method at zero temperature, by introducing discrete frequencies $i\omega_n=2ni\pi/\beta$ [or $(2n+1)i\pi/\beta$]. Such a modification causes complexities when we identify amplitudes given by Matsubara Green's functions to those of actual processes which are specified by (continuous) real energy variables but not by discrete imaginary energies. In fact, the analytic properties of N-point functions are usually quite involved unless $N\le 2$. Therefore an analysis by the Matsubara Green's functions is often indirect.

Contrary to the Matsubara method, TFD is formulated with the *real time* variable from the beginning. The time variable and temperature are treated entirely on a different basis. The ground state is identified as the temperature-dependent vacuum. Any ensemble average is estimated as an expectation value on this vacuum, and almost all operator formalisms at zero temperature can be extended straightforwardly.

It has been shown by use of the interaction representation that the Feynman diagram method in TFD and that in the Matsubara's formalism are related through a certain analytic continuation.[5] The relation between the axiomatic statistical mechanics (the C^*-algebra approach) and TFD was discussed in Ref. 10. The extension to gauge theory is found in Refs. 10 and 11. The renormalization scheme in TFD was studied in Ref. 6, in which the scheme was shown to be a straightforward extension of zero-temperature quantum field theory. Owing to the formulation of TFD, one can easily define temperature-dependent renormalized masses and coupling constants which are directly related to the real observations, without analytic continuations. Therefore TFD proves powerful in analyzing many dynamical systems at finite temperature.

In the renormalization theory of thermo field dynamics, the arbitrariness of the renormalization point for momentum may lead to the same renormalization group as the one at zero temperature. However, a new situation arises from the fact that the renormalization procedure also includes temperature; namely, one should choose a temperature-renormalization point T_0, at which the renormalized parameters are determined. Any physical amplitude at temperature T is expressed in terms of these renormalized parameters. This is possible without changing the fundamental form of physical amplitudes as long as a system belongs to the same phase at T and T_0. It has been shown that the procedure to write amplitudes at T in terms of parameters at T_0 is a finite renormalization.[6] Because of these two types of freedoms of the choice for renormalization points (momentum and temperature), the renormalization group becomes a two-parameter Abelian group. If one combines this renormalization group with the scale transformation (i.e., dimensionality), the momentum change and the temperature change are performed through suitable renormalization. To clarify this point and to formulate the renormalization group at finite temperature are the main purposes of this paper.

This paper is organized as follows. For simplicity, we use the ϕ^4 model as an example. In the next section, the renormalization scheme at finite temperature is presented. In Sec. III, the renormalization group at finite temperature is obtained. Its Lie differential form leads to two types of renormalization-group equations in the manner of Callan and Symanzik, which define two types of sets of renormalization-group coefficients. In Secs. IV and V, several applications are presented. It will be shown in Sec. IV that the deep-inelastic limit (the leading high-momentum behavior) is not modified by temperature, that the high-temperature behavior is related to the deep-inelastic behavior, and that the low-temperature behavior is determined by low excitation modes. In Sec. V we analyze how the critical behavior is related to the low-momentum behavior. The scaling hypothesis[12] [that the critical behavior is controlled by the long-range correlation length, $\xi(T)\to\infty$ $(T\to T_c)$], is not a hypothesis any more, but a special result of the present analyses. The relations between the renormalization-group coefficients and the critical exponents will be shown. The final section is devoted to the conclusion.

II. RENORMALIZATION AT FINITE TEMPERATURE

For definiteness we choose a simple model of the scalar ϕ^4 model, in which the Lagrangian density is given by

$$\mathscr{L}=\tfrac{1}{2}(\partial\phi)^2-\tfrac{1}{2}m_0{}^2\phi^2-\frac{g_0}{4!}\phi^4 . \tag{2.1}$$

In the thermo field dynamics,[4] each field has double components:

$$\phi(x)=\begin{bmatrix}\phi^1(x)\\ \phi^2(x)\end{bmatrix} . \tag{2.2}$$

The field components $\phi^\alpha(x)$ $(\alpha=1,2)$ are considered to be independent operators and form the thermal doublet $\phi(x)$. The superscript α will be called the thermal index. The thermal Lagrangian density is given by

$$\hat{\mathscr{L}}=\sum_{\alpha=1}^{2}\epsilon^\alpha\left[\tfrac{1}{2}(\partial\phi^\alpha)^2-\tfrac{1}{2}m_0{}^2(\phi^\alpha)^2-\frac{g_0}{4!}(\phi^\alpha)^4\right] \tag{2.3}$$

with $\epsilon^1=+1$ and $\epsilon^2=-1$.

We consider the phase $\langle 0(\beta)\,|\,\phi(x)\,|\,0(\beta)\rangle=0$, where $|\,0(\beta)\rangle$ is the temperature-dependent vacuum and $\langle 0(\beta)\,|\cdots|\,0(\beta)\rangle$ is equal to the usual thermal average. In the perturbation theory of TFD, the thermal Lagrangian is divided into the unperturbed part $\hat{\mathscr{L}}_0$ and the interaction part $\hat{\mathscr{L}}_I$:

$$\hat{\mathscr{L}}_0=\sum_{\alpha=1}^{2}\epsilon^\alpha[\tfrac{1}{2}(\partial\phi_R^\alpha)^2-\tfrac{1}{2}m^2(\phi_R^\alpha)^2] , \tag{2.4a}$$

$$\hat{\mathscr{L}}_I=\sum_{\alpha=1}^{2}\epsilon^\alpha\left[-\frac{g}{4!}(\phi_R^\alpha)^4+(Z-1)[\tfrac{1}{2}(\partial\phi_R^\alpha)^2-\tfrac{1}{2}m^2(\phi_R^\alpha)^2]\right.$$
$$\left.-\,\delta m^2(\phi_R^\alpha)^2-\frac{g}{4!}(Z_1-1)(\phi_R^\alpha)^4\right] \tag{2.4b}$$

with

$$\phi=Z^{1/2}\phi_R , \tag{2.5a}$$

$$g_0=gZ_1Z^{-2} , \tag{2.5b}$$

$$\delta m^2=(m_0{}^2-m^2)Z^{-1} . \tag{2.5c}$$

The perturbation is carried out by the vertices given by $\hat{\mathscr{L}}_I$ and the propagator

$$\langle 0(\beta)\,|\,T\phi_R^\alpha(x)\phi_R^\gamma(y)\,|\,0(\beta)\rangle =i\int\frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\Delta^{\alpha\gamma}(k) , \tag{2.6}$$

in which

$$\Delta^{\alpha\gamma}(k)=[U_B(\,|\,k_0\,|\,)\Delta_0(k)U_B(\,|\,k_0\,|\,)]^{\alpha\gamma} \tag{2.7}$$

with

$$\Delta_0(k)=\tau(k^2-m^2+i\epsilon\tau)^{-1} , \tag{2.8}$$

$$\tau=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} , \tag{2.9}$$

$$U_B(\omega)=\frac{1}{(e^{\beta\omega}-1)^{1/2}}\begin{bmatrix}e^{\beta\omega/2} & 1\\ 1 & e^{\beta\omega/2}\end{bmatrix} . \tag{2.10}$$

The propagator $\Delta(k)$ is also written as

$$\Delta(k)=\Delta_0(k)+\Delta_\beta(k) , \tag{2.11}$$

where

$$\Delta_\beta(k)=-2\pi i\delta(k^2-m^2)f_B(\,|\,k_0\,|\,) , \tag{2.12}$$

$$f_B(\omega)=\frac{1}{e^{\beta\omega}-1}\begin{bmatrix}1 & e^{\beta\omega/2}\\ e^{\beta\omega/2} & 1\end{bmatrix} . \tag{2.13}$$

(Examples of perturbation calculations are found in the third and fourth articles of Refs. 4 and in Refs. 6 and 13.)

As was discussed in Ref. 6 in detail, several theorems related to the renormalizability are easily proven; since Δ_0 has the same structure as that of zero temperature and there is no divergence caused by Δ_β, a theory is renormalizable if it is renormalizable at zero temperature. Also all divergences are renormalized by the temperature-independent counterterms. Therefore the renormalization at different temperatures is related through finite renormalization. The same counterterms renormalizing Green's functions with identical thermal indices can renormalize those with mixed thermal indices. Since the renormalization schemes and the topological structure of the Feynman diagrams are not changed from those at zero temperature, the multiplicative renormalization of quantum field theory is also true at finite temperature.

For the renormalization conditions, we shall consider the following two schemes.

A. T-renormalization scheme

We can define the renormalized coupling constant $g(T)$ and the renormalized mass $m(T)$ at each temperature T. This renormalization scheme will be called the T-renormalization scheme. A renormalized N-point proper vertex function is denoted by $\hat{\Gamma}^{(N)}(p,T;g(T),m(T),\mu)$, where p is the abbreviation of the set of external momenta $(p_1,\ldots,p_N)$ and μ denotes the renormalization point of momentum. $\hat{\Gamma}^{(N)}$ carries also the thermal indices $(\alpha_1\cdots\alpha_N)$ with $\alpha_i=1$ or 2, which are omitted for convenience. The vertex function $\hat{\Gamma}^{(N)}$ is related to the unrenormalized one, $\Gamma_u^{(N)}$, through the relation

$$\hat{\Gamma}^{(N)}(p,T;g(T),m(T),\mu)=Z(T)^{N/2}\Gamma_u^{(N)}(p,T;g_0,m_0,\Lambda)\ , \tag{2.14}$$

where Λ is a cutoff momentum which will be taken infinite ($\Lambda\rightarrow\infty$), $Z(T)$ is the wave-function renormalization factor

$$Z(T)=Z(g(T),m(T),\mu,T;\Lambda) \tag{2.15}$$

and (g_0,m_0) and $(g(T),m(T))$ are related through

$$g_0=g(T)Z_g(g(T),m(T),\mu,T;\Lambda)\ , \tag{2.16}$$

$$m_0=m(T)Z_m(g(T),m(T),\mu,T;\Lambda)\ . \tag{2.17}$$

The renormalization conditions used in this section and Secs. III and IV are

$$\mathrm{Re}\hat{\Gamma}^{(2)}(p,T;g(T),m(T),\mu)\,|_{p=p(\mu)}=-\mu^2-m(T)^2\ , \tag{2.18}$$

$$\frac{\partial}{\partial\vec{p}^{\,2}}\mathrm{Re}\hat{\Gamma}^{(2)}(p,T;g(T),m(T),\mu)\bigg|_{p=p(\mu)}=-1\ , \tag{2.19}$$

and

$$\mathrm{Re}\hat{\Gamma}^{(4)}(p,T;g(T),m(T),\mu)\,|_{p_i=p_i(\mu)}=-g(T)\ , \tag{2.20}$$

where

$$p(\mu)=(0,\vec{p}(\mu))\ ,\quad \vec{p}(\mu)^2=\mu^2\ , \tag{2.21}$$

and $p_i(\mu)$ denotes a certain set of momenta to renormalize $\hat{\Gamma}^{(4)}$. In (2.18)—(2.20), all the thermal indices of each $\hat{\Gamma}^{(N)}$ should be equal to 1; for example, $\hat{\Gamma}^{(2)}$ in (2.18) means $\hat{\Gamma}^{(2)}_{11}$. Any other $\hat{\Gamma}^{(N)}$ with a thermal index 2 is renormalized by the same counterterms determined by the above renormalization conditions.[6] The real parts are used since $\hat{\Gamma}^{(N)}$ usually has an imaginary part because of the thermal effects.

B. T_0-renormalization scheme

One can express amplitudes at T in terms of the renormalized parameters at T_0. This renormalization scheme will be called the T_0-renormalization scheme. A renormalized N-point proper vertex function is denoted by $\Gamma^{(N)}(p,T;g,m,\mu,T_0)$, where g and m are renormalized coupling constant and mass at a renormalization point $(p(\mu),T_0)$. The vertex function $\Gamma^{(N)}$ is related to the unrenormalized one through the relation

$$\Gamma^{(N)}(p,T;g,m,\mu,T_0)=Z^{N/2}\Gamma_u^{(N)}(p,T;g_0,m_0,\Lambda)\ , \tag{2.22}$$

where Z is the wave-function renormalization factor at T_0. The renormalized parameters g and m, and Z are related to the bare parameters g_0 and m_0, through the relations (2.15)—(2.17) with T replaced by T_0. Therefore the renormalization conditions are

$$\mathrm{Re}\Gamma^{(2)}(p,T;g,m,\mu,T_0)\,|_{\substack{p=p(\mu)\\T=T_0}}=-\mu^2-m^2\ , \tag{2.23}$$

$$\frac{\partial}{\partial\vec{p}^{\,2}}\mathrm{Re}\Gamma^{(2)}(p,T;g,m,\mu,T_0)\bigg|_{\substack{p=p(\mu)\\T=T_0}}=-1\ , \tag{2.24}$$

and

$$\mathrm{Re}\Gamma^{(4)}(p,T;g,m,\mu,T_0)\,|_{\substack{p_i=p_i(\mu)\\T=T_0}}=-g\ . \tag{2.25}$$

The relation between $\hat{\Gamma}^{(N)}$ and $\Gamma^{(N)}$ will be clarified in the next section. Since $g(T)$ and $m(T)$ are directly related to the coupling constant and mass at the temperature T, $\hat{\Gamma}^{(N)}$ is convenient for investigating its properties of parameter dependence. For example, the condition $m(T)=0$ for $\mu=0$ indicates the existence of a massless mode at the temperature T.

III. THE RENORMALIZATION GROUP AT FINITE TEMPERATURE

Let us start from the proper vertex function $\Gamma^{(N)}(p,T;g,m,\mu,T_0)$ of the T_0-renormalization scheme. The scaling of the renormalization point into $(\mu s,T_0t)$ leads to the new renormalized parameters $(g(s,t),m(s,t))$:

$$(\mu,T_0)\rightarrow(\mu s,T_0t)\ , \tag{3.1}$$

$$(g,m)\rightarrow(g(s,t),m(s,t))\ . \tag{3.2}$$

Note that $g=g(1,1)$ and $m=m(1,1)$. With finite wave-function renormalization, we have a renormalization-group equation:

$$\Gamma^{(N)}(p,T;g,m,\mu,T_0)=\rho(s,t)^{-N/2}\Gamma^{(N)}(p,T;g(s,t),m(s,t),\mu s,T_0t)\ , \tag{3.3}$$

where $\rho(s,t)$ is given by

$$\rho(s,t)=\lim_{\Lambda\rightarrow\infty}\frac{Z(g(s,t),m(s,t),\mu s,T_0t;\Lambda)}{Z(g,m,\mu,T_0;\Lambda)}\ . \tag{3.4}$$

The renormalized parameters (g,m) and $(g(s,t),m(s,t))$ are related to bare parameters $g_0,m_0)$ through relations similar to (2.16) and (2.17):

$$g_0 = gZ_g(g,m,\mu,T_0;\Lambda)$$
$$= g(s,t)Z_g(g(s,t),m(s,t),\mu s,T_0 t;\Lambda)\ , \quad (3.5)$$
$$m_0 = mZ_m(g,m,\mu,T_0;\Lambda)$$
$$= m(s,t)Z_m(g(s,t),m(s,t),\mu s,T_0 t;\Lambda)\ . \quad (3.6)$$

Therefore $g(s,t)$, $m(s,t)$, and $\rho(s,t)$ are functions of (g,m,μ,T_0) and $(\mu s,T_0 t)$:

$$g(s,t) = \mathscr{G}(g,m,\mu,T_0;\mu s,T_0 t)\ , \quad (3.7)$$
$$m(s,t) = \mathscr{M}(g,m,\mu,T_0;\mu s,T_0 t)\ , \quad (3.8)$$

and

$$\rho(s,t) = \mathscr{R}(g,m,\mu,T_0;\mu s,T_0 t)\ . \quad (3.9)$$

Because of the multiplicative renormalizability, $\mathscr{G}$, $\mathscr{M}$, and $\mathscr{R}$ satisfy the relations

$$\mathscr{G}(g,m,\mu,T_0;\mu s_1 s_2,T_0 t_1 t_2) = \mathscr{G}(g(s_1,t_1),m(s_1,t_1),\mu s_1,T_0 t_1;\mu s_1 s_2,T_0 t_1 t_2)\ , \quad (3.10)$$
$$\mathscr{M}(g,m,\mu,T_0;\mu s_1 s_2,T_0 t_1 t_2) = \mathscr{M}(g(s_1,t_1),m(s_1,t_1),\mu s_1,T_0 t_1;\mu s_1 s_2,T_0 t_1 t_2)\ , \quad (3.11)$$

and

$$\mathscr{R}(g,m,\mu,T_0;\mu s_1 s_2,T_0 t_1 t_2) = \mathscr{R}(g,m,\mu,T_0;\mu s_1 T_0 t_1)\mathscr{R}(g(s_1,t_1),m(s_1,t_1),\mu s_1,T_0 t_1;\mu s_1 s_2,T_0 t_1 t_2)\ . \quad (3.12)$$

It can be shown, in a similar way as in the case of zero temperature, that transformations $R_{s,t}$ defined by

$$R_{s,t}(\mu,T_0) = (\mu s,T_0 t)\ , \quad (3.13)$$
$$R_{s,t}(g,m) = (g(s,t),m(s,t))\ , \quad (3.14)$$

and

$$R_{s,t}\Gamma^{(N)} = \rho(s,t)^{-N/2}\Gamma^{(N)} \quad (3.15)$$

form a group:

$$R_{s_1 s_2,t_1 t_2} = R_{s_1,t_1}R_{s_2,t_2}\ . \quad (3.16)$$

Transformations $R_{s,t}$ define a two-parameter Abelian renormalization group. If we combine the scaling transformation (dimensionality)

$$\Gamma^{(N)}(p,T;g,m,\mu,T_0) = l^{D_N}\Gamma^{(N)}\left[\frac{p}{l},\frac{T}{l};g,\frac{m}{l},\frac{\mu}{l},\frac{T_0}{l}\right] \quad (3.17)$$

with $D_N = 4-N$ being the dimensionality of $\Gamma^{(N)}$, we have the relation

$$\Gamma^{(N)}(p,T;g,m,\mu,T_0) = l^{D_N}\rho(s,t)^{-N/2}\Gamma^{(N)}\left[\frac{p}{l},\frac{T}{l};g(s,t),\frac{m(s,t)}{l},\frac{\mu s}{l},\frac{T_0 t}{l}\right]\ . \quad (3.18)$$

Various asymptotic limits can be discussed by use of this relation.

The differentiation of (3.3) with respect to s and t leads to two types of renormalization-group equations in the manner of Callan and Symanzik:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_s\frac{\partial}{\partial g} + \theta_s m\frac{\partial}{\partial m} - N\gamma_s\right]\Gamma^{(N)}(p,T;g,m,\mu,T_0) = 0\ , \quad (3.19)$$
$$\left[T_0\frac{\partial}{\partial T_0} + \beta_t\frac{\partial}{\partial g} + \theta_t m\frac{\partial}{\partial m} - N\gamma_t\right]\Gamma^{(N)}(p,T;g,m,\mu,T_0) = 0\ , \quad (3.20)$$

where the renormalization-group coefficients are defined by

$$\beta_s = s\frac{d}{ds}g(s,t)\Big|_{s=t=1}\ , \quad \theta_s = \frac{1}{m}s\frac{d}{ds}m(s,t)\Big|_{s=t=1}\ ,$$
$$\gamma_s = \tfrac{1}{2}s\frac{d}{ds}\ln\rho(s,t)\Big|_{s=t=1}\ , \quad (3.21)$$
$$\beta_t = t\frac{d}{dt}g(s,t)\Big|_{s=t=1}\ , \quad \theta_t = \frac{1}{m}t\frac{d}{dt}m(s,t)\Big|_{s=t=1}\ ,$$
$$\gamma_t = \tfrac{1}{2}t\frac{d}{dt}\ln\rho(s,t)\Big|_{s=t=1}\ .$$

These quantities are functions of g,m,μ, and T_0. For finite s and t, $g(s,t)$, for example, satisfies the relations as follows:

$$s\frac{d}{ds}g(s,t)=\beta_s(g(s,t),m(s,t),\mu s,T_0 t)\ , \tag{3.22a}$$

$$t\frac{d}{dt}g(s,t)=\beta_t(g(s,t),m,(s,t),\mu s,T_0 t)\ . \tag{3.22b}$$

The $g(s,t)$ and $m(s,t)$ are the generalizations of the running coupling constant and mass, respectively.

By use of the renormalization conditions (2.18)—(2.20), we can have several relations among the renormalization-group coefficients. From the definition of the running parameters, we have

$$-\mu^2 s^2-m(s,t)^2=\rho(s,t)\mathrm{Re}\Gamma^{(2)}(p,T_0t;g,m,\mu,T_0)\,|_{p=p(\mu s)}\ , \tag{3.23}$$

$$-1=\rho(s,t)\frac{\partial}{\partial\vec{p}^{\,2}}\mathrm{Re}\Gamma^{(2)}(p,T_0t;g,m,\mu,T_0)\Big|_{p=p(\mu s)}\ , \tag{3.24}$$

$$-g(s,t)=\rho(s,t)^2\mathrm{Re}\Gamma^{(4)}(p,T_0t;g,m,\mu,T_0)\,|_{p_i=p_i(\mu s)}\ . \tag{3.25}$$

Differentiating (3.23)—(3.25) with respect to s and t at $s=t=1$, we have

$$\theta_s=\gamma_s\left[1+\frac{\mu^2}{m^2}\right]\ , \tag{3.26}$$

$$\beta_s=4g\gamma_s-\mu\frac{dp_i(\mu)}{d\mu}\left[\frac{\partial}{\partial p_i}\mathrm{Re}\Gamma^{(4)}(p,T;g,m,\mu,T_0)\right]_{\substack{p_i=p_i(\mu)\\ T=T_0}}\ , \tag{3.27}$$

$$\theta_t=\gamma_t\left[1+\frac{\mu^2}{m^2}\right]-\frac{1}{2m^2}\left[T\frac{\partial}{\partial T}\mathrm{Re}\Gamma^{(2)}(p,T;g,m,\mu,T_0)\right]_{\substack{p=p(\mu)\\ T=T_0}}\ , \tag{3.28}$$

and

$$\beta_t=4g\gamma_t-\left[T\frac{\partial}{\partial T}\mathrm{Re}\Gamma^{(4)}(p,T;g,m,\mu,T_0)\right]_{\substack{p_i=p_i(\mu)\\ T=T_0}}\ . \tag{3.29}$$

Functions $\hat{\Gamma}^{(N)}$ and $\Gamma^{(N)}$ are related through the following equations:

$$\Gamma^{(N)}(p,T_0;g,m,\mu,T_0)=\hat{\Gamma}^{(N)}(p,T_0;g(T_0),m(T_0),\mu)\ , \tag{3.30}$$

$$\Gamma^{(N)}(p,T;g,m,\mu,T_0)=\rho(T)^{-N/2}\hat{\Gamma}^{(N)}(p,T;g(T),m(T),\mu) \tag{3.31}$$

with [cf. (3.5)—(3.7)]

$$\rho(T)=\rho\left[1,\frac{T}{T_0}\right]=\mathscr{R}(g,m,\mu,T_0;\mu,T)\ , \tag{3.22}$$

$$g(T)=\mathscr{G}(g,m,\mu,T_0;\mu,T)\ , \tag{3.33}$$

and

$$m(T)=\mathscr{M}(g,m,\mu,T_0;\mu,T)\ . \tag{3.34}$$

IV. ASYMPTOTIC BEHAVIOR

In this section, we apply the renormalization-group equation (3.18) to discuss the several asymptotic limits. Equation (3.18), i.e.,

$$\Gamma^{(N)}(p,T;g,m,\mu,T_0)=l^{D_N}\rho(s,t)^{-N/2}\Gamma^{(N)}\left[\frac{p}{l},\frac{T}{l};g(s,t),\frac{m(s,t)}{l},\frac{\mu s}{l},\frac{T_0 t}{l}\right]\ , \tag{4.1}$$

indicates that the change of the temperature is related to the change of the momentum scale[14] with suitable renormalization of the coupling constant, and of the mass, and vice versa. The physical intuition of the above fact is that the thermal excitation modifies clouds around particles, inducing the renormalization of parameters, and of the range of the correlation. Therefore the asymptotic limits of the momentum variables and those of the temperature are expected to have a close relationship.

A. High-momentum limit

The high-momentum limit can be obtained as usual by scaling p into λp and by choosing $l=\lambda$, $s=\lambda$, and $t=1$ (i.e., t fixed):

$$\Gamma^{(N)}(\lambda p,T;g,m,\mu,T_0) = \lambda^{D_N}\rho(\lambda,1)^{-N/2}\Gamma^{(N)}\left[p,\frac{T}{\lambda};g(\lambda,1),\frac{m(\lambda,1)}{\lambda},\mu,\frac{T_0}{\lambda}\right] \underset{\lambda\to\infty}{\longrightarrow} \lambda^{D_N}\rho(\lambda,1)^{-N/2}\Gamma^{(N)}(p,0;g(\lambda,1),0,\mu,0)\ , \tag{4.2}$$

where we assumed

$$m(\lambda,1)/\lambda \underset{\lambda\to\infty}{\longrightarrow} 0\ . \tag{4.3}$$

Equation (4.2) indicates that, at the limit $\lambda\to\infty$, the explicit dependence of m, T, and T_0 disappears. The result shows that the leading behavior of the asymptotic limit $p\to\infty$ is independent of the temperature.

B. High-temperature behavior

Choose $T=\lambda T_0$, $l=\lambda$, $s=\lambda$, and $t=\lambda$ in (4.1). We have

$$\Gamma^{(N)}(p,\lambda T_0;g,m,\mu,T_0) = \lambda^{D_N}\rho(\lambda,\lambda)^{-N/2}\hat{\Gamma}^{(N)}\left[\frac{p}{\lambda},T_0;g(\lambda,\lambda),\frac{m(\lambda,\lambda)}{\lambda},\mu\right]\ , \tag{4.4}$$

where (3.30) was used. In order to investigate the λ dependence of ρ, g, and m, we use the transformation properties (3.10)–(3.12). We have

$$g(\lambda,\lambda)=\mathcal{G}(g(\lambda,1),m(\lambda,1),\mu\lambda,T_0;\mu\lambda T_0\lambda)\ , \tag{4.5}$$

$$m(\lambda,\lambda)=\mathcal{M}(g(\lambda,1),m(\lambda,1),\mu\lambda,T_0;\mu\lambda,T_o\lambda)\ , \tag{4.6}$$

and

$$\rho(\lambda,\lambda)=\mathcal{R}(g,m,\mu,T_0;\mu\lambda,T_0) \times\mathcal{R}(g(\lambda,1),m(\lambda,1),\mu\lambda,T_0;\mu\lambda,T_0\lambda)\ . \tag{4.7}$$

By considering the dimensionality, in the limit $\lambda\to\infty$, we have

$$g(\lambda,\lambda)=\mathcal{G}\left[g(\lambda,1),\frac{m(\lambda,1)}{\lambda},\mu,\frac{T_0}{\lambda};\mu,T_0\right] \underset{\lambda\to\infty}{\longrightarrow} \mathcal{G}(g(\lambda,1),0,\mu,0;\mu,T_0) \equiv\mathcal{G}^*(g(\lambda,1))\ , \tag{4.8}$$

$$m(\lambda,\lambda) \underset{\lambda\to\infty}{\longrightarrow} \lambda\mathcal{M}(g(\lambda,1),0,\mu,0;\mu,T_0) \equiv\lambda\mathcal{M}^*(g(\lambda,1))\ , \tag{4.9}$$

and

$$\rho(\lambda,\lambda) \underset{\lambda\to\infty}{\longrightarrow} \rho(\lambda,1)\mathcal{R}(g(\lambda,1),0,\mu,0;\mu,T_0) \equiv\rho(\lambda,1)\mathcal{R}^*(g(\lambda,1))\ , \tag{4.10}$$

where we have assumed again $m(\lambda,1)/\lambda\to 0$ as $\lambda\to\infty$. Equations (4.4) and (4.8)–(4.10) show that, in the limit $\lambda\to\infty$, $\Gamma^{(N)}$ approaches

$$\Gamma^{(N)}(p,\lambda T_0;g,m,\mu,T_0) \underset{\lambda\to\infty}{\longrightarrow} \lambda^{D_N}\rho(\lambda,1)^{-N/2}\mathcal{R}^*(g(\lambda,1)) \times\hat{\Gamma}^{(N)}(0,T_0;\mathcal{G}^*(g(\lambda,1)),\mathcal{M}^*(g(\lambda,1)),\mu)\ , \tag{4.11}$$

in which $\mathcal{R}^*$, $\mathcal{G}^*$, and $\mathcal{M}^*$ are functions of $g(\lambda,1)$. Apart from the $g(\lambda,1)$ dependence, the scaling factor of the high-temperature limit is the same as the one of the high-momentum limit. Therefore when $g(\lambda,1)$ approaches a constant as $\lambda\to\infty$, the leading behavior of the high-temperature and the high-momentum limits are the same. This is physically understandable since the high-temperature behavior is controlled by high-energy excitation.

C. Low-temperature behavior

Low-temperature behavior is closely related to the low-momentum behavior. Examples of this are discussed in the textbook of Ref. 4. A systematic investigation of the relations between low-energy behavior restricted due to symmetries (i.e., restrictions from Ward-Takahashi relations), and the low-temperature behavior, are found in a recent paper of Ref. 15.

To study the low-temperature behavior, it is convenient to use renormalized parameters at zero temperature. Then the temperature dependence appears in the Feynman diagrams through propagators $\Delta(k)$, each of which is separated in a boson case as (2.11):[6,11]

$$\Delta(k)=\Delta_0(k)+\Delta_\beta(k)\ . \tag{4.12}$$

When the mass m of zero temperature is finite, the temperature-dependent part Δ_β in (4.12) damps faster than $\exp[-\beta m/2]$ as $\beta\to\infty$ $(T\to 0)$. Therefore the low-temperature behavior is controlled by the lowest excitation mode. Special interest lies in the case of a massless mode, which is a possible lowest mode. We write $\Gamma^{(N)}$ in the following form:

$$\Gamma^{(N)}(p,\lambda T_0;g,m,\mu,0) = \lambda^{D_N}\rho(s,\ )^{-N/2}\Gamma^{(N)}\left[\frac{p}{\lambda},T_0;g(s,\ ),\frac{m(s,\ )}{\lambda},\frac{\mu s}{\lambda},0\right], \quad (4.13)$$

where $\rho(s,\ )$, $g(s,\ )$, and $m(s,\ )$ indicate that they depend only on the scale parameter s of the momentum renormalization group.

We assume that the boson mass m vanishes at zero temperature, which leads to the condition

$$m(s,\ )\underset{s\to 0}{\longrightarrow} 0 . \quad (4.14)$$

The massless particle may interact even in the zero-energy limit and therefore

$$g(s,\ )\underset{s\to 0}{\longrightarrow} g^* , \quad (4.15)$$

with g^* being a certain nonvanishing constant.

When $m(s,\ )/s$ goes to m^* (finite) as $s\to 0$, choose $s=\lambda$ in (4.14). We have

$$\Gamma^{(N)}(p,\lambda T_0;g,m,\mu,0) \underset{\lambda\to 0}{\longrightarrow} \lambda^{D_N}\rho(\lambda,\ )^{-N/2}\Gamma^{(N)}\left[\frac{p}{\lambda},T_0;g^*,m^*,\mu,0\right] . \quad (4.16)$$

Therefore the momentum-independent part of $\Gamma^{(N)}$ behaves like $\lambda^{D_N}\rho(\lambda,\ )^{-N/2}$ at low temperature.

When $m(s,\ )/s$ goes to infinity as $s\to 0$, we can always choose s as

$$m(s,\ )/\lambda = m^* \quad (4.17)$$

for small λ, where m^* is a certain mass parameter. We denote s which satisfies (4.17) by $s(\lambda)$. Then we have

$$\Gamma^{(N)}(p,\lambda T_0;g,m,\mu,0) \underset{\lambda\to 0}{\longrightarrow} \lambda^{D_N}\rho(s(\lambda),\ )^{-N/2}\Gamma^{(N)}\left[\frac{p}{\lambda},T_0;g^*,m^*,0,0\right] , \quad (4.18)$$

where we have used

$$\frac{\mu s}{\lambda}=\frac{\mu m^* s}{m(s,\ )}\underset{s\to 0}{\longrightarrow} 0 . \quad (4.19)$$

The momentum-independent part of $\Gamma^{(N)}$ behaves as $\lambda^{D_N}\rho(s(\lambda),\ )^{-N/2}$ at low temperature. These results show that the low-temperature behavior of physical quantities which are independent of momentum (such as the coefficients of momentum expansions) is controlled by the behavior of the low-excitation mode of $m(s,\ )$; specifically, a massless mode.

V. CRITICAL PHENOMENA

In order to discuss the critical behavior,[16] we modify the definition of $\Gamma^{(N)}$ by use of the temperature measured from the critical temperature T_c. The critical temperature T_c is defined by

$$\Gamma^{(2)}(0,T_c;g,m,\mu,T_0)=0 , \quad (5.1)$$

which gives T_c as a function of g,m,μ, and T_0:

$$T_c=T_c(g,m,\mu,T_0) . \quad (5.2)$$

Note that Eq. (5.1) defines T_c independently of the renormalization procedure. Because of the definition of T_c given by (5.1), the phase transition we consider is the second-order one. Owing to (5.1), one can define the most fundamental critical exponents, η and γ, as

$$\mathrm{Re}\Gamma^{(2)}(p,T_c;g,m,\mu,T_0)\,|_{p_0=0} \underset{\vec p\to 0}{\propto} |\vec p\,|^{2-\eta} , \quad (5.3)$$

$$\mathrm{Re}\Gamma^{(2)}(0,T;g,m,\mu,T_0) \underset{T\to T_c}{\propto} |T-T_c|^{\gamma} . \quad (5.4)$$

It is convenient to use the following renormalization conditions which are different from those of the previous sections:

$$\mathrm{Re}\Gamma^{(2)}(0,T_0;g,m,\mu,T_0)=-m^2 , \quad (5.5)$$

$$\mathrm{Re}\frac{\partial}{\partial \vec p^{\,2}}\Gamma^{(2)}(p,T_0;g,m,\mu,T_0)\Big|_{p=0}=-1 , \quad (5.6)$$

and

$$\mathrm{Re}\Gamma^{(4)}(p,T_0;g,m,\mu,T_0)\,|_{p_i=p_i(\mu)}=-g . \quad (5.7)$$

We define τ and τ_0 by

$$\tau=T-T_c ,\quad \tau_0=T_0-T_c . \quad (5.8)$$

Since T_c is a function of (g,m,μ,T_0), $\Gamma^{(N)}$ can be also considered as a function of (g,m,μ,τ,τ_0):

$$\Gamma^{(N)}(p,T_c+\tau;g,m,\mu,T_c+\tau_0)\equiv\widetilde\Gamma^{(N)}(p,\tau;g,m,\mu,\tau_0) . \quad (5.9)$$

By following an argument similar to the one in Sec. III and by taking account of the renormalization-point independence (μ,T_0 independence) of T_c, we have a renormalization-group equation similar to (3.17):

$$\widetilde\Gamma^{(N)}(p,\tau;g,m,\mu,\tau_0) = l^{D_N}\rho(\ ,t)^{-N/2}\widetilde\Gamma^{(N)}\left[\frac{p}{l},\frac{\tau}{l};g(s,t),\frac{m(\ ,t)}{l},\frac{\mu s}{l},\frac{\tau_0 t}{l}\right] . \quad (5.10)$$

In the renormalization scheme of (5.5)–(5.7), the mass parameter and the wave-function renormalization factor are defined at a fixed momentum $p=0$. Therefore the renormalization-dependent quantities, ρ and m, depend only on t. The correlation length is defined by

$$\xi(\tau)^{-2}=\frac{\mathrm{Re}\widetilde\Gamma^{(2)}(p,\tau;g,m,\mu,\tau_0)}{(\partial/\partial\vec p^{\,2})\mathrm{Re}\widetilde\Gamma^{(2)}(p,\tau;g,m,\mu,\tau_0)}\Bigg|_{p=0} , \quad (5.11)$$

which is renormalization-scheme independent. The definitions (5.5) and (5.6) indicate that

$$m(\ ,t)^2=\xi(\tau)^{-2} \quad (5.12)$$

with $t=\tau/\tau_0=(T-T_c)/(T_0-T_c)$. It should be insisted that the critical temperature T_c is independent of the renormalization procedure and that $m(s,t)$ does not depend on s, and therefore

$$T_c = T_c(g(s,t), m(\ ,t), \mu s, \tau_o t) \ . \tag{5.13}$$

The Lie differentiation of (5.10) leads to two types of sets of renormalization-group coefficients $(\beta_s, \theta_s, \gamma_s)$ and $(\beta_t, \theta_t, \gamma_t)$, among which $\theta_s = \gamma_s = 0$ in the renormalization scheme (5.5)—(5.7). (Although we have been using the variables τ and τ_0 instead of T and T_0, hereafter we use the same notation as before for the renormalization-group coefficients, etc., in order to avoid notational complexities.)

The condition for the critical point (5.1) leads to the condition

$$m(\ ,0)^2 = 0 \ . \tag{5.14}$$

Then we can define another critical exponent, ν, by

$$m(\ ,t)^2 \underset{t\to 0}{\longrightarrow} m^{*2} t^{2\nu} \ . \tag{5.15}$$

We first assume that $\nu \le 1$.

The low-momentum behavior of $\tilde{\Gamma}^{(N)}$ at the critical temperature ($\tau=0$) is obtained in the following way. In (5.10), replace p by λp and choose $l = s = \lambda$. Also choose $t = t(\lambda)$, where $t(\lambda)$ satisfies

$$m(\ ,t(\lambda)) = \mu^* \lambda \tag{5.16}$$

with μ^* being a certain mass parameter. Note that

$$t(\lambda) \underset{\lambda\to 0}{\longrightarrow} t_0 \lambda^{1/\nu} \tag{5.17}$$

with a constant t_0 given by $(\mu^*/m^*)^{1/\nu}$ and

$$\frac{t(\lambda)}{\lambda} \underset{\lambda\to 0}{\longrightarrow} \begin{cases} 0 & (\nu<1) \\ t_0 & (\nu=1) \ . \end{cases} \tag{5.18}$$

Then we have

$$\tilde{\Gamma}^{(N)}(\lambda p, 0; g, m, \mu, \tau_0)\,|_{p_0=0} \underset{\lambda\to 0}{\longrightarrow} (\rho_0 t_0)^{-N/2} \lambda^{D_N-(N/2)\gamma_t^*/\nu} \tilde{\Gamma}^{(N)}(p, 0; g^*, \mu^*, \mu, \{{}^{0}_{\tau_0 t_0}\})\,|_{p_0=0} \ \text{for} \begin{cases} \nu<1 \\ \nu=1 \end{cases}, \tag{5.19}$$

where

$$\lim_{\lambda\to 0} g(\lambda, t(\lambda)) = g^* \ , \tag{5.20}$$

$$\rho(\ ,t) \underset{t\to 0}{\longrightarrow} \rho_0 t^{\gamma_t^*} \ , \tag{5.21}$$

with γ_t^* being the limiting value of $\gamma_t(\ ,t)/2$. The infrared stability (5.20) is physically reasonable since the renormalized coupling constant should be well defined even in the low-momentum limit. The coefficient $\tilde{\Gamma}^{(N)}$ on the right-hand side of (5.19) is nonvanishing since $p \neq 0$.

Let us now consider the $\tau \to 0$ limit for static and homogeneous quantities, i.e., $p_i = 0$ for all i. By choosing $\tau = \tau_0 t$, $l = s$, we have

$$\begin{aligned} \tilde{\Gamma}^{(N)}(0, \tau_0 t; g, m, \mu, \tau_0) &= s^{D_N} \rho(\ ,t)^{-N/2} \tilde{\Gamma}^{(N)}\left[0, \frac{\tau_0 t}{s}; g(s,t), \frac{m(\ ,t)}{s}, \mu, \frac{\tau_0 t}{s}\right] \\ &= s^{D_N} \rho(\ ,t)^{-N/2} \hat{\Gamma}^{(N)}\left[0, \frac{\tau_0 t}{s}; g(s,t), \frac{m(\ ,t)}{s}, \mu\right] \ . \end{aligned} \tag{5.22}$$

The function $\hat{\Gamma}^{(N)}$ in (5.22) is the renormalized vertex function of the T-renormalization scheme (see Sec. II), and therefore $g(s,t)$ and $m(\ ,t)/s$ are considered to be a coupling constant and a mass at temperature $\tau_0 t/s$. Now choose $s = s(t)$, where $s(t)$ satisfies

$$m(\ ,t) = \mu^* s(t) \ . \tag{5.23}$$

Note that

$$s(t) \underset{t\to 0}{\longrightarrow} s_0 t^{\nu} \ , \tag{5.24}$$

where $s_0 = t_0^{-\nu} = m^*/\mu^*$. Then we have

$$\tilde{\Gamma}^{(N)}(0, \tau_0 t; g, m, \mu, \tau_0) \underset{t\to 0}{\longrightarrow} s_0^{D_N} \rho_0^{-N/2} t^{\nu D_N - (N/2)\gamma_t^*} \hat{\Gamma}^{(N)}(0, \{{}^{0}_{\tau_0/s_0}\}; g^*, \mu^*, \mu) \ \text{for} \begin{cases} \nu<1 \\ \nu=1 \end{cases}, \tag{5.25}$$

where use was made of (5.21) and (5.20):

$$\lim_{t\to 0} g(s(t), t) = g^* \ . \tag{5.26}$$

We can make $\hat{\Gamma}^{(N)}$ on the right-hand side of (5.25) nonvanishing by choosing μ^* ($\mu^* = s_0{}^{-1} m^* = t_0{}^{\nu} m^*$) suitably.

The λ behavior of $t(\lambda)$ in (5.18) or the t behavior of $s(t)$ in (5.23) originates from the critical behavior of the correlation length (5.15). Therefore (5.19) and (5.25) indicate that the critical behavior of static quantities is controlled by that of the correlation length $\xi(\tau)$, which proves the scaling hypothesis.

From (5.19) and (5.25), it is easy to obtain the scaling law

$$(2-\eta)=\gamma/\nu \tag{5.27}$$

by identifying that

$$\eta=\gamma_t^*/\nu=\gamma_t^*/\theta_t^* \;, \tag{5.28}$$

$$\gamma=2\theta_t^*-\gamma_t^* \tag{5.29}$$

with θ_t^* being the limiting value of $\theta_t(\ ,t)$. From the derivation of the critical exponents in this section, it is obvious that they are closely related to the low-momentum behavior of physical quantities, rather than the higher-momentum behavior, in the present formalism. The critical behavior strongly depends on the appearance of a massless mode at T_c. Therefore we may expect that the scaling law such as (5.27) continues to hold in each of various theories even below critical temperature as far as massive partners of Goldstone modes incline to masslessness.

When $\nu>1$, the low-momentum behavior at T_c is given by a relation similar to (5.19):

$$\tilde{\Gamma}^{(N)}(\lambda p,0;g,m,\mu,T_0)$$
$$=\lambda^{D_N}\rho(\ ,\lambda)^{-N/2}\tilde{\Gamma}^{(N)}\left[p,0;g(\lambda,\lambda),\frac{m(\ ,\lambda)}{\lambda},\mu,\tau_0\right]$$
$$\underset{\lambda\to 0}{\longrightarrow}\rho_0{}^{-N/2}\lambda^{D_N-(N/2)\gamma_t^*}\tilde{\Gamma}^{(N)}(p,0;g^*,0,\mu,\tau_0)\;. \tag{5.30}$$

However, since the $\tau\to 0$ limit at $p_i=0$ is obtained from

$$\tilde{\Gamma}^{(N)}(0,t\tau_0;g,m,\mu,\tau_0)$$
$$=\rho(\ ,t)^{-N/2}s^{D_N}\hat{\Gamma}^{(N)}\left[0,\frac{\tau_0 t}{s};g(s,t),\frac{m(\ ,t)}{s},\mu\right]\;, \tag{5.31}$$

there is no choice of s, which makes both t/s and $m(\ ,t)/s$ finite simultaneously with the condition $\lim_{t\to 0}m(\ ,t)/s\neq 0$. Therefore one needs to know the $m\to 0$ behavior of $\hat{\Gamma}^{(N)}(0,\tau_0;g^*,m,\mu)$; this limit usually induces an infrared catastrophe.

V. CONCLUSION

In this paper, we have studied the renormalization group in quantum field theory at finite temperature and presented several applications. Thermo field dynamics is a powerful ingredient in the analyses since the structure of Feynman diagrams is not changed and therefore the freedom of the temperature and momentum renormalization point comes in the theory naturally. Because of this freedom, we can relate the momentum change to the temperature change, which makes it possible to discuss, for example, high-momentum and high-temperature behaviors and the critical behavior in the renormalization-group approach. In thermo field dynamics, the frequency dependence is also included in the theory. Therefore no modification of the theory is required to analyze the dynamical critical behavior. Examples of explicit calculations of critical behavior in the present formulation and the extension to the dynamical critical behavior will be discussed elsewhere.

ACKNOWLEDGMENTS

This work was supported by the Natural Sciences and Engineering Research Council, Canada, and the Dean of the Faculty of Science, The University of Alberta, Edmonton, Canada.

[1]E. C. G. Stueckelberg and A. Peterman, Helv. Phys. Acta 26, 499 (1953); M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954); N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959), Chap. VIII.

[2]See, for example, D. J. Gross, in *Method in Field Theory,* edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).

[3]K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974); E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6.

[4]L. Leplae, F. Mancini, and H. Umezawa, Phys. Rep. 10C, 151 (1974); Y. Takahashi and H. Umezawa, Collect. Phenom. 2, 55 (1975); H. Matsumoto, Fortschr. Phys. 25, 1 (1977); H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North-Holland, Amsterdam, 1982).

[5]H. Matsumoto, Y. Nakano, H. Umezawa, F. Mancini, and F. Marinaro, Prog. Theor. Phys. 70, 599 (1983).

[6]H. Matsumoto, I. Ojima, and H. Umezawa, Ann. Phys. (N.Y.) (to be published).

[7]See, for example, A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Method of Quantum Field Theory in Statistical Physics* (Pergamon, Oxford, 1965).

[8]C. Bernard, Phys. Rev. D 9, 3312 (1974); L. Dolan and R. Jackiw, *ibid.* 9, 3320 (1974); S. Weinberg, *ibid.* 9, 3357 (1974).

[9]See, for a review and references, D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. 54, 43 (1981).

[10]I. Ojima, Ann. Phys. (N.Y.) 137, 1 (1981).

[11]H. Matsumoto, Y. Nakano, and H. Umezawa, Phys. Rev. D 28, 1931 (1983).

[12]See, for example, S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1975).

[13]A. J. Niemi and G. W. Semenoff, MIT Report No. MIT-CTP-1092 (unpublished).

[14]The relation between the scale changes of temperature and another dimensionful variable (e.g., chemical potential, particle density, etc.) was discussed also in the zero-temperature-renormalization scheme by J. C. Collins and M. J. Perry, Phys. Rev. Lett. 34, 1353 (1975).

[15]J. Whitehead, H. Matsumoto, and H. Umezawa, Phys. Rev. B 29, 423 (1984).

[16]A different method from the present one, in which temperature-dependent vertex functions are defined, has been attempted by G. Jona-Lasinio, in *Critical Phenomena in Alloys, Magnets, and Superconductors,* edited by R. C. Mills, E. Ascher, and R. I. Jaffee (McGraw-Hill, New York, 1971).